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FACTORIAL MOMENT - GENERATING FUNCTION AND THE PASCAL DISTRIBUTION

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ABSTRACT

Given a distribution, the cumulants or factorial moments can be used to obtain the skewness and kurtosis which in turn are used to determine the normal approximation of the given distribution. It is shown in this paper that for the Pascal Distribution, the factorial moment generating function provides a simpler technique. It is also shown that the probability that at the xth trial, we will obtain the kth success can be obtain easily from the factorial moment generating function of the Pascal distribution.

INTRODUCTION

The rth moment of a distribution can be computed directly or by using the moment- generating function. It is observed that for some distributions the moment - generating function provides an easier method of obtaining the moments than direct computation.

The factorial moment generating function offers even a further simplification. The cumulants or factorial moments- can be used to obtain the Skewness or kurtosis of a distribution, which are used to determine the normal approximation of the given distribution. Cumulants are obtained from the moment-generating function while factorial moments are obtained from the factorial moment- generating function. This paper presents both the cumulant and factorial moment- generating function approaches for obtaining the Skewness of the pascal distribution.

In calculating the probability associated with a random variable having a Pascal distribution, information can be obtained from the factorial moment- generating function of the distribution.

The paper also shows the relationship that exists between the rth factorial moment of the pascal distribution denoted by $\mu_{(r)\ \text{pascal}}$ and the rth factorial moment of the binomial distribution $\mu_{(r)\ \text{Binomial}}$. Closely related to the Binomial distribution is the pascal distribution . While the binomial distribution is concerned with the number of successes x in n trials, the pascal distribution also called the negative binomial distribution or the binomial waiting time distribution focuses on the positional occurrence of a success. Their probability distributions are then used to calculate the probabilities associated with the trials. Press et al.(1992) worked on the negative binomial distribution (pascal distribution) and obtained the moment-generating function, the mean, variance, Skewness and Kurtosis. Bruce, Richard (1997) gave the normal approximation of the binominal distribution giving the interval for the probability of success on each trial. Shangodoyin et al. (2002) obtained three types of moments, namely, moment about the origin, moment about the mean and factorial moments for both discrete and continuous random variables. Rao (1965) derived the moment-generating function, and probability generating function for the binomial distribution. upon which the pascal distribution is built.

The probability that the kth success occurs in the xth trial is given as

$$f(x,k,\theta)=\binom{x-1}{k-1}\theta^{k}\left(1-\theta\right)^{x-k}$$

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Where θ is the probability that the kth success occurs in the xth trial.

Skewness = $\frac{\mu_{(3)}}{\sigma^3} = \frac{k_3}{\sigma^3}$

Where $\mu_{(r)}$ is the rth factorial moment and K_r is the rth cumulant.

Cumulants of the Pascal Distribution

The moment- generating function of the Pascal Distribution can be obtained as follows:

$$M_x^{(t)} = E\left(e^{tx}\right) = \sum_{x=0} e^{tx} f(x)$$

$$= \sum_{\alpha=0} e^{t\alpha} \binom{x-1}{k-1} \theta^k (1-\theta)^{x-k} \quad = \sum_{\alpha=0} e^{tx} \binom{\beta}{\alpha} \theta^k \gamma^{\beta-\alpha}$$

$$= \sum_{x=0}^{\beta} e^{t\alpha} \binom{\beta}{\alpha} \left(e^t \gamma^{(\beta/\alpha-1)}\right)^{\alpha} \theta^k$$

$$\sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \left(e^t \gamma^{(\beta/\alpha-1)}\right)^{\alpha} \theta^{\beta-\alpha} \theta^{2\alpha-\beta+1}$$

(by letting $x = k - 1 and \beta = x - 1$)

$$= \left(\theta + e^t \gamma\left(\beta/_{\alpha} - 1\right)\right)^{\beta} \theta^{2x-\beta+1}$$

Thus the moment- generating function of the Pascal distribution is given as

$$M_x(t) = \left(e^t \gamma\left(\beta/_{x} - 1\right) + \theta\right)^{\beta} \theta^{2x-\beta+1}$$

The rth cumulant

Freund (1992) stated that the coefficient of $\frac{t^r}{r!}$ in the Maclaurin's series of

$K_x^{(t)} = \ln M_x(t)$ is called the rth cumulant denoted by K_r.

Thus the cumulant – generating function associated with the random variable is

$$k_x(t) = \ln M_x(t) = \ln \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta}$$

$$= (2\alpha - \beta + 1) \ln \theta + \beta \ln \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)$$

$$K_r = k_x^{(r)}(t)\Big|_{t=0}$$

$$k_x^{(1)}(t) = \beta \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 1} e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} \theta^{2\alpha - \beta + 1}$$

$$= \beta \theta^{2x - \beta + 1} \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 1} e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)}$$

$$= \beta \theta^{2x - \beta + 1} \left(\gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 1} \gamma^{\left(\frac{\beta}{\alpha} - 1\right)}$$

$$k_x^{(2)}(t) = (\beta - 1)\beta \theta^{2\alpha - \beta + 1} \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 2} e^t \gamma^{\left(\frac{\beta}{2} - 1\right)}$$

$$+ \beta \theta^{2x - \beta + 1} \left(e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 1} e^t \gamma^{\left(\frac{\beta}{\alpha} - 1\right)}$$

$$k_2 = (\beta - 1)\beta \theta^{2\alpha - \beta + 1} \left(\gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 2} \gamma^{\left(\frac{\beta}{\alpha} - 1\right)}$$

$$+ \beta \theta^{2x - \beta + 1} \left(\gamma^{\left(\frac{\beta}{\alpha} - 1\right)} + \theta \right)^{\beta - 1} \gamma^{\left(\frac{\beta}{\alpha} - 1\right)}$$

It can be seen that the cumulant approach is quite complex and labourious.

