

NEW THIRD ORDER EXPLICIT RUNGE-KUTTA METHOD WITH EQUAL NODES

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ABSTRACT

In the light of [4] and [5], a new third order explicit Runge-Kutta Method with equal nodes and its computer implementation were highlighted. The method compared efficiently with the existing third order explicit Runge-Kutta methods proposed by Kutta, Ralston, Heun and Nystrom respectively.

Keywords: Explicit Runge-Kutta methods, stepsize, order conditions, and parameters.

INTRODUCTION

Virtually every branch of physics is concerned with the rate at which something changes. Hence, the study of differential equations and the behaviour of their solutions have always been of central interest to physical scientists [1].

However, most differential equations arising in scientific modeling do not have closed-form solutions. For this reason, we will concentrate on initial value problem of the form:

$$y'(x) = F(x, y(x)) \dots\dots\dots (1.1)$$

with initial condition:

$$y(x_0) = y_0 \dots\dots\dots (1.2)$$

We obtain the numerical solution of (1.1) and (1.2) by exploring the s-stage explicit Runge-kutta method of order p [2] defined as:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \dots\dots\dots (1.3)$$

such that the increment function

$$\phi(x_n, y_n; h) = \sum_{i=1}^s b_i k_i \dots\dots\dots (1.4)$$

with the constraint

$$\sum_{i=1}^s b_i = 1 \dots\dots\dots (1.5)$$

Equation (1.5) ensures consistency of the method. For $i = 1, 2, \dots, s$; the slopes k_i are

$$k_i = F\left(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j\right) \dots\dots\dots (1.6)$$

with constraint

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad i = 1, 2, \dots, s \quad \dots \dots \dots (1.7)$$

Equation (1.7) confirms the method explicitly, while the parameters c_i , b_i and h are the nodes, weights and stepsize respectively.

In a more compact form, [2] described the s -stage explicit Runge-Kutta method by the tableau below:

$$\begin{array}{c|ccc}
 0 & & & \\
 c_2 & a_{21} & & \\
 c_3 & a_{31} & a_{32} & \\
 \vdots & \vdots & \vdots & \\
 \vdots & \vdots & \vdots & \\
 c_s & a_{s1} & a_{s2} & \dots & a_{ss-1} \\
 \hline
 & b_1 & b_2 & \dots & b_s
 \end{array} \quad \dots \dots \dots (1.8)$$

Order Conditions: According to [3] and [4], the order conditions for a system of differential equations (1.1) and (1.2) to have order 3 are:

$$\sum_{i=1}^3 b_i c_i^k = \frac{1}{k+1}, \quad k = 0, 1, 2 \quad \dots \dots \dots (2.1)$$

and

$$\sum_{i=1}^3 b_i a_{ij} c_j = \frac{1}{6} \quad \dots \dots \dots (2.2)$$

For the purpose of derivation of third order explicit Rung-kutta method with equal nodes, we shall introduce the following parameters:

$$\alpha_i = \sum_{j=1}^{i-1} a_{ij} c_j - \frac{c_i^2}{2}; \quad i = 1, 2, 3 \quad \dots \dots \dots (2.3)$$

and

$$\beta_j = \sum_{i=j+1}^3 b_i a_{ij} - b_j (1 - c_j), \quad j = 1, 2, 3 \quad \dots \dots \dots (2.4)$$

We found that the parameters α_i and β_i satisfy the following order conditions:

$$\sum_{i=1}^3 b_i c_i^k \alpha_i = 0; \quad k = 0, 1 \quad \dots \dots \dots (2.5)$$

$$\sum_{i=1}^3 \beta_i c_i^k = 0; \quad k = 0, 1, 2 \quad \dots \dots \dots (2.6)$$

$$\sum_{i=1}^3 \beta_i \alpha_i = 0 \dots\dots\dots(2.7)$$

and

$$\sum_{i=1}^3 b_i \alpha_i^2 = 0 \dots\dots\dots(2.8)$$

We shall derive a new third order explicit Runge-Kutta Method whose parameters will satisfy all order conditions except equations (2.5) and (2.7) respectively.

