GLOBAL JOURNAL OF MATHEMATICAL SCIENCES, VOL 2, NO. 1, 2003: 1-7
 ISSN 1596-6208

TWO-STEP RATIONAL CANONICAL FUNCTION IN THE NUMERICAL INTEGRATION OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

M. R. ODEKUNLE

(Received 8 March 2002; Revision accepted 30 May 2002).

ABSTRACT

This work concerns the approximation of the numerical solution of Initial Value Problem (IVP) by rational interpolants. Canonical polynomials were used as the rational interpolants. By collocation, an explicit nonlinear two-step scheme is obtained.
Numerical examples are provided to demonstrate the performance of the scheme. The results obtained were found to be quite comparable with those by existing schemes.

Key Words: Collocation, two-step scheme.

INTRODUCTION

Formulas based on polynomial interpolation like Linear Multistep Methods (LMMs) are in general known to perform poorer than those based on non-polynomial interpolating functions like rational functions. This has encouraged many authors like Lambert and Shaw (1965), Luke et al. (1975), Fatunla (1982) and Niekerk (1987) to investigate the use of rational function methods. While Lambert and Shaw made use of $(b+x)$, as a divisor, b a constant, Luke et al. made use of the generalized rational function

$$y(x) = \frac{N_s(x)}{D_t(x)} \tag{1a}$$

where

$$N_s(x) = \sum_{r=0}^{s} a_r x^r, \qquad D_t(x) = 1 + \sum_{r=1}^{t} b_r x^r \tag{1b}$$

This was specifically used to handle singularity problems in which case the singularities are specified by the zeros of $D_t(x)$.

One thing is common to all their proposals and that is the fact that they all obtained their schemes from the error function $E_{s,t}(x)$ where

$$E_{s,t}(x) = D_t(x)y(x) - N_s(x) \tag{2}$$

Instead of this, we shall use direct polynomial collocation as used by authors like De Boor and Swartz (1973), Hall and Watt (1976), which leads to the new scheme hereby discussed.

M. R. ODEKUNLE, Department of Mathematics and Computer Science, Fed. University of Technology, Yola, Nigeria

CANONICAL POLYNOMIALS

Let L be a differential operator defined as

$$L = \frac{d}{dx} + 1 \qquad (3)$$

and

$LQ_i = x^i$

then,

$$Lx^j = \frac{d}{dx}\left(x^j\right) + x^j = jx^{j-1} + x^j$$

$$= jLQ_{j-1}(x) + LQ_j(x)$$

That is,

$$Q_j(x) = x^j - jQ_{j-1}(x) \qquad j > 0 \qquad (4)$$

which gives

$Q_0(x) = 1$

$Q_1(x) = x - 1$ (5)

$Q_2(x) = x^2 - 2x + 2$ e.t.c.

THE METHOD

We shall solve

$y'(x) = f(x,y), \quad y(a) = \propto, \quad x \in [a, b]$ (6)

Let us assume that

$$y(x) \approx y_1(x) = \frac{a_0Q_0(x) + a_1Q_1(x)}{1 + b_0Q_0(x) + b_1Q_1(x)}$$

where Q_0 and Q_1 are as defined in (5) and the a_0, a_1, b_0 and b_1 are to be determined.

By (5),

$$y_1(x) = \frac{a_0 + a_1(x-1)}{1 + b_0 + b_1(x-1)} \qquad (7)$$

Let us assume that

$$y_1'(x) = f(x, y_1), \qquad y_1(a) = \alpha, \qquad x \in [a, b] \tag{8}$$

Collocate (8) at $x = x_{n-1}$ and $x = x_n$ to obtain

$$y_1'(x_{n-1}) = f(x_{n-1}, y_1(x_{n-1})) = f_{n-1}, \quad y_1(x_{n-1}) = y_{n-1}, \quad x \in [x_{n-1}, x_n] \tag{9a}$$

$$y_1'(x_n) = f(x_n, y_1(x_n)) = f_n, \quad y_1(x_n) = y_n, \quad x \in [x_n, x_{n+1}] \tag{9b}$$

From (7),

$$y_1'(x) = \frac{a_1(1+b_0) - a_0 b_1}{[1 + b_0 + b_1(x-1)]^2} \tag{10}$$

So that by (9),

$$\left[\frac{1 + b_0 + b_1(x_{n-1} - 1)}{1 + b_0 + b_1(x_n - 1)}\right]^2 = \left(\frac{f_n}{f_{n-1}}\right) \tag{11}$$

