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FINITE ELEMENT SOLUTION OF MICROWAVE HEATING EQUATIONS

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ABSTRACT

A system of equation describing the microwave heating of biologic tissues is considered, with an attempt to providing a solution for the equation. Use is made of the Finite Element Method to provide an approximate solution

KEYWORDS: Microwave Heating, Biologic Tissues, Cancerous Cells, Finite Element Method

1. INTRODUCTION

The effect of microwave heating on cancerous cells on the surface of the skin has been investigated by a number of researchers. Early works in this area were those of Bush in 1866 as reported by Adebile (2004). He noticed that cancerous cells on the surface of the skin of a patient were no longer present after temperature increase. A lot of work on the investigation of the effect of heat deposition and consequent rise in temperature in an attempt of using heat to destroy or control the growth of cancerous cells, has been carried out.

The physical properties of tissues can lead to formation or hindrance of hot spot as reported by Smyth (1992). Saxena and Arya (1981) investigated the steady state temperature distribution on human skin and sub-dermal tissues exposed to a dry and cool environment with negligible perspiration. El-dabe et al (2003) studied the effects of microwave heating on the thermal states of biological tissues They used finite-difference method to predict the effects of thermal physical properties on the transient temperature of biological tissues Adebile (2006) investigated this same problem analytically Recently Jiang et al (2002) discussed the effects of thermal properties and geometrical dimension on the skin burns

Adebile (2004) established qualitatively that solutions exists, and gave conditions for uniqueness of solution for an equation describing the microwave heating of biologic tissues Adebile et al (2006) presented an investigation into the existence and uniqueness of solution of self-similar solution for the coupled Maxwell and Pennes Bio-heat equations for a generalized body heating coefficient of the form $Q(T) = T^m$

Also, a research by Ayeni et al (2006), shows how the choice of perfusion coefficient could lead to more than one temperature fields which could lead to an undesired result. He inferred that to avoid adverse consequences during microwave heating of biologic tissues and an even m is preferred

The purpose of this paper is to provide a mathematical model for the microwave heating of biologic tissues using the finite element method.

This work is presented in 6 sections. Section 1 contains the literature review, in section 2 mathematical formulation is presented, method of solution is outlined in section 3, the exact solution is obtained in section 4, the results are presented in section 5 and these results are discussed in section 6.

2. MATHEMATICAL FORMULATION

Adebile (2003) in his work gave an equation for microwave heating presented in equation 1

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x,t,u,\theta) \quad (1)$$

$$u(x,0) = u_0(x) \qquad x \in \{0,1\} \quad (2)$$

$$u(0,t) = u(1,t) = 0 \qquad t \geq 0 \quad (3)$$

Where

$u(x,t)$ = Temperature at time 't' and space 'x'.

$$q = -\alpha \frac{\partial u(x)}{\partial x} = \text{Heat flux from Fourier's law} \quad (4)$$

$$\alpha = \text{Thermal diffusivity} = \frac{k}{\rho cp} \quad (5)$$

$f(x,t,u) =$ Function describing heat source

The following simplifying assumptions are made-

1 $\alpha = 0.021$

2. Heat source $f(x,t,\ u,) = [x(1-x)]^d e^{-mx}$ (6)

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then equation 1 becomes

$$\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}=[x(1-x)]^d e^{-mx} \quad (7)$$

With initial condition

$u(x,0)=u_0(x)$

And boundary condition

$u(0,t)=u(1,t)=0$

3. METHOD OF SOLUTION

We provide a solution for equation 7 using the finite element method (F.E.M.) a single element is considered and used to develop a 4-element (9-nodes) model.

For a single element (e), weak formulation

$$\int_{x_1}^{x_2} w\left[\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-[x(1-x)]^d e^{-mx}\right]dx=0 \quad (8)$$

Where

w = Interpolation function

Integrating $w\left[-\alpha\frac{\partial^2 u}{\partial x^2}\right]$ by parts

$$\int_{x_1}^{x_2} w\left[-\alpha\frac{\partial^2 u}{\partial x^2}\right]dx=w\left[-\alpha\frac{\partial^2 u}{\partial x^2}\right]_{x_1}^{x_2}+\int_{x_1}^{x_2}\left[\frac{\partial w}{\partial x}\alpha\frac{\partial u}{\partial x}\right]dx \quad (9)$$

