GLOBAL JOURNAL OF ENGINEERING RESEARCH VOL. 6 NO. 2 2007: 101 - 105
 ISSN 1596 – 292X

# A SIMPLIFIED APPROACH TO THE GEOMETRIC DESIGN OF PARABOLIC HIGHWAY VERTICAL CURVES

**YAW A. TUFFOUR**

(Received 2 April, 2007; Revision Accepted 3 September, 2007)

**ABSTRACT**

A simplified approach to the geometric design of symmetric parabolic highway vertical curve to connect two intersecting tangent grade lines is presented. Unlike the traditional tangent offset method which uses the initial curve parameters to evaluate a number of additional parameters associated with the curve and tangent grade lines before the curve profile can be defined, the proposed method uses only the initial curve parameters to define the curve profile directly. The method can be applied to the design of asymmetric parabolic curves by treating such curves as a composite of two consecutive symmetric parabolae joined by a common tangent point. The proposed method is simpler and time-saving as it results in less design computations than the traditional tangent offset method. A numerical example is presented to illustrate the computational efficiency of the method.

**KEYWORDS**: Curve elevation, symmetric parabola, tangent offset, vertical curve.

## 1. INTRODUCTION

In the geometric design of roads, the horizontal alignment represents the plan view and the vertical alignment the vertical profile of the road. The design of the vertical profile is essentially the determination of the appropriate elevations of all stations and critical points along the roadway centerline on the horizontal alignment. During design, the vertical profile of the terrain along the route alignment is first approximated by a series of tangent grade lines (with acceptable slopes) which are then connected with vertical parabolic curves to provide a gradual and smooth transition from one grade line to the other. Conventionally, the design method uses tangent offsets from the intersecting grade lines to determine the profile of the parabolic curve. Generally, the parabolic curves are designed to be symmetric in order to facilitate the computations associated with the design (Kavanagh and Bird, 1990). However, occasionally, it may be necessary to design the parabola as asymmetric (with unequal tangent lengths) when, for example, large amounts of cut and/or fill are to be avoided or a particular site condition is to be met (Papacostas, 1990; Uren and Price, 1995).

Designing by the tangent offset approach requires the curve length and grade line parameters as inputs but several additional parameters characterising the curve also have to be evaluated before curve elevations at the required chainages and critical points on the horizontal alignment can be computed. These include;

- the elevation of the point of curvature (beginning of curve),
- the elevation of the point of tangency (end of curve),
- mid-chord elevation;
- the external distance or maximum offset at the point of grade line intersection, etc.

The design process is organised in sequential steps so that the output of one step becomes the input of the next. The calculations associated with each step and the design process as a whole result in a considerable volume of computations and make the process lengthy. In this paper, a simplified approach to the design of highway vertical curves which evaluates curve elevations directly using only the input curve length and grade line parameters is presented.

## 2. Characteristics of a symmetric parabolic vertical curve

Fig. 1 (not drawn to scale) shows the geometric features of a symmetric parabolic highway vertical curve and the definition of some of the design parameters.

BVC = origin (beginning) of the parabolic vertical curve;
PVI = point of vertical intersection of grade lines;

EVC = end of the vertical curve;
$L$ = length of curve on the horizontal alignment (horizontal projection);
P = a general point on the curve with coordinates $(x,y)$ with respect to point BVC;
$G_1$ = slope (in %) of lower chainage grade line;
$G_2$ = slope (in %) of higher chainage grade line.

