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ON THE SEGMENTATION OF THE RESPONSE SURFACES FOR SUPER CONVERGENT LINE SERIES' OPTIMAL SOLUTIONS OF CONSTRAINED LINEAR AND QUADRATIC PROGRAMMING PROBLEMS

P. E. CHIGBU and T. A. UGBE

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ABSTRACT

The solutions of Linear and Quadratic Programming Problems using Super Convergent Line Series involving the Segmentation of the Response Surfaces are presented in this paper. It is verified that the number of segments, S, for which optimal solutions of these problems selected for verification are obtained are 2 and 4 for the Linear and Quadratic Programming Problems, respectively. On the whole, the line search exchange algorithm is exploited.

Key Words : Exchange Algorithm, Response Surfaces, Segmentation, Super Convergent Line Series, Support points

INTRODUCTION

Response Surface Methodology offers effective means for using experimental design principles to determine the Optimizer of a real-valued function and embodies techniques for performing experiments sequentially: see, for example, Onukogu (1997). Response Surface Methodology (RSM), therefore, can be seen as a bridge linking the subject of experimental design with the subject of Unconstrained Optimization.

Umoren (1996) also showed that Response Surface Methodology linked experimental design with constrained optimization. An algorithm known as the Maximum Norm Exchange Algorithm which is made on the basis of the maximum norm of the support points used to form the initial design matrix was developed in Umoren (1999). Besides finding the optimizer, response surface methodology also deals with the exploration of the surface around the optimum point. The aim is to gain better understanding of the optimal region and, if possible, determine alternative optimal solutions that could be technically or economically more feasible than the initial one found.

Linear Programming Problems belong to a class of constrained convex optimization problems which have been widely discussed by several authors; see, for example, Dantzig (1963) and Philip, Walter and Wright (1981). The commonly used algorithm for solving linear programming problems are: the simplex method and active set method which require the use of artificial and slack variables. Umoren (1996) developed some line search techniques approach which have been considered useful in Optimal experimental design. His approach avoids the use of surplus (slack) and artificial variables.

Quadratic Programming Problems, on the other hand, make use of Kuhu- Tucker conditions and Frank – Wolf Algorithm for obtaining optimal solutions: see, for example, Taha (1987) and Hillier and Lieberman (1995). In this work, line search techniques via Super Convergent Line Series have been made use of to solve Linear and Quadratic Programming Problems via the Segmentation of the Response Surfaces. Also, this work is about verifying and establishing the number of segments, S, for which optimal solutions are obtained. In this regard, the theoretical framework, illustration of segmentation as well as illustrative examples are presented in subsequent sections.

P. E. CHIGBU, Department of Statistics, University of Nigeria, Nsukka, Enugu State, Nigeria
T. A. UGBE, Department of Statistics, University of Nigeria, Nsukka, Enugu State, Nigeria
Present Address: Department of Mathematics, Statistics, and Computer Science, University of Calabar Calabar, Nigeria

PRELIMINARIES

Some Definitions

Definition 1: $\underline{x}_i$, $i = 1, \ldots, n$; are n support points for an n-point design measure.
Definition 2: ζ_N is an N- point design measure whose support points may or may not have equal weights.

Definition 3: $\bar{\zeta}_N$; is an N- point design measure whose support points may or may not have equal weights but are usually applicable when ζ_N cannot attain optimum solution(s) in the current move.
Definition 4: $\tilde{X}$ is the experimental space of the response surface that can be partitioned into segments, every support point in the segments is a subset of $\tilde{X}$.

THEORETICAL FRAMEWORK

Over the years, a variety of line search algorithms have been employed in locating the local optimizers of Response Surface Methodology (RSM) problems: see, for example, Wilde & Beightler (1967) and Myers (1971). Similarly, the active set and simplex methods which are available for solving Linear Programming Problems also belong to the class of line search exchange algorithms. A Line search exchange algorithm is defined by the following steps:

(a) Select N_k support points from the k- th segment, hence make up an N- point design,

$$\zeta_N = \begin{Bmatrix} \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \ldots, \underline{x}_N \\ w_1, w_2, \ldots, w_n, \ldots w_N \end{Bmatrix}; \; N = \sum_{k=1}^{s} N_k.$$

(b) Compute the vectors $\underline{\bar{x}}^*, \underline{d}^*$ and the step-length ρ^*.

