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A NECESSARY AND A SUFFICIENT CONDITION FOR THE EXISTENCE OF OPTIMIZERS OF LINEAR PROGRAMMING PROBLEMS

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ABSTRACT

Umoren (2000) has given the Linear Exchange Algorithm (LEA) for solving certain class of LP problems using the principles of optimal experimental design. This work sets out to establish, first, a first – order necessary condition for the existence of optimizers (minimizers) of certain class of LP problems. That is, if the d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space, then the d-function at the minimizer $\underline{x}^*$ is the maximum of the minimum d-functions for k different iterations. Second, we also establish the fact that the differential d-functions between the starting point $\underline{x}_{ok}$ and the end point $\underline{x}_k$ of the kth iteration in the LEA is a non-increasing function. This is a sufficient condition for the existence optimizers of LP problems.

Key Word: Linear Exchange Algorithm, Experimental Design, Linear Programming Problems, d-function.

INTRODUCTION

Discussions on some optimality conditions for the existence of optimizers in LP problems are already given in Umoren (2001). We shall in this work specifically consider the necessary and sufficient conditions for the existence of optimizers of certain class of LP problems, namely, the class for which the objective function $f(x) \geq 0$. Each of the conditions (necessary or sufficient) is based on the Linear Exchange Algorithm (LEA) itself, a line search algorithm, which makes use of the principles of optimal experimental design, the sequence and operation which are already given in Umoren (2000). We have also shown that the LEA has a down-hill property (see Umoren 2001). Thus the value of the objective function at the (k + 1) th iteration is less than its value at the kth iteration for the minimization problem. Furthermore, we have shown that the d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space (see Umoren 2001).

First – Order Necessary Condition

We shall now state the first - order necessary condition for the existence of optimizers of LP problems. In particular, we establish the fact that if the d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space S_x, the d-function at the minimizer $\underline{x}^*$ is the maximum of the minimum d- functions for k different iterations. But before that let us state a lemma which is fundamental to the proof of the theorem that follows.

Lemma 1

Given a line search equation

$$\underline{x}_{j+1} = \underline{x}_j + \alpha_j d_j ; \qquad d_j d_j = 1$$

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where $\underline{d}_j$ and α_j are respectively the direction of search and step length at the jth iteration. Let M_j, $M_{j+1} \in M^{mxm}$, be information matrices at the jth and (j+1)th iteration respectively; M^{mxm} is the set of all information matrices. Then

(i) $\det(M_{j+1}) = \det(M_j)(1 + n_o\alpha_j^2 \underline{d}'_j M_j^{-1} \underline{d}_j)$ and

$$\frac{\det(M_{j+1})}{\det(M_j)} = 1 + w_j, \quad \text{where}$$

$$w_j = n_o\alpha_j^2 \underline{d}'_j M_j^{-1} \underline{d}_j$$

(ii) $M_{j+1}^{-1} - z_j z_j'$; $z_j = n_o^{1/2} \alpha_j (1 + w_j)^{1/2} M_j^{-1} d_j$

Proof: The proof is similar to the one given in Umoren (2001) and is here omitted.

Theorem 1

Given $\underline{x}^*$ to be a minimizer of an LP problem. Then,

$$\underline{x}^* M_*^{-1} \underline{x}^* = \max \min \{\underline{x}' M_k^{-1} \underline{x}, \underline{x} \in \tilde{X}\}$$

is a first – order necessary condition to be satisfied by $\underline{x}^*$; M_* is the information matrix at the point where $\underline{x}_k$ converges to $\underline{x}^*$; $\underline{x}_k$ and M_k are the end point and information matrix at the kth iteration respectively.

Proof:

Let $\lim_{k \to \infty} M_k = M_*$

i.e. $|M_1| \geq |M_2| \geq \ldots \geq |M_*|$ and let

$$M_k = M_* + d_k d'_k, \; d_k \geq 0$$

Then from Lemma 1

$$M_k^{-1} = M_*^{-1} - v_k v'_k$$

$$v_k = n_o^{1/2} \alpha_{ok}(1 + w_k)^{1/2} M_*^{-1} \underline{d}_k; \; w_k = n_o \alpha_{ok}^2 \underline{d}'_k M_*^{-1} \underline{d}_k.$$