Derivation of the New Third Order Explicit Runge-Kutta Method With Equal Nodes

According to [5] and [7], we define the new third order explicit Runge-Kutta Method with equal nodes, δ as:

0				
δ	δ			
δ	a_{31}	a_{32}		
	b_1	b_2	b_3	

.....(3.1)

We choose the equal nodes, $c = [0, \delta, \delta]^T$, where $\delta \in \mathbb{R}$. Consequently, the values of α_i and β_i are obtained in terms of δ and the weights. That is:

$$\alpha_1 = 0 \dots\dots\dots(3.2)$$

$$\alpha_2 = -\frac{\delta^2}{2} \dots\dots\dots(3.3)$$

$$\alpha_3 = \frac{1 - 3\delta^2 b_3}{6b_3} \dots\dots\dots(3.4)$$

$$\beta_1 = \frac{1}{6} - b_1 \dots\dots\dots(3.5)$$

$$\beta_2 = \frac{(2 - 3b_1)}{6} \dots\dots\dots(3.6)$$

$$\beta_3 = -b_3(1 - \delta) \dots\dots\dots(3.7)$$

The parameters α_3 and β_3 are non-zero if and only if $\delta \neq 1$ or $b_3 \neq 0$. Hence, in line with the work of Heun

(1900): $c_2 + c_3 = 1$ (3.8)

Which implies that:

$$c_2 = c_3 = \delta = \frac{1}{2} \quad \dots\dots\dots(3.9)$$

for equal nodes.

Substituting α_1 , α_2 and α_3 into equation (2.8), we found that

$$\frac{b_2}{64} + b_3 \left[\frac{4 - 3b_3}{24b_3} \right]^2 = 0 \quad \dots\dots\dots(3.10)$$

Adopting equation (2.1) for $k = 1$ into equation (3.10), we obtain the following results:

$$b_3 = \frac{16}{15}$$

$$b_2 = -\frac{1}{15}$$

$$b_1 = 0$$

Consequently, we found that

$$\alpha_2 = -\frac{1}{8}$$

$$\alpha_3 = \frac{1}{32}$$

$$\beta_1 = \frac{1}{6}$$

$$\beta_2 = \frac{11}{30}$$

$$\beta_3 = -\frac{16}{30}$$

$$a_{21} = \frac{1}{2}$$

$$a_{31} = \frac{3}{16}$$

$$a_{32} = \frac{5}{16}$$

Hence, the new third order explicit Runge-Kutta Method with equal nodes is described by the tableau

$$\begin{array}{c|cc}
 0 & & \\
 \frac{1}{2} & \frac{1}{2} & \\
 \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\
 \hline
 & 0 & \frac{1}{5} & \frac{10}{15}
 \end{array} \dots\dots\dots(3.11)$$

Existing Third Order Explicit Runge-kutta Methods

A survey of third order explicit Rung-kutta Method are given below:

Heun's Method

$$\begin{array}{c|cc}
 0 & & \\
 \frac{1}{3} & \frac{1}{3} & \\
 \frac{1}{3} & 0 & \frac{2}{3} \\
 \hline
 & \frac{1}{4} & 0 & \frac{3}{4}
 \end{array}$$

Kutta's Method

$$\begin{array}{c|cc}
 0 & & \\
 \frac{1}{2} & \frac{1}{2} & \\
 1 & -1 & 2 \\
 \hline
 & \frac{1}{6} & \frac{4}{3} & \frac{1}{6}
 \end{array}$$

Ralston's Method

$$\begin{array}{c|cc}
 0 & & \\
 \frac{1}{2} & \frac{1}{2} & \\
 \frac{3}{4} & 0 & \frac{3}{4} \\
 \hline
 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9}
 \end{array}$$

Nystrom's Method

$$\begin{array}{c|cc}
 0 & & \\
 \frac{1}{3} & \frac{1}{3} & \\
 \frac{1}{3} & 0 & \frac{2}{3} \\
 \hline
 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8}
 \end{array}$$

Computer Implementation of The New Third Order Explicit Runge-Kutta Method

Numerical Test: For computer implementation of the method, we shall consider the initial value problem [6]:

$$y' = -10(y-1)^2$$

with the initial condition.

$$y(0) = 2; 0 \leq x \leq 1; h = 0.1$$

Exact solution of the initial value problem: $y(x) = 1 + \frac{1}{1+10x}$

Computer Algorithm:

Initialize $x \rightarrow x_0, y \rightarrow y_0$

Input $h, c_i, a_{ij},$ and b_i

Declare the initial value problem: $yp \rightarrow y' \rightarrow F(x, y)$

Declare the exact solution: $yE \rightarrow y(x)$

Do 10 $i = 1, 3$

$x \rightarrow x_{i-1}$

$y \rightarrow y_{i-1}$

$yp \rightarrow y(x_{i-1})$

compute $x \rightarrow x_{i-1} + c_i h$

$y \rightarrow y_{i-1} + h \text{ sum}$

$k_i \rightarrow F(x, y)$

10 continue

$k \rightarrow \sum_{i=1}^3 k_i$

$y \rightarrow y + hk$

$x \rightarrow x + h$

$yE \rightarrow y(x + h)$

Print $x; y, yE$

End

Computer Program: For the purpose of proper comparison of existing third order explicit Runge-Kutta Method with the Method (3.11) proposed, a computer programm "STERK" was developed based on the computer algorithm described above. The program language adopted was "FORTRAN"