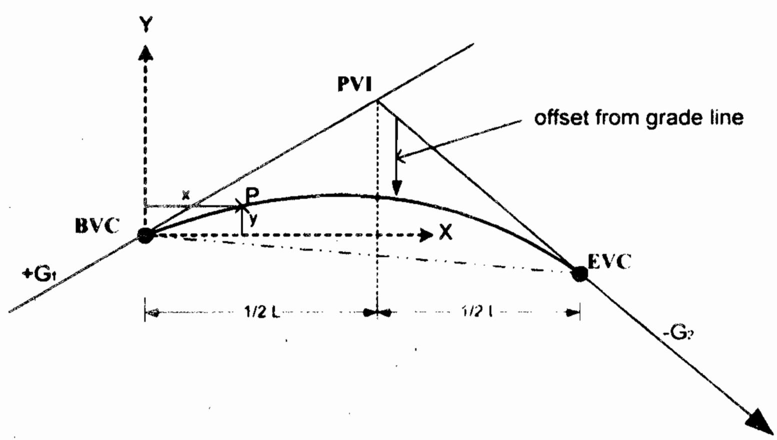


Fig. 1: Geometric features of a symmetric parabolic vertical curve

**Yaw A. Tuffour**, Civil Engineering Department, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

## 3. Overview of vertical curve design by tangent offsets

To bring into focus the parameters and computations involved in the design of vertical curves by the tangent offset approach, an overview of the design process is presented. The computation of the curve and grade line parameters required for the evaluation of curve elevations is organized in a sequential process so that the output of a step becomes the input of the next. For each step, the expression required for evaluating the relevant parameter associated with the step is given. For a symmetric parabola, the beginning and end of curve are located half a curve length before and after the point of grade line intersection, respectively. The steps in the sequential design process are as detailed below.

***Parameter definitions:***

*elevPVI* = elevation of PVI
*elevBVC* = elevation of point BVC
*elevEVC* = elevation of point EVC
*elevGradeline* = elevation of gradeline
*elevMidchord* = elevation of midpoint of chord joining points BVC and EVC
*elevP*$_x$ = elevation of curve at a distance *x* from BVC
$d_x$=tangent offset at a station distance *x* from BVC
*d*=tangent offset or external distance at PVC

Step 1: Supply initial curve parameters; $G_1$, $G_2$, *L*, *elevPVI*

Step 2: Compute the elevations of points BVC and EVC.

$$elevBVC = elevPVI - \frac{G_1 L}{200};$$

$$elevEVC = elevPVI + \frac{G_2 L}{200} \quad (1)$$

Step 3: Compute the elevation of midpoint of the chord joining BVC and EVC

$$elevMidchord = \frac{elevBVC + elevEVC}{2} \quad (2)$$

Step 4: Compute the tangent offset or external distance at PVI

$$d = \frac{elevPVI - elevMidchord}{2} \quad (3)$$

Step 5: Compute the tangent offset at a station distance *x* from BVC

$$d_x = \left(\frac{x}{L/2}\right)^2 d, \quad \text{when } x \le \tfrac{1}{2}L \text{ or}$$

$$d_x = \left(\frac{L-x}{L/2}\right)^2 d, \quad \text{when } x > \tfrac{1}{2}L \quad (4)$$

Step 6: Compute the grade line elevation at the station distance *x* from BVC.

$$elevGradeline = elevBVC + \frac{G_1 x}{100}, \quad \text{when } x \le \tfrac{1}{2}L \quad (5a)$$

$$elevGradeline = elevEVC - \frac{G_2(L-x)}{100}, \quad \text{when } x > \tfrac{1}{2}L \quad (5b)$$

Step 7: Compute the curve elevation at the station distance *x* from BVC

$$elev\ P_x = \text{Output of Step 5} + \text{Output of Step 6} \quad (6)$$

The design process returns to *Step 5* until the curve elevations at all required stations have been computed.

## 4. Simplified design method

### 4.1 Theoretical consideration

The general form of the symmetric vertical axis parabola used in the geometric design of highway vertical curves is given by the equation,

$$y = ax^2 + bx + c \quad (7)$$

where *a*, *b*, and *c* are constants and *y* is the elevation above datum of a point on the curve located at a horizontal distance *x* from the curve origin.