Thus,

$$\underline{x}'_k M_k^{-1} \underline{x}_k = \underline{x}'_k [M_*^{-1} - v_k v'_k] \underline{x}_k$$

$$= \underline{x}'_k M_*^{-1} \underline{x}_k - \underline{x}'_k v_k \underline{v}'_k \underline{x}_k$$

$$\Rightarrow \quad \underline{x}'_k M_*^{-1} \underline{x}_k = \underline{x}'_k M_k^{-1} \underline{x}_k + \underline{x}'_k v_k \underline{v}'_k \underline{x}_k.$$

$$\Rightarrow \quad \underline{x}'_k M_*^{-1} \underline{x}_k \geq \underline{x}'_k M_k^{-1} \underline{x}_k. \text{ since } \underline{x}'_k v_k \underline{v}'_k \underline{x}_k \geq 0$$

$$\Rightarrow \quad \underline{x}^{*\,\prime} M_*^{-1} \underline{x}^* \geq \underline{x}'_k M_k^{-1} \underline{x}_k.$$

And from the fact that

$\underline{x}'_{k+1} M_k^{-1} \underline{x}_{k+1} \geq \underline{x}'_k M_k^{-1} \underline{x}_k$. (See Umoren, 2001)

we have

$$\underline{x}^{*\prime} M_*^{-1} \underline{x}^* = \max \{\underline{x}'_k M_k^{-1} \underline{x}_k\}$$

But it is well known (see Umoren 2001) that

$\underline{x}_k' M_k^{-1} \underline{x}_k = \min\{\underline{x}' M^{-1} \underline{x}\}, \underline{x} \in S_x$

Therefore

$$\underline{x}^{*\prime} M_*^{-1} \underline{x}^* = \max \min\{\underline{x}' M_k^{-1} \underline{x} \,;\, \underline{x} \in \tilde{X} \}$$

for the minimization problem.

Using the fact that min f (x)), = max(-f(x)), we have

$$\underline{x}^{*\prime} M_*^{-1} \underline{x}^* = \min \max \{\underline{x}' M_k^{-1} \underline{x} \,;\, \underline{x} \in \tilde{X}\}$$

for the maximization problem. Thus, the optimizer of an LP problem has both the maximin and minmax properties since it could be reached either through minimization or maximization.

A Sufficient Condition for the Existence of Optimizers

We now give a sufficient condition for the existence of optimizers of LP problems, namely, the fact that the differential d-functions between the starting point $\underline{x}_{ok}$ and the end point $\underline{x}_k$ of the kth iteration in the LEA is a non-increasing function. But before that, we shall state the Kolmogorov's condition for convergence which is fundamental to the proof of the theorem that follows.

Kolmogorov's Criterion for Absolute Convergence (see Feller, 1966).

A sufficient condition for a series of independent random variables $\{\underline{x}_j\}$ with finite variance $\{\sigma_j^2\}$ to be convergent with probability one is for.

$$\sum \left(\frac{\sigma_j^2}{j} \right) < \infty$$

Theorem 2

Given the line search sequence

$$\underline{x}_k = \overline{\underline{x}}_{ok} - \alpha_{ok} \underline{g} \,;\, \overline{\underline{x}}_{ok} = X'_{ok} \underline{1}/n_o,$$

$$\overline{X}_{ok} M_k^{-1} \overline{X}_{ok} - X'_k M_k^{-1} X_k = q_k$$

$$= 2\alpha_{ok} g' M_k^{-1} \overline{X}_{ok} - \alpha_{ok}^2 g' M_k^{-1} g$$

and

$$\overline{X}'_{o(k+1)} M_{k+1}^{-1} \overline{X}_{o(k+1)} - X'_{k+1} M_{k+1}^{-1} X_{k+1} = q_{k+1}$$

$$= 2\alpha_{o(k+1)} g' M_{k+1}^{-1} \overline{X}_{o(k+1)} - \alpha_{o(k+1)}^2 g' M_{k+1}^{-1} g$$

where $\overline{X}_{ok} \geq \overline{X}_{o(k+1)}$

A sufficient condition for $q_k \geq q_{k+1}$ is that
$\alpha_{ok} \geq \alpha_{o(k+1)}$.