If we define

$$F_{n-1} = \left(\frac{f_n}{f_{n-1}}\right)^{\frac{1}{2}} \tag{12}$$

then from (11),

$$b_1 = \frac{(1+b_0)(1 - F_{n-1})}{F_{n-1}(x_n - 1) - (x_{n-1} - 1)} \tag{13}$$

From the initial conditions:

$$y_1(x_{n-1}) = y_{n-1}$$
$$y_1(x_n) = y_n$$

we have that,

$$a_1 = \frac{(1+b_0)(y_n - F_{n-1} y_{n-1})}{F_{n-1}(x_n - 1) - (x_{n-1} - 1)} \tag{14}$$

and

$$a_0 = \frac{(1+b_0)[F_{n-1} y_{n-1}(x_n - 1) - y_n(x_{n-1} - 1)]}{F_{n-1}(x_n - 1) - (x_{n-1} - 1)} \tag{15}$$

Substitute (13), (14) and (15) in (7) we have,

$$y_1 = \frac{F_{n-1}y_{n-1}(x-x_n) - y_n(x-x_{n-1})}{F_{n-1}(x-x_n)-(x-x_{n-1})} \tag{16}$$

Equation (16) is the continuous form of our scheme (Taiwo and Onumanyi, 1991).

If we collocate at $x = x_{n+1}$ then, we obtain the scheme

$$y_{n+1} = \frac{F_{n-1}y_{n-1} - 2y_n}{F_{n-1} - 2} \tag{17}$$

which is a two-step scheme.

NUMERICAL EXPERIMENT AND DISCUSSION

Problem 1: $y' = y$, $y(0) = 1$, $y_{exact} = e^x$

Problem 2: $y' = -5xy^2 + \frac{5}{x} - \frac{1}{x^2}$, $y(1) = 1$, $y_{exact} = \frac{1}{x}$

Problem 3: $y' = 1 + y^2$, $y(0) = 1$, $0 \le x \le 1$, $y_{exact} = \tan(x + \frac{\pi}{4})$

In all the results obtained, the extra step y_1 needed to start the scheme is obtained from the exact value of the respective problem.

Tables 1 and 2 show the performance of the scheme for problem 1 for h = 0.05 and h = 0.01. The

Table 1: Numerical result of problem 1 when h = 0.05

h = 0.05 / x	$y_{exact}(x)$	$Y_{n+1}(x)$ Eqn. (17)	Error = $\lvert y_{exact}(x) - Y_{n+1}(x) \rvert$	
			Eqn. (17)	Oladele (1997)
0.10	1.105171	1.105205	3.400000E-05	9.666681E-04
0.20	1.221403	1.221632	2.290000E-04	2.135634E-03
0.30	1.349859	1.350493	6.340000E-04	3.538847E-03
0.40	1.491825	1.493136	1.311000E-03	5.212307E-03
0.50	1.648721	1.651053	2.332000E-03	7.197380E-03
0.60	1.822119	1.825905	3.786000E-03	8.432121E-03
0.70	2.013753	2.019532	5.779000E-03	9.013782E-03
0.80	2.225541	2.233978	8.437000E-03	2.007813E-02
0.90	2.459603	2.471514	1.191100E-02	2.063418E-02
1.00	2.718282	2.734660	1.637800E-02	2.368140E-02

Table 2: Numerical result of problem 1 when h = 0.01

h = 0.01 / x	$y_{exact}(x)$	$Y_{n+1}(x)$ Eqn. (17)	Error = $\mid y_{exact}(x) - Y_{n+1}(x) \mid$ Eqn. (17)	Oladele (1997)
0.01	1.010050	1.010050	-	1.704693E-05
0.02	1.020201	1.020202	1.000000E-06	3.433228E-05
0.03	1.030455	1.030455	0.000000E-00	5.197525E-05
0.04	1.040811	1.040812	1.000000E-06	7.009506E-05
0.05	1.051271	1.051274	3.000000E-06	8.845329E-05
0.06	1.061837	1.061840	3.000000E-06	1.072884E-04
0.07	1.072508	1.072514	6.000000E-06	1.263618E-04
0.08	1.083287	1.083295	8.000000E-06	1.457930E-04
0.09	1.094174	1.094184	1.000000E-05	1.657009E-04
0.10	1.105171	1.105183	1.200000E-05	1.859665E-04
...	...	...	...	...
1.00	2.718282	2.721555	3.273000E-03	4.570246E-03

results obtained were compared with that of Oladele (1997), which is even an implicit two-step method. Implicit methods are generally known to perform better than explicit ones Lambert (1973). Table 3 shows the performance of the scheme for the non-linear problem 2 and the effect of the step-lengths on the result obtained. Clearly from the table, smaller step-lengths do not necessarily imply better approximation. The best result is obtained when h = 0.2. As h decreases, the result becomes poorer. The same applies when h ≥ 0.3. This is an indication of stability problem. That is, there is a range of h for which the scheme is stable outside which the results obtained become unreasonable for every particular problem. This is still being investigated. Notwithstanding, the results obtained by this new scheme is quite comparable with that by Lambert (1973), (page 100), and with less computational effort.