(c) Move to the point $\underline{x}^* = \underline{\bar{x}}^* - \rho^* \underline{d}^*$.

(d) Is $\underline{x}^* = \underline{x}_f$ [where $\underline{x}_f^*$ is the minimizer of f (•)]

Yes, stop, No, replace $\underline{x}_m$ in ζ_N with $\underline{x}^*$ and thus define a new measure

$$\bar{\zeta}_N = \begin{Bmatrix} \underline{x}_1, \underline{x}_2, \ldots, \underline{x}^*, \ldots, \underline{x}_N \\ w_1, w_2, \ldots, w^*, \ldots w_N \end{Bmatrix}$$

(e) Is $N_k \geq n+1 \; \forall \; k = 1, \ldots, S$?

Yes: go to step (b) above.

No: adjust the segment boundaries or take extra support points so that $N_k \geq n+1 \; \forall k$ and go to step (b) above.

In the algorithm above, $\underline{x}_m$ is the point of maximum variance; i.e.

$$\underline{x}'_m M^{-1}(\zeta_N)\underline{x}_m = \underset{x}{Max}\; \underline{x}' M^{-1}(\zeta_N)\underline{x};$$

where $M^{-1}(\zeta_N)$ is the inverse of information matrix of ζ_N; see Onukogu (1997).

SEGMENTATION AND IMPLICATIONS

The space $\tilde{X}$ is partitioned into S non-overlapping segments and we let the design measure in the k-th segment be defined by $\zeta_k = N \left\{ \begin{matrix} x_1, x_2, \ldots, x_{Nk} \\ w_1, w_2, \ldots, w_{Nk} \end{matrix} \right\}$

$W_i = \frac{r_i}{N_k}$ and $\sum_{i=1}^{n} r_i = N_k$.

The regression model is $y_k(\bullet) = \underline{C}'_k \underline{x}_k + \underline{e}_k(.)$

where, $\underline{C}'_k = (C_{ok}, C_{1k}, \ldots, C_{nk})$ is an n+1 – component vector of parameters, x_k is a point in ζ_{Nk}, $\underline{e}_k(\bullet)$ is a vector of random errors. We have that in the segments, we define the direction vector by $\underline{d}_o = {}_k\underline{C}_k$; $\sum_{k=1}^{s} \theta_k = 1$, $_k \geq 0$. Some of the illustrative ways of partitioning $\tilde{X} = \{x_1, x_2; -1 \leq x_1, x_2 \leq 1\}$ into segments are:

Segmentation with Common Boundaries (illustrated by Figure 1)

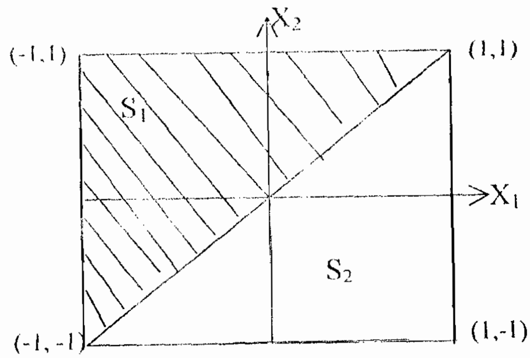
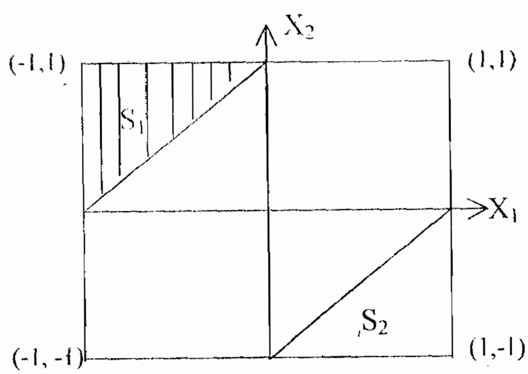


Figure 1

$\tilde{X} = \{x_1, x_2; -1 \leq x_1, x_2 \leq 1\}$

Segmentation without Common Boundaries (illustrated by Figure 2)



$X = \{x_1, x_2; -1 \leq x_1, x_2 \leq 1\}$

Figure 2

Notice, as in Figure 2, that the segments, S_1 and S_2 need not have common boundary nor do they need cover the entire space, $\tilde{X}$; i.e. $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 < X$: see Onukogu *et al* (2000) and Onukogu a Chigbu (2002). From the above illustration, it means that the segments can be picked at random provid there is no overlap between or among them. For instance, if there are four segments, S_1, S_2, S_3 and S_4 c. be randomly picked in the entire space provided

$S_1 \cap S_2 \cap S_3 \cap S_4 = \emptyset$ and $S_1 \cup S_2 \cup S_3 \cup S_4 \leq \tilde{X}$.