Proof:
$q_k \geq q_{k+1} \Rightarrow 2\alpha_{ok} g' M_k^{-1} \overline{X}_{ok} - \alpha_{ok}^2 g' M^{-1} g$
$\geq 2\alpha_{o(k+1)} g' M_{k+1}^{-1} \overline{X}_{o(k+1)} - \alpha^2_{o(k+1)} g' M_{k+1}^{-1} g$ ... ... (i)

A sufficient condition for (i) above to hold is that

(a) $\alpha^2_{ok} g' M_k^{-1} g \leq \alpha^2_{o(k+1)} g' M_{k+1}^{-1} g$

(b) $2\alpha_{ok} g' M_k^{-1} \overline{X}_{ok} \geq 2\alpha_{o(k+1)} g' M_{k+1}^{-1} \overline{X}_{o(k+1)}$

Consider the transformation
$r_k = \alpha^2_{ok} M_k^{-1} g$, $r_{k+1} = \alpha^2_{o(k+1)} M_{k+1}^{-1} g$ ... ... (ii)

Then the inequality (a) above implies

$\alpha_{ok}^{-2} r'_k M_k r_k \leq \alpha_{o(k+1)}^{-2} r'_{k+1} M_{k+1} r_{k+1}$ ... ... (iii)

From the transformation we get
$g = \alpha_{ok}^{-2} M_k r_k = \alpha_{o(k+1)}^{-2} M_{k+1} r_{k+1}$ ... ... (iv)

$$\Rightarrow \quad r_k = \left(\frac{\alpha_{ok}}{\alpha_{o(k+1)}}\right)^2 M_k^{-1} M_{k+1} r_{k+1}$$

$$\Rightarrow \quad r_k' M_k r_k = \left(\frac{\alpha_{ok}}{\alpha_{o(k+1)}}\right)^2 r'_{k+1} M'_{k+1} M_k^{-1} M_k M_k^{-1} M_{k+1} r_{k+1}$$

$$= \left(\frac{\alpha_{ok}}{\alpha_{o(k+1)}}\right)^2 r'_{k+1} M_{k+1} M_k^{-1} M_{k+1} \underline{r}_{k+1}$$

Therefore, (iii) above becomes

$$\frac{1}{\alpha^2_{0(k+1)}} \; r'_{k+1} M_{k+1} M_k^{-1} M_{k+1} \underline{r}_{k+1} \;\le\; \frac{1}{\alpha^2_{0(k+1)}} \; r'_{k+1} M_{k+1} M_{k+1} \underline{r}_{k+1}$$

$$\Rightarrow M_{k+1} M_k^{-1} M_{k+1} \le M_{k+1}$$

or

$$M_k^{-1} M_{k+1} \le I$$

which proves (a) above.
Similarly,

$$2\alpha_{ok} g' M_k^{-1} \overline{\underline{x}}_{ok} \ge 2\alpha_{0(k+1)} \underline{g}' M_{k+1} \overline{x}_{o(k+1)}$$

From (iv) above we get

$$\frac{\alpha_{ok}}{\alpha^2_{ok}} \underline{r}_k \overline{\underline{x}}_{ok} \ge \frac{\alpha_{o(k+1)}}{\alpha^2_{ok}} \underline{r}'_k M_k M^{-1}_{K+1} \overline{\underline{x}}_{o(k+1)}$$

$$\Rightarrow \frac{\alpha_{o(k+1)}}{\alpha^2_{ok}} M_k M^{-1}_{K+1} = I$$

$$\Rightarrow \left(M_k M^{-1}_{k+1} - \frac{\alpha_{ok}}{\alpha_{0(k+1)}}\right) = 0$$

$$\Rightarrow \frac{\alpha_{ok}}{\alpha_{o(dk+1)}} \text{ is a root of } M_k^{-1} M_{k+1}$$

which is positive. Therefore

$$\alpha_{ok} \ge \alpha_{o(k+1)}$$

which is a sufficient condition for $q_k \ge q_{k+1}$. This satisfies Kolmogorov's condition for absolute convergence. That is, $q_k \ge q_{k+1}$ is a sufficient condition for the sequence to converge absolutely. Thus, the differential d-functions between the starting point $\underline{x}_{ok}$ and end point $\underline{x}_k$ of the sequence is a non-increasing function as was required to be shown.

Numerical Illustration

We remind that the maxmin property of the d-function at the minimizer of an LP problem implies that the optimizer (minimizer) occurs at the point which the d-function is the maximum of the minima for k iterations. The d- function at the kth iteration is defined by $d_k(x, \xi_n) = x'_k M_k^{-1} \underline{x}_k$. We shall demonstrate this condition or property with the numerical example 1.

minimize $\quad f(x) = 3x_1 + 2x_1 \ldots \quad \ldots \quad (1)$

subject to $2x_1 + x_2 \geq 6$
$x_1 + x_2 \geq 4$
$x_1 + 2x_2 \geq 6$
$x_1, x_2 \geq 0$

Umoren (2000) has given the solution to this problem using the linear Exchange Algorithm (LEA). Our interest in this paper is to show that the optimizer occurs at the point where the d-function is the maximum of the minima for k different iterations. Table I gives the d-functions with the corresponding values of the objective function at four different iterations of the LEA for example 1.