We shall give proper definition of some vital variables to make the program explicit. The variables are as follows:

N : Order of Runge-Kutta Method

TT : Step size

X : Independent Variable

Y : Numerical Solution

NJ-1 : Number of Iterations
 YE : Exact Solution
 ABS : Absolute Value
 B(I) : Weights of the Runge-Kutta Method
 C(I) : Nodes of the Runge-Kutta Method
 D(I) : Slopes of the Runge-kutta Method
 A(I, J) : Elements of the Matrix $[a_{ij}]$
 YP : Differential Equation
 XEND : Last value of Independent Variable
 T : Initialized Value
 ERR : Actual Error

Program:

```

Double Precision T(12), B, C, D, H, A, TT, X, S
Double Precision YE, SUM, YP, SUM2, SUM3, SUM4, ERR, SUM1
INTEGER N, MN, NJ, JK, I, L, J, KK, II, KI, LI, K2
DIMENSION B(5), C(5), A(5,5), H(12), D(12)
OPEN (UNIT=6, FILE = 'STERK I.OUT', STATUS = 'NEW')
    N = 3
    TT = 0.1
    X = 0.0
    Y = 2.0
    XEND = 1.0
    NJ = 11
    YE = 2.0
    T(1) = 0.0
DO 5 I = 2, NJ
    K2 = I-1
    T(I) = T(K2) + TT
5 Continue
DO 100 I = 2, NJ
    II = I-1
    H(II) = T(I) - T(II)
    
```

$$S = H(I)$$

$$X = T(I)$$

C Input the values of B(I), C(I), A(L,J)

$$\text{SUM } 1 = X$$

$$D(1) = -10.0 * ((y-1.0)**2)$$

DO 250 L = 2, N

$$\text{SUM } 2 = X + C(L) * S$$

$$\text{SUM } 3 = 0.0$$

$$KK = L-1$$

DO 300 J = 1, KK

$$\text{SUM } 3 = \text{SUM } 3 + (S * D(J) * A(L, J))$$

300 Continue

$$\text{SUM } 4 = Y + \text{SUM } 3$$

$$YP = -10.0 * ((\text{SUM } 4 - 1.0)**2)$$

$$D(L) = YP$$

250 Continue

$$\text{SUM} = 0.0$$

DO 200 JK = 1, 3

$$\text{SUM} = \text{SUM} + (D(JK) * B(JK))$$

200 Continue

$$Y = Y + (\text{SUM} * S)$$

$$X = X + S$$

$$W = 1.0 + (10.0 * x)$$

$$YE = 1.0 + \left(\frac{1.0}{w} \right)$$

$$\text{ERR} = \text{DABS}(YE - Y)$$

Write (6, 20) X, Y, YE, ERR

20 Format (5X, F7.3, IP3D15.6)

100 Continue

END

CONCLUSION

The new third order explicit Runge-Kutta Method proposed seems to be more efficient than the existing third order explicit Runge-Kutta Methods (see OUTPUT 6). Hence, the method compares well with the existing methods.

Consequently, the following verification:

$$\sum_{i=1}^3 b_i c_i^k \alpha_i = \begin{cases} 0.0708333 & \text{if } k = 1 \\ 0.0416666 & \text{if } k = 0 \end{cases}$$

$$\sum_{i=1}^3 \beta_i \alpha_i = -0.0625000$$

confirms the method proposed.

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OUTPUT 1: NEW METHOD

COMPUTER OUTPUT

Number Of Iterations	Independent Variable X	Numerical solution Y	Exact Solution YE	Actual Error ERR
1	0.100000	1.441406 D+00	1.500000 D+00	5.859400 D-02
2	0.200000	1.36789 D+00	1.333333 D+00	3.456400 D-02
3	0.300000	1.268501 D+00	1.250000 D+00	1.850100 D-02
4	0.400000	1.211633 D+00	1.200000 D+00	1.163300 D-02
5	0.500000	1.174681 D+00	1.166667 D+00	8.014000 D-03
6	0.600000	1.148721 D+00	1.142857 D+00	5.864000 D-03
7	0.700000	1.129478 D+00	1.125000 D+00	4.478000 D-03
8	0.800000	1.114644 D+00	1.111111 D+00	3.533000 D-03
9	0.900000	1.102859 D+00	1.100000 D+00	2.859000 D-03
10	1.000000	1.093270 D+00	1.090909 D+00	2.361000 D-03