Substituting equation 9 in equation 8 we have

$$\int_{x_1}^{x_2}\left[\frac{\partial w}{\partial x}\alpha\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial t}-[x(1-x)]^d e^{-mx}\right]dx+w\left[-\alpha\frac{\partial u}{\partial x}\right]_{x_1}^{x_2}=0 \quad (10)$$

$$\int_{x_1}^{x_2}\left[\frac{\partial w}{\partial x}\alpha\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial t}-[x(1-x)]^d e^{-mx}\right]dx-q_2 w(x_2)-q_1 w(x_1)=0 \quad (11)$$

Where

$$q_2=\left[-\alpha\frac{\partial u}{\partial x}\right]_{x_2} \quad (12)$$

$$q_1=\left[-\alpha\frac{\partial u}{\partial x}\right]_{x_1} \quad (13)$$

Now let

$$u(x,t)=\sum_{j=1}^{n}u_j(t)\phi_j(x) \quad (14)$$

then

$$\frac{\partial u}{\partial t}=\sum_{j=1}^{n}\frac{\partial u_j(t)}{\partial t}\phi_j(x) \quad (15)$$

$$\frac{\partial u}{\partial x}=\sum_{j=1}^{n}u_j(t)\frac{\partial \phi_j(x)}{\partial x} \quad (16)$$

$$w=\phi_i(x) \quad (17)$$

Substituting equation 14, 15, 16 and 17 in equation 11 we have

$$\int_{x_1}^{x_2}\left[\alpha\frac{\partial \phi_i}{\partial x}\left(\sum_{j=1}^{n}u_j(t)\frac{\partial \phi_j(x)}{\partial x}\right)+\phi_i\left(\sum_{j=1}^{n}\frac{\partial u_j(t)}{\partial t}\phi_j(x)\right)-\phi_i\left([x(1-x)]^d e^{-mx}\right)\right]dx-q_2\phi_i(x_2)-q_1\phi_i(x_1)=0 \quad (18)$$

$$\int_{x_1}^{x_2}\left[\alpha\frac{\partial \phi_i}{\partial x}\left(\sum_{j=1}^{n}u_j(t)\frac{\partial \phi_j(x)}{\partial x}\right)+\phi_i\left(\sum_{j=1}^{n}\frac{\partial u_j(t)}{\partial t}\phi_j(x)\right)\right]dx=q_2\phi_i(x_2)+q_1\phi_i(x_1)+\phi_i\left([x(1-x)]^d e^{-mx}\right)$$

$$\Rightarrow [K]\{U\}+[M]\{\dot{U}\}=\{F\} \tag{19}$$

Where

$$K_{ij}=\int_{x_1}^{x_2}\alpha\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} \tag{20}$$

$$M_{ij}=\int_{x_1}^{x_2}\phi_i(x)\phi_j(x) \tag{21}$$

$$F_i=f_i+q_i \tag{22}$$

$$f_i=\int_{x_1}^{x_2}\phi_i\left([x(1-x)]^d e^{-mx}\right)dx \tag{23}$$

$$q_i=\left[-\alpha\frac{\partial u}{\partial x}\right]_{x_i} \tag{24}$$

Now let us make use of Lagrange quadratic interpolation functions

$$\psi_1^e=\left(1-\frac{x}{h}\right)\left(1-\frac{2x}{h}\right),\ \psi_2^e=4\frac{x}{h}\left(1-\frac{x}{h}\right)\ \text{and}\ \psi_3^e=-\frac{x}{h}\left(1-\frac{2x}{h}\right) \tag{25}$$

$$\text{let } h=\text{length of element}=\frac{1}{4}$$

Then

$$M_{11}=\int_0^{1/4}\psi_1\psi_1dx=\frac{1}{30},\ M_{12}=\int_0^{1/4}\psi_1\psi_2dx=\frac{1}{60},\ M_{13}=\int_0^{1/4}\psi_1\psi_3dx=\frac{-1}{120}$$

$$M_{21}=\int_0^{1/4}\psi_2\psi_1dx=\frac{1}{60},\ M_{22}=\int_0^{1/4}\psi_2\psi_2dx=\frac{2}{15},\ M_{23}=\int_0^{1/4}\psi_2\psi_3dx=\frac{1}{60}$$

$$M_{31}=\int_0^{1/4}\psi_3\psi_1dx=\frac{-1}{120},\ M_{32}=\int_0^{1/4}\psi_3\psi_2dx=\frac{1}{60}\ \text{and}\ M_{33}=\int_0^{1/4}\psi_3\psi_3dx=\frac{1}{30}$$

We consider here, a case in which $\alpha=0.021$, $m=1$ and $d=0$
Therefore, the stiffness coefficients are:

$$K_{