Pazman (1986) has established a functional relationship between the square of the response function and the variance of the response function in Response Surface Methodology (RSM). Following from this, Umoren(2000) has established a functional relationship between the d-function of the objective function and the squuare of the objective function in Linear Programming Problems. Thus table 1 was obtained by employing this relationship which has been shown to be algebraically definite.

Table I: d-functions and values of the objective function at four different iterations of the Linear Exchange Algorithm for example

	x_1	x_2	$d_1(x,\xi n)$	$d_2(x, ,\xi n)$	$d_3(x, ,\xi n)$	$d_4(x, ,\xi n)$	f(x)
	0	6	0.3124	0.3752	0.4308	0.4495	12.00
	0.25	5.50	0.2995	0.3597	0.4130	0.4309	11.75
	0.50	5.00	0.2869	0.3446	0.3957	0.4128	11.5
	0.75	4.50	0.2745	0.3297	0.3786	0.3950	11.25
	1.00	4.00	0.2625	0.3152	0.3620	0.3777	11.00
	1.50	3.00	0.2392	0.2872	0.3298	0.3376	10.4
	1.75	2.50	0.2291	0.2752	0.3160	0.3297	10.28
	2.50	1.75	0.2625	0.3152	0.3620	0.3777	11.28
	3.00	1.50	0.3124	0.3752	0.4308	0.4495	12.00
	3.50	1.25	0.3666	0.4403	0.5056	0.5275	13.00
	4.00	1.00	0.4252	0.5106	0.5964	0.6118	14.00
	4.50	0.75	0.4881	0.5862	0.6731	0.7023	15.00
	5.00	0.50	0.5553	0.6670	0.7659	0.7991	16.00
	5.50	0.25	0.6269	0.7529	0.8646	0.9021	1700
	6.00	0	0.7028	0.8441	0.9693	1.0113	18.00
$\underline{x}_1$	2.24	1.88	0.2382*	0.2861	0.3256	0.3428	10.48
$\underline{x}_{01}$	2.67	2.17	0.3309				152.5225
$\underline{x}_2$	1.85	2,31		0.2695*	0.3094	0.3228	10.17
$\underline{x}_{02}$	2.08	2.46		0.3245			124.5456
$\underline{x}_3$	1.65	2.70			0.3204*	0.3344	10.35
$\underline{x}_{03}$	1.70	2.75			0.3361		112.36
$\underline{x}_4$	1.87	2.27				0.3216*	10.15
$\underline{x}_{04}$	1.91	2.30				0.3331	106.7089
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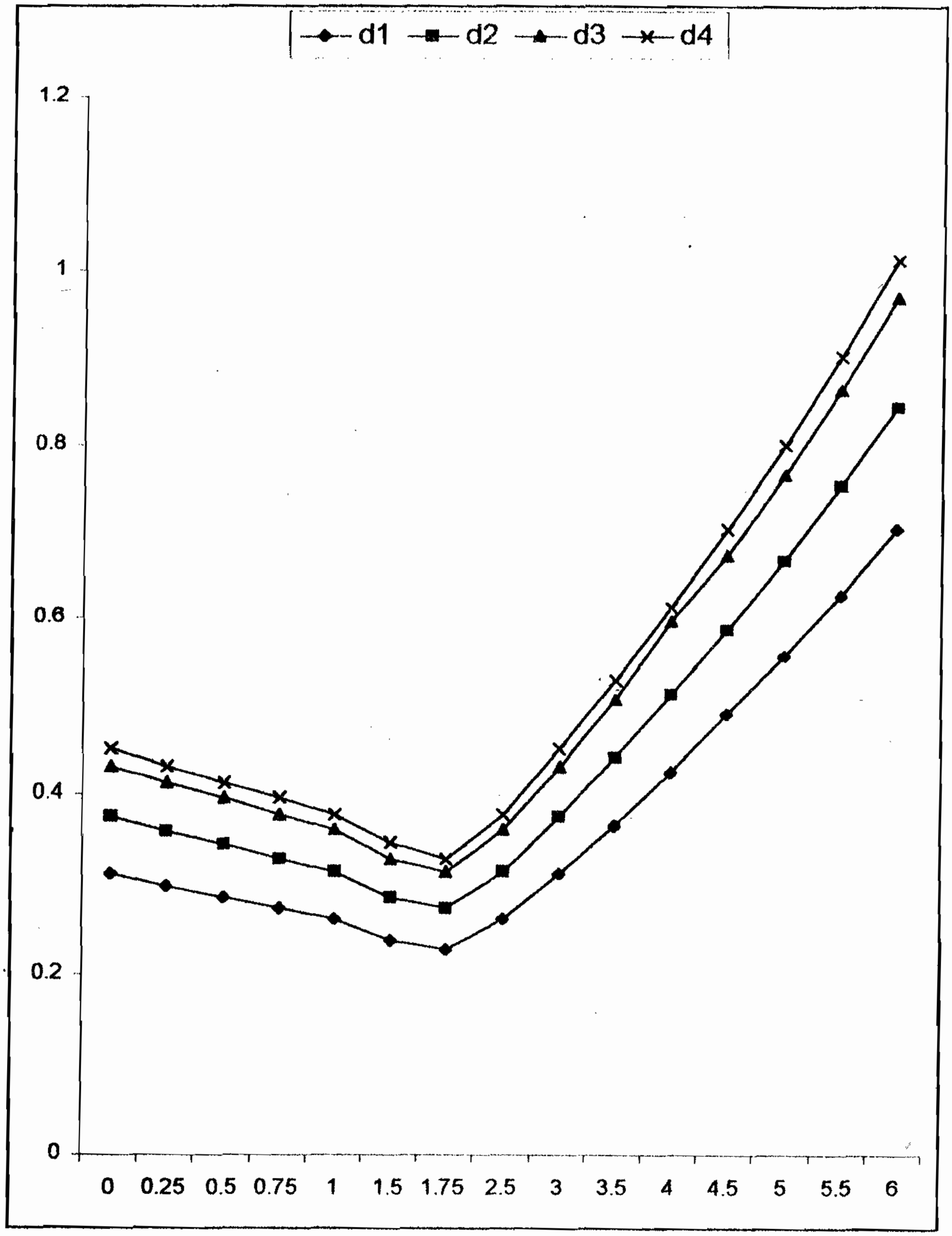