Consider the case of a symmetric parabolic curve joining two adjacent tangents with slopes $G_1$% and $G_2$% and having its origin coincident with the point BVC as shown in Fig. 1. The equation defining the elevation above point BVC of such a curve is given as;

$$y = ax^2 + bx \quad (8)$$

where *y*, in this case, represents the elevation with respect to point BVC of a general point P located on the curve at a horizontal distance *x* from the curve origin (BVC). The first derivative of Eq. (8) gives the slope of the tangent to the curve at any point, that is.,

$$\frac{dy}{dx} = 2ax + b \quad (9)$$

By selecting the origin (point BVC) and the end (point EVC) of the curve where the grade lines are tangential, the constants *a* and *b* in the parabolic equation may be determined. From Eq. (8),

$$\frac{dy}{dx} = b, \text{ at } x = 0; \qquad \frac{dy}{dx} = 2aL + b, \text{ at } x = L \quad (10)$$

At *x* = 0, and *x* = *L*, the slopes are $\frac{G_1}{100}$ and $\frac{G_2}{100}$ respectively, and these lead to;

$$b = \frac{G_1}{100} \text{ and } a = \frac{G_2 - G_1}{200L} \quad (11)$$

The general expression for the parabolic equation defined by Eq. (8), therefore, becomes

$$y = \frac{G_2 - G_1}{200L}x^2 + \frac{G_1}{100}x \quad (12)$$

### 4.2 Curve elevations

The elevation above datum of the general point P with coordinates (*x*, *y*) on the parabola is obtained by simply adding the value of the point's *y*-coordinate to the curve elevation at the origin (point BVC). Let *elev BVC* and *elev P*$_x$ be the elevation above datum of point BVC and the general point P, respectively. Then,

$$elevP_x = \frac{A}{200L}x^2 + \frac{G_1}{100}x + elevBVC \quad (13)$$

where, $A=(G_2-G_1)$ [-ve for crest curves, +ve for sag curves]

Using the lower chainage grade line, it can be shown by trigonometric considerations that for a symmetric parabolic curve,

$$elevBVC = elevPVI - \frac{G_1 L}{200} \quad (14)$$

Hence Eq. (13) finally becomes

$$elevP_x = \frac{A}{200L}x^2 - \frac{G_1}{200}(L-2x) + elevPVI \quad (15)$$

Eq. (15) is the general expression forming the basis of the proposed simplified approach to the design of symmetric parabolic vertical curves. The equation provides the curve elevations directly for any station within the interval $0 \le x \le L$. It is seen from the expression that to design a curve, the only input parameters required are *elev PVI*, $G_1$, $G_2$, $L$ and the chainage of BVC for distance ($x$) or station referencing. These are exactly the same input parameters required for the tangent offset method. However, unlike the tangent offset approach, the simplified approach does not require additional parameters related to the grade lines and the curve to be evaluated before computations of the curve elevations at the various chainages can be made.

**5. Simplified method applied to asymmetric curves**

Asymmetric parabolic curves have unequal tangent lengths and, therefore, Eq. (15) cannot be applied directly to the design of such curves. According to Uren and Price (1995), such curves are composed essentially of two consecutive symmetric parabolic curves having a common tangent point. This means that Eq. (15) can be applied to each of the component curves separately provided they can be defined in order to determine the elevations of the total curve. Fig. 2 (not drawn to scale) is an asymmetric parabolic curve having line BCD tangential to it at point C.