Factorial Moment -Generating Function

The Factorial Moment- Generating Function $F_x(t)$ can be stated as

$$F_X(t) = E\left(t^X\right) = \sum_x t^x f(x) \qquad \text{(discrete case)}$$

with the rth factorial moment of x given as:

$$\mu'_{(r)} = E\left[x(x-1)(x-2)\ldots(x-r+1)\right]$$

$$= \frac{d^r F_X}{dt^r}(t)\Big|_{t=1}$$

or $F_X(t) = E(1+t)^X = \sum (1+t)^x f(x)$

with $\mu'_{(r)} = \frac{d^r F(t)}{dt^r}\Big|_{t=0}$

The fmgf of the pascal distribution is given as

$F_x(t) = E(1+t)^x = E(1+t)^{k-1}$

$$= \sum_{\alpha=0} (1+t)^\alpha \binom{\beta}{\alpha} \theta^k (1-\theta)^{x-k} \qquad (\text{with } \alpha = k-1, \beta = x-1)$$

$$= \sum_{\alpha=0} (1+t)^\alpha \binom{\beta}{\alpha} \theta^\alpha \theta^{-1} (1-\theta)^{x-k}$$

$$= \sum_{\alpha=0} (\theta(1+t))^\alpha \binom{\beta}{\alpha} \theta^{-1} \gamma^{x-k}$$

$$= \sum_{\alpha=0} (\theta(1+t))^\alpha \binom{\beta}{\alpha} \theta^{-1} \gamma^{\beta-\alpha}$$

$$= (\theta(1+t))^0 \binom{\beta}{0} \theta^{-1} \gamma^{\beta-0}$$

$$+ (\theta(1+t))^1 \binom{\beta}{1} \theta^{-1} \gamma^{\beta-1} + (\theta(1+t))^2 \binom{\beta}{2} \theta^{-1} \gamma^{\beta-2} + \ldots$$

$$+ (\theta(1+t))^3 \binom{\beta}{3} \theta^{-1} \gamma^{x-4} + \ldots + (\theta(1+t))^m \binom{\beta}{\beta} \theta^{-1} \gamma^{\beta-\beta} = \theta^{-1}(\theta(1+t)+\gamma)^\beta$$

$$\theta^{-1}(1-\theta+\theta(1+t))^\beta = \theta^{-1}(1-\theta+\theta+\theta t)^\beta = \theta^{-1}(1+\theta t)^\beta$$

The rth factorial moment

$$\mu_{(r)} = \sum_{x=0} (x)_r f^{(x)} = E[(x)_r]$$

where $(x)_r = x(x-1) \ldots\ldots(x-r+1)$

Factorial moments are obtained from factorial moment- generating functions as follows:

$$\mu_{(r)} = F_x^{(r)}\Big|_{t=0}$$

where

$$F^{(r)} = \frac{d^r f}{dt^r}$$

$F(t)= \theta^{-1} (1+ \theta)^{\beta}$

$$\mu(1) = F^{(1)}(t)\Big|_{t=0} = \beta\theta^{-1}(1+\theta t)^{\beta-1}\theta\Big|_{t=0}$$

$$= \beta(1+\theta t)^{\beta-1}\Big|_{t=0} = \beta$$

$$F^{(2)}(t) = \beta(\beta-1)(1+\theta t)^{\beta-2}\theta$$

$$= \theta\beta(\beta-1)(1+\theta t)^{\beta-2}$$

$$\mu_{(2)} = F^{(2)}(t)\Big|_{t=0} = \theta\beta(\beta-1)$$

$$F^{(3)}(t) = \theta^2\beta(\beta-1)(\beta-2)(1+\theta t)^{\beta-3}$$

$$\mu(3) = F^{(3)}\Big|_{t=0} = \theta^2\beta(\beta-1)(\beta-2)$$

$$\mu_{(r)} = \theta^{r-1}\beta(\beta-1)(\beta-2)\ldots(\beta-r+1) = F^{(r)}(0)$$

$$= \beta^{(r)}\theta^{r-1}$$

where $\beta^{(r)} = (\beta-1)\beta-2\ldots(\beta-r+1)$

This can be compared with the rth factorial moment of the binomial distribution given as

$$\mu_{(r)} = \beta^{(r)}\theta^r$$

We can now state the relationship between the binomial distribution and the pascal distribution as follows:

$\mu_{(r)pascal} = \theta^{-1} \mu_{(r)Binomial}$

PROBABILITY APPLICATION

The probability that at the xth trial, we will obtain the kth success can be obtained very easily from the factorial moment - generating function of the pascal distribution.

ILLUSTRATION

The probability that on tossing a fair die the face "six" will occur the 4th time at the tenth trial can be obtained from the factorial moment-generating function given as

$$F_x(t) = 6\left(1+\frac{1}{6}t\right)^9$$

Thus P(xth trial =10, kth success =4)

$$= \binom{9}{3}\left(\frac{1}{6}\right)^4\left(1-\frac{1}{6}\right)^{10-4} = 0.022$$

CONCLUSION

Either the rth factorial moment or the rth cumulant of the Pascal distribution can be used to obtain the Skewness of the distribution. It has been shown in this paper that the factorial moment - generating function provides a simpler technique over the derivation of the rth cumulant. It is therefore recommended that the factorial moment should always be used in the computation of the kurtosis of the Pascal distribution if none of them is given. The relationship between the rth moment of the Pascal distribution and the binomial distribution is given as

$$\mu_{(r)Pascal} = \theta^{-1}\ \mu_{(r)\ Binomial}$$

where θ is the probability of a success on the xth trial.

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