The number of support points, N_k in any k-th segment according to Pazman (1987) should not exceed p(p+1) + 1, where p is the number of variables in the polynomial of interest. Of course, $p \leq N_k \leq \frac{1}{2}p(p+1)+$ Also, the number of support points per segment, N_k, according to Onukogu (1997) is given by $n+1 \leq N_k \leq$ n(n+1) + 1, where n is the number of variables in the regression model of interest. The support points pe segment are also randomly picked provided they satisfy the constraint equations or do not lie outside th feasible region. It is also important to note that not all the support points that satisfy the constraint equatior will be picked but then random selection will be done as long as $n+1 \leq N_k \leq \frac{1}{2}n(n+1)+1$.

Illustrative Examples

Example 1 is used to illustrate what happens in a Linear Programming Problem (LPP) whil Example 2 is presented for the Quadratic Programming Problems (QPP). It is nice to note that differer segments were used for investigating the optimal number of segments for each of the Linear Programmin Problems and Quadratic Programming Problems. Indeed, the following number of segments: 2,3 and 4 are considered for the Linear Programming Problems while the following number of segments: 2,3,4,5 and 6 are considered for the Quadratic Programming Problem.

Example 1 [Hillier & Lieberman (1995) , Chapter 5, Pp. 183]

Maximize $Z = 3x_1 + 2x_2$

s.t. $2x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Using 2 segments (S = 2)

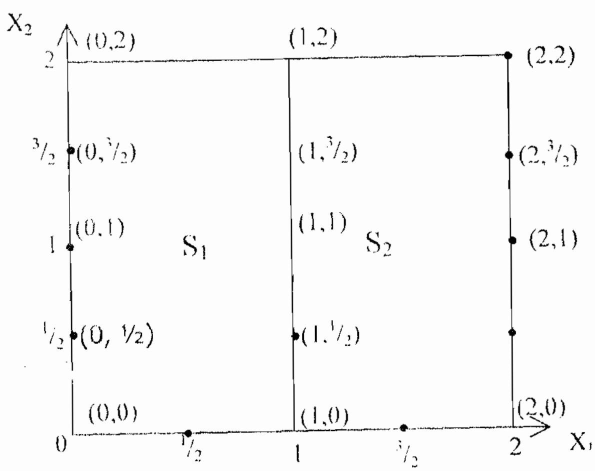


Figure 3

We define the segments by:

$S_1 = \{x_1, x_2;\ 0 \le x_1 \le 1,\ 1 \le x_2 \le 2\}$

$S_2 = \{x_1, x_2;\ 1 \le x_1 \le 2,\ 0 \le x_2 \le 2\}$

The design matrices are:

$$X_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},\quad X_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}$$

The inverses of the information matrices are: $(X'_1X_1)^{-1} = \begin{pmatrix} 0.9091 & \text{SYM} \\ -0.2727 & 0.1818 \end{pmatrix}$,

$(X'_2X_2)^{-1} = \begin{pmatrix} 0.1818 & \text{SYM} \\ -0.1364 & 0.2273 \end{pmatrix}$, where 'SYM' means that $(X'_1X_1)^{-1}$ and $(X'_2X_2)^{-1}$ are symmetric matrices.

The matrices of coefficient of convex combination of the inverses of the information matrices are: H_1 = diag {0.1667, 0.5556}, H_2 = diag {0.8333, 0.4444}. These matrices are normalized to give
H_1^* = diag {0.1962, 0.7809}, H_2^* = diag {0.9805, 0.6246}.

The direction vector, $\underline{d} = \begin{pmatrix} 3.0006 \\ 1.9998 \end{pmatrix}$; this is normalized to give $d^* = \begin{pmatrix} 0.8321 \\ 0.5546 \end{pmatrix}$

$\bar{\underline{x}}^* = \sum w_i x_i = \begin{pmatrix} 0.7517 \\ 0.8651 \end{pmatrix}$, the step-length, $\rho^* = -1.6367$, $\underline{x}^* = \begin{pmatrix} 2.1 \\ 1.8 \end{pmatrix}$.