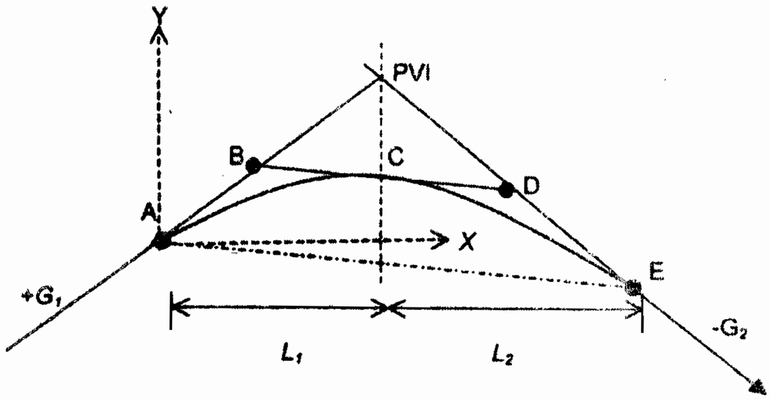


**Fig. 2:** Parabolic curve with unequal tangent lengths (asymmetric curve)

Point C which lies directly below PVI is the common point which divides the curve into two consecutive symmetric parabolae. The lower chainage symmetric curve is defined by tangent lines AB and BC with point B serving as the shadow point of vertical intersection. Similarly, the higher chainage curve is defined by lines CD and DE with point D serving as the shadow point of vertical intersection. The elevations of points B and D (*elevB* and *elevD*, respectively) may be evaluated as follows;

$$elevB = elevPVI - \frac{G_1 L_1}{200} \quad (16)$$

$$elevD = elevPVI + \frac{G_2 L_2}{200} \quad (17)$$

It can be shown by trigonometric considerations (Uren and Price, 1995) that line BD is always parallel to chord AE and has a gradient $G_{BD}$ (in %) given by

$$G_{BD} = \frac{L_1 G_1 + L_2 G_2}{L_1 + L_2} \quad (18)$$

For the design of the lower chainage parabola, application of Eq.(15) results in the following expression:

$$elevP_x = \frac{A}{200L_1}x^2 + \frac{G_1}{200}(2x - L_1) + elevB \quad (19)$$

where $A = G_{BD} - G_1$

By making the appropriate substitution for *elevB* and simplifying, Eq.(19) finally becomes

$$elevP_x = \frac{A}{200L_1}x^2 - \frac{G_1}{100}(L_1 - x) + elevPVI,$$

$$\text{for } x \le L_1 \quad (20)$$

Similarly, to apply Eq 6 to the higher chainage curve, the curve length in the equation must be changed to $L_2$ and the parameter $A$ evaluated as

$$A = G_2 - G_{BD} \quad (21)$$

In addition, the station distance variable $x$ which is measured from the beginning of the point of curvature of the global curve must be replaced by $x$-$L_1$ to reflect the fact that the higher chainage parabola begins at point C which is at a distance of $L_1$ from the beginning of the global curve. By making the appropriate substitution for the elevation of the shadow point of vertical intersection for the curve (point D) and simplifying, the design equation becomes:

$$elevP_x = \frac{A}{200L_2}(x - L_1)^2 + \frac{G_{BD}}{100}(x - L_1) + \frac{AL_2}{200} + elevPVI, \text{ for}$$

$$x \ge L_1 \quad (22)$$

At point C where $x = L_1$ both Eqs.(20) and (22) should yield the same result for the elevation of that point; this serves as a useful check on the computations.

**6. Comparison of Tangent Offset and Simplified Design Methods**

In Table 1, a summary comparison between the two design approaches has been made in terms of the parameters required for curve computations.

Table 1: Computational requirements of the offset and simplified design methods

| | TANGENT OFFSET METHOD | SIMPLIFIED METHOD |
|---|---|---|
| Input Parameters | $G_1$, $G_2$, $L$, *elev PVI*, chainage of PVI, chainage of BVC | $G_1$, $G_2$, $L$, *elev PVI*, chainage of PVI, chainage of BVC. |
| Computational Parameters | 1. Elevation of BVC<br>2. Elevation of EVC<br>3. Mid-chord elevation<br>4. Tangent offset at PVI or external distance<br>5. Tangent grade-line elevation at individual stations<br>6. Tangent offset at individual stations | |
| Output (curve elevations) | Tangent offset + Tangent grade-line elevation | Direct application of Eq.(15) or Eqs.(20) and (22). |

It is to be noted that despite the differences in their procedural details, the two design methods require exactly the same initial input data. The numerical example that follows serves to provide verification of the proposed design approach.