Figure 1:Plot of the d-function of the objective function at different iterations of the Linear Exchange Algorithm for Example 1.

Figure 1 is a plot of the d-functions of the objective function at four different iterations of the Linear Exchange Algorithm for example 1 which show that the minimizer $\underline{x}^*$ is at the point where the d-function is the maximum of the minimum d-functions of the four different iterations. This is a graphical illustration of the maxmin property of the minmizer $\underline{x}^*$ when the Linear Exchange Algorithm is employed.

Also, table 2 gives the differential d-functions between the starting point and the end point of the kth iteration for the four different iterations of the LEA.

Table 2: Differential d-functions of the objective function for k = 4 iterations of example I

k	$d(\overline{\underline{x}}_{ok}, \xi_n)$	$d(\underline{x}_k; \xi_n)$	$d(\overline{\underline{x}}_{ok}, \xi_n) - d(\underline{x}_k; \xi_n)$	α_{ok}
1	0.3309	0.2382	0.0927	0.1443
2	0.3245	0.2695	0.0550	0.0775
3	0.3361	0.3204	0.0157	0.01635
4	0.3331	0.3216	0.0115	0.0150

Results show that the differential d-function is a non-creasing function as the step-length α_{ok} of the search technique decreases. This agrees with the algebraic result of the sufficiency theorem for the existence of optimizers (minimizers) of LP problems. Thus, we have established in this work

(i) a necessary condition for the existence of optimizers of LP problems, namely, the maxmin property of the d-function of the objective function at $\underline{x}^*$.

(ii) a sufficient condition for the existence of minimizers of LP problems, namely that the differential d-function at the starting point and at the end point of the k iterations is a non-increasing function and this is accomplished with a decreasing step-length from one iteration to another.

REFERENCES

Feller, W., 1966. An introduction to Probability Theory and its Applications. 3rd ed . John Wiley and Sons, Inc. 509 P.

Pazman, A., 1986. Foundations of Optimum Experimental Design D-Reidal publishing Company. Boston, 225p.

Umoren, M. U., 2000. Application of optimal design theory to the solution of a constrained optimization problem. Journal of Natural and Applied Sciences. 1: 78 – 89.

Umoren, M. U., 2001. Some optimality conditions for the existence of optimizers of certain class of linear programming problems. Global Journal of Pure and Applied Sciences 7(2): 389 – 398.