Therefore, Max Z = 9.9.

This value is close to the optimum value, got by Hillier & Lieberman (1995), Chapter 5, Pp. 183, as Max Z = 10.0. However, when 3 and 4 segments were used, the maximum values of Z for the problem of Example 1 are 12.4 with corresponding values, (x_1, x_2) = (2.20, 2.90) and 10.8 with corresponding values, (x_1, x_2) = (2.48, 1.8). The optimal values got in 3 and 4 segments though, greater than that got in 2 segments but their corresponding x- values do not satisfy constraint equations. Therefore, optimal solutions in 3 and 4 segments are rejected.

Example 2 [Taha (1987), Chapter 19, Pp. 795]

Maximize $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

s.t. $x_1 + 2x_2 \le 2$

$x_1, x_2 \ge 0$

We define the segments by:

$S_1 = \{x_1, x_2;\ 0 < x_1 \le {}^1/_8,\ {}^1/_2 \le x_2 \le 1\}$, $S_2 = \{x_1, x_2;\ {}^1/_2 \le x_1 \le {}^3/_4,\ {}^1/_2 \le x_2 \le {}^5/_8\}$,

$S_3 = \{x_1, x_2;\ 0 \le x_1 \le {}^1/_4,\ 0 \le x_2 \le {}^1/_2\}$, $S_4 = \{x_1, x_2;\ {}^1/_2 \le x_1 \le {}^3/_4,\ 0 \le x_2 \le {}^1/_2\}$.

Using 4 Segments (S = 4)

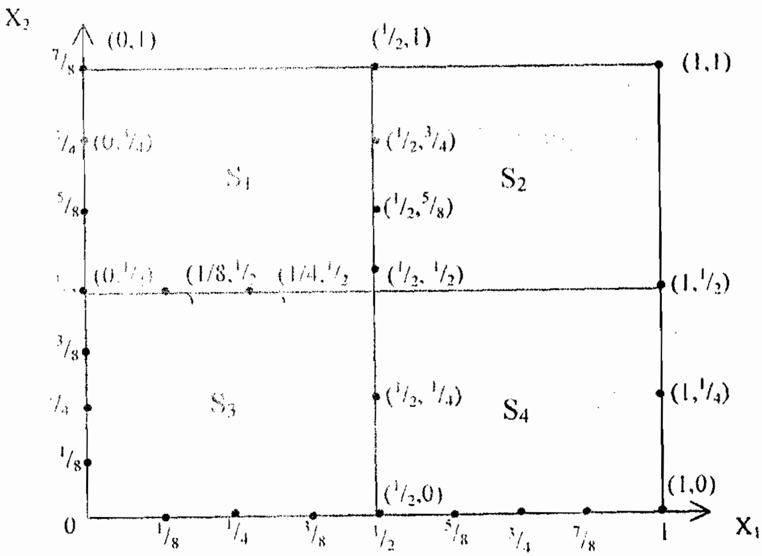


Figure 4

The design and bias matrices are:

$$X_1=\begin{pmatrix}1&0&1\\1&0&\frac{1}{2}\\1&0&\frac{3}{4}\\1&\frac{1}{8}&\frac{1}{2}\end{pmatrix},\ X_2=\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}\\1&\frac{1}{2}&\frac{3}{4}\\1&\frac{3}{4}&\frac{1}{2}\\1&\frac{1}{8}&\frac{5}{8}\end{pmatrix},\ X_3=\begin{pmatrix}1&0&0\\1&\frac{1}{8}&0\\1&0&\frac{1}{2}\\1&\frac{1}{4}&0\end{pmatrix},\ X_4=\begin{pmatrix}1&\frac{3}{4}&\frac{1}{2}\\1&\frac{1}{2}&0\\1&\frac{1}{2}&\frac{1}{4}\\1&\frac{1}{8}&\frac{1}{2}\end{pmatrix},$$