Numerical example

The vertical curve data are as follows:
$G_1$ = +3%; $G_2$ = -1%; $L$ = 360m
PVI at Km 6+480.314 and elevation 235.881m above datum.
A symmetric parabola is to join the grade lines; station BVC and EVC are, respectively, at Km 6+300.314 and Km 6+660.314. Compute the elevation of the high point and 50-m even stations.

*I.* **Solution by Tangent Offsets**

Using the procedure detailed above for the tangent offset method, the computed parameters required for the evaluation of curve elevations are as follows:

1. *External distance, d* = -1.800m
2. *elev BVC* = 230.481m
3. *elev EVC* = 234.081m
4. *elevMidchord* = 232.281m

Subsequent computations and the resulting curve elevations at the required stations have been detailed in Table 2.

Table 2: Parabolic curve computations by tangent offset approach

| Station | $X$ (m) | Tangent Elevation (m) | Tangent Offset (m) | Curve Elevation (m) |
|---|---|---|---|---|
| 6+300.314 (BVC) | 0.000 | 230.481 | 0.000 | 230.481 |
| 6+350 | 49.686 | 231.972 | -0.137 | 231.835 |
| 6+400 | 99.686 | 233.472 | -0.552 | 232.920 |
| 6+450 | 149.686 | 234.972 | -1.245 | 233.727 |
| 6+500 | 199.686 | 235.684 | -1.428 | 234.256 |
| 6+550 | 249.686 | 235.184 | -0.676 | 234.508 |
| 6+570.314 (high point) | 270.000 | 234.981 | -0.450 | 234.531 |
| 6+600 | 299.686 | 234.684 | -0.202 | 234.482 |
| 6+650 | 349.686 | 234.184 | -0.006 | 234.178 |
| 6+660.314 (EVC) | 360.000 | 234.081 | 0.000 | 234.081 |

*II.* **Solution by Simplified Method**

The curve elevations at the required stations or chainages are obtained by direct application of Eq.(15). The resulting output is as detailed in Table 3.

Table 3: Parabolic curve computations by simplified approach

| Station | $X$ (m) | Curve Elevation (m) |
|---|---|---|
| 6+300.314 (BVC) | 0.000 | 230.481 |
| 6+350 | 49.686 | 231.834 |
| 6+400 | 99.686 | 232.920 |
| 6+450 | 149.686 | 233.727 |
| 6+500 | 199.686 | 234.256 |
| 6+550 | 249.686 | 234.508 |
| 6+570.314 (high point) | 270.000 | 234.531 |
| 6+600 | 299.686 | 234.482 |
| 6+650 | 349.686 | 234.178 |
| 6+660.314 (EVC) | 360.000 | 234.081 |

It is seen that the output (Curve Elevation) for each station is the same as in Table 2, however, there are fewer computations associated with the simplified method than with the conventional tangent offset approach. This indeed is a clear demonstration of the simplicity of the proposed method and the reduction in design computations and time.

## 7. CONCLUSION

This paper has presented a simplified method for the design of symmetric parabolic highway vertical curves. Unlike the traditional tangent offset approach which uses the initial curve parameters to determine several other parameters before curve elevations can be determined, the simplified method uses only the initial curve parameters to arrive at curve elevations directly. The method may be applied to the design of asymmetric parabolic curves by treating such curves as a composite of two consecutive symmetric curves joined by a common tangent point. It is demonstrated through a numerical example that the proposed method results in less computations than the traditional tangent offset approach. It is, therefore, expected that the method will result in savings in design time when used for the design of highway vertical curves.

## REFERENCES

Kavanagh, B. F. and Bird, S. J. G., 1990. Surveying: Principles and Applications, Prentice Hall, 2nd edition.

Papacostas, C. S., 1990. Fundamentals of Transportation Engineering, Prentice Hall, Eastern economy edition.

Uren, J. and Price, W. F., 1995. Surveying for Engineers, Macmillan, 3rd edition.