OUTPUT 2 HEUN'S METHOD

Number of Iterations N	Independent Variable X	Numerical Solution Y	Theoretical Solution YE	Actual Error ER R
1	0.100000	1.378601 D+00	1.500000 D+00	1.213992 D-01
2	0.200000	1.272705 D+00	1.333333 D+00	6.062856 D-02
3	0.300000	1.213839 D+00	1.250000 D+00	3.616131 D-02
4	0.400000	1.176027 D+00	1.200000 D+00	2.397291 D-02
5	0.500000	1.149623 D+00	1.166667 D+00	1.704338 D-02
6	0.600000	1.130124 D+00	1.142857 D+00	1.273329 D-02
7	0.700000	1.115128 D+00	1.125000 D+00	9.872098 D-03
8	0.800000	1.103234 D+00	1.111111 D+00	7.876636 D-03
9	0.900000	1.093570 D+00	1.100000 D+00	5.871475 D-03
10	1.000000	1.085561 D+00	1.090909 D+00	4.880622 D-03

OUTPUT 3 KUTTA'S METHOD

Number of Iterations N	Independent Variable X	Numerical Solution Y	Theoretical Solution YE	Actual Error ERR
1	0.100000	1.291667 D+00	1.500000 D+00	2.083334 D-01
2	0.200000	1.225476 D+00	1.333333 D+00	1.078574 D-01
3	0.300000	1.183901 D+00	1.250000 D+00	6.609933 D-02
4	0.400000	1.155302 D+00	1.200000 D+00	4.469776 D-02
5	0.500000	1.134412 D+00	1.166667 D+00	3.225492 D-02
6	0.600000	1.118479 D+00	1.142857 D+00	2.437806 D-02
7	0.700000	1.105925 D+00	1.125000 D+00	1.907487 D-02
8	0.800000	1.095778 D+00	1.111111 D+00	1.533353 D-02
9	0.900000	1.087405 D+00	1.100000 D+00	1.259521 D-02
10	1.000000	1.080378 D+00	1.090909 D+00	1.053063 D-02

OUTPUT 4 RALSTON'S METHOD

Number of Iterations N	Independent Variable X	Numerical Solution Y	Theoretical Solution YE	Actual Error ERR
1	0.100000	1.401042 D+00	1.500000 D+00	9.895834 D-02
2	0.200000	1.284395 D+00	1.333333 D+00	4.893853 D-02
3	0.300000	1.221035 D+00	1.250000 D+00	2.896527 D-02
4	0.400000	1.180902 D+00	1.200000 D+00	1.909848 D-02
5	0.500000	1.153142 D+00	1.166667 D+00	1.352460 D-02
6	0.600000	1.132783 D+00	1.142857 D+00	1.352460 D-02
7	0.700000	1.117207 D+00	1.125000 D+00	7.792749 D-02
8	0.800000	1.104905 D+00	1.111111 D+00	6.206169 D-03
9	0.900000	1.094941 D+00	1.100000 D+00	5.058745 D-03
10	1.000000	1.086707 D+00	1.090909 D+00	4.202265 D-03

OUTPUT 6: COMPARISON OF EXISTING EXPLICIT RUNGE-KUTTA METHODS AND THE NEW METHOD WITH REFERENCE TO EXACT SOLUTION, YE

INDEPENDENT VARIABLE X	EXACT SOLUTION YE	NEW METHOD Y	HEUN'S METHOD Y	KUTTA'S METHOD Y	RALSTON'S METHOD Y	SYSTEM'S METHOD Y
0.100000	1.500000 D+00	1.441406 D+00	1.378601 D+00	1.291667 D+00	1.401042 D+00	1.386831 D+00
0.200000	1.333333 D+00	1.367897 D+00	1.272705 D+00	1.225476 D+00	1.284395 D+00	1.277211 D+00
0.300000	1.250000 D+00	1.268501 D+00	1.213839 D+00	1.183901 D+00	1.221035 D+00	1.216676 D+00
0.400000	1.200000 D+00	1.211633 D+00	1.176027 D+00	1.153302 D+00	1.180902 D+00	1.177972 D+00
0.500000	1.166667 D+00	1.174681 D+00	1.149623 D+00	1.134412 D+00	1.153142 D+00	1.151038 D+00
0.600000	1.142857 D+00	1.148721 D+00	1.130124 D+00	1.118479 D+00	1.132783 D+00	1.131198 D+00
0.700000	1.125000 D+00	1.129478 D+00	1.115128 D+00	1.105925 D+00	1.117207 D+00	1.115971 D+00
0.800000	1.111111 D+00	1.114644 D+00	1.103234 D+00	1.095778 D+00	1.104905 D+00	1.103914 D+00
0.900000	1.100000 D+00	1.102859 D+00	1.093570 D+00	1.087407 D+00	1.094941 D+00	1.094129 D+00
1.000000	1.090909 D+00	1.093270 D+00	1.085561 D+00	1.080378 D+00	1.086707 D+00	1.086028 D+00