$$X_{1B}=\begin{pmatrix}0&0&1\\0&0&\frac{1}{4}\\0&0&\frac{9}{16}\\\frac{1}{16}&\frac{1}{16}&\frac{1}{4}\end{pmatrix},\ X_{2B}=\begin{pmatrix}\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\\frac{3}{8}&\frac{1}{4}&\frac{9}{16}\\\frac{3}{8}&\frac{9}{16}&\frac{1}{4}\\\frac{5}{16}&\frac{1}{4}&\frac{25}{64}\end{pmatrix},\ X_{3B}=\begin{pmatrix}0&0&0\\0&\frac{1}{64}&0\\0&0&\frac{1}{4}\\0&\frac{1}{16}&0\end{pmatrix},\ X_{4B}=\begin{pmatrix}\frac{3}{8}&\frac{9}{16}&\frac{1}{4}\\0&\frac{1}{4}&0\\\frac{1}{8}&\frac{1}{4}&\frac{1}{16}\\\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\end{pmatrix}.$$

The vector of the biasing parameters is $\underline{g}_2 = \begin{bmatrix}-2\\-2\\-2\end{bmatrix}$

The mean square error matrices are:

$$\overline{M}_1=\begin{pmatrix}5.9186&&\text{SYM}\\-16.3161&119.8398&\\-9.1254&20.7496&17\end{pmatrix},\ \overline{M}_2=\begin{pmatrix}35.6262&&\text{SYM}\\-33.1938&41.8755&\\-35.1096&28.3953&44.25\end{pmatrix},$$

$$\overline{M}_3=\begin{pmatrix}0.8334&&\text{SYM}\\-4.0052&32.25&\\-1.6773&8.5104&8.3753\end{pmatrix},\ \overline{M}_4=\begin{pmatrix}9.3906&&\text{SYM}\\-19.0419&42.7773&\\-0.75&-0.6668&12\end{pmatrix},$$

where SYM' means that the mean square error matrices M_1, M_2, M_3 and M_4 are symmetric matrices
The matrices of coefficient of convex combination of the mean square error matrices are:
H_1 = diag {0.1814, 0.1807, 0.1470}, H_2 = diag {0.0161, 0.5062, 0.2083},
H_3 = diag {0.6882, 0.1769, 0.5421}, H_4 = diag {0.1143, 0.5062, 0.2083}; these matrices are normalized to give:
H_1^* = diag {0.2516, 0.3105, 0.2419}, H_2^* = diag {0.0223, 02340, 0.1688},
H_3^* = diag {0.9545, 0.3039, 0.8919}, H_4^* = diag {0.1585, 0.8698, 0.3427}.

The direction vector, $\underline{d} = \begin{pmatrix} 1.1172 \\ 6.4751 \end{pmatrix}$. This is normalized to give $\underline{d}^* = \begin{pmatrix} 0.1700 \\ -0.9854 \end{pmatrix}$

$$\bar{\underline{X}}^* = \sum w_i x_i = \begin{pmatrix} 0.2854 \\ 0.3771 \end{pmatrix}$$, the step-length, $\rho^* = -0.4486$.

$$\underline{x}^* = \bar{\underline{x}}^* - \rho^* \underline{d}^* = \begin{pmatrix} 0.36 \\ 0.82 \end{pmatrix}$$

Therefore, Max Z = 4.17.
This value is close to the value got by Taha (1987), Chapter 19, Pp 795, which is
Max Z = 4.16 for $(x_1, x_2) = (0.33, 0.83)$. The maximum values of Z for this problem by using 2,3,5 and 6 segments are 4.35 for $(x_1, x_2) = (0.76, 0.97)$, 4.14 for $(x_1, x_2) = (0.49, 0.76)$, 4.13 for $(x_1, x_2) = (0.5, 0.75)$ and 4.12 for $(x_1, x_2) = (0.53, 0.74)$. These values are not optimal because they do not satisfy the equations and do not compare favorably with existing solution got by Hillier and Lieberman (1995).

CONCLUSION

From the foregoing, optimal solutions are attained by using two segments (S =2) for Linear Programming Problems and by using four segments (S = 4) for Quadratic Programming Problems. As earlier highlighted, Ugbe (2001) considered two, three and four segments for Linear Programming Problems and two, three, four, five and six segments for Quadratic Programming Problems. Three problems of Linear programming and three of Quadratic programming were used to investigate and demonstrate the above results.

The examples used for verification were chosen because we can easily illustrate how to partition the response surface into segments (as shown in figures 3 and 4 of this paper) and then select support points. Furthermore, the problems consist of two unknown variables (x_1, x_2). Therefore, we can easily select support points that satisfy constraint equations from the response surface for verification.

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