Table 3: Errors obtained in problem 2 for various values of h

x	h = 0.01	h = 0.03	h = 0.05	h = 0.10	h = 0.15	h = 0.20
1	0	0	0	0	0	0
21	1.781740E-07	6.655906E-08	3.053061E-08	1.403575E-08	1.034665E-08	7.384455E-09
41	2.254667E-07	7.399683E-08	1.243452E-08	1.314471E-08	1.169800E-09	1.471540E-09
61	2.919526E-08	6.548147E-08	4.845556E-09	3.741270E-09	7.088517E-09	2.538451E-09
81	1.089750E-08	3.133546E-08	1.260990E-08	5.848469E-09	5.626469E-09	3.152430E-09
91	1.998246E-08	2.666174E-08	1.635994E-08	7.782513E-09	4.433212E-09	4.622860E-09
100	1.430909E-09	4.680764E-08	3.027414E-09	1.418280E-09	4.941693E-09	1.981413E-09

Table 4: Errors obtained in problem 3 for h = 0.05 using the new scheme and implicit Trapezoidal scheme

X	Equation (17)	Trapezoidal
0.10	3.110266E-04	5.0E-04
0.20	2.434277E-03	2.0E-03
0.30	80472958E-03	4.0E-03
0.40	2.412355E-02	1.0E-02
0.50	6.809590E-02	3.0E-02
0.60	2.265051E-01	2.0E-01
0.70	1.341315E-00	3.4E-00
0.75	7.4536[illegible]3E-00	Unreasonable result

Table 5: Problem 3 at the point where $x = 0.75$, $y_{exact}(x) = 28.238253$

h	y_{n+1}	Error = $\lvert y_{exact}(x) - Y_{n+1}(x) \rvert$
0.05	20.784640	7.453613
0.025	23.754970	4.483283
0.0125	25.740430	2.497823
0.01	26.192190	2.046063
0.00625	26.901280	1.336973
0.003125	27.589940	0.648313
0.0015625	27.848810	0.389443

Problem 3 is singular with simple pole at $x = \pi/4$. Table 4 gives the errors for h = 0.05. The results obtained close to the singularity point and after are poor and inconsistent – in fact, the errors after the point of singularity were diverging. This is expected due to F_{n-1} as there is a change of sign in the value of the true solution, which the scheme was not able to keep track of. The results in Table 4 shows the comparison with that of implicit Trapezoidal scheme that enjoys one function evaluation as our scheme although implicit as described by Lambert. For this problem, decrease in the step length really improves the result as shown in Table 5.

CONCLUSION

In general, the results obtained were very comparable to that obtained by other existing even implicit two-step schemes with one function evaluation at each step using the problems in the experiment although our scheme is an explicit two – step scheme.

REFERENCES

De Boor, C., and Swartz, B., 1973. Collocation at Gaussian Points. SIAM J. Numer. Analy. 10: 582-606

Fatunla, S. O., 1982. Non-linear Multistep Methods for Initial Value Problems. Computers and Mathematics with Applications 8. 231-239

Hall, H., and Watt. J.M., 1976. Modern Numerical Methods for ODEs. Oxford Univ. Press. London 336pp

Lambert, J.D., 1973. *Computational Methods in ODEs*. New York: John Wiley.

Lambert, J.D., and Shaw, B., 1965. On the Numerical Solution of $y' = f(x,y)$ by a class of Formulae based on Rational Approximations. Mathematics and Computation 19: 456-462

Luke, Y.L.,Fair, W., and Wimp, J., 1975. Predictor Corrector formulas Based on Rational Interpolants. J. on Computer and Mathematics with Applications 1 : 3-12

Niekerk, F.D.D., 1987. Nonlinear One-step Methods for IVPs. Computers and Mathematics with Applications 4, 367-371

Oladele, J.O., 1997. An Implicit Runge-Kuttta Method for IVPs in ODEs. Abacus 25(2): 482 - 489

Taiwo, O. and Onumanyi, P., 1991. A Collocation Approximation of Singularly Perturbed Second Order O.D.E. Intern. J. Computer Math. 39: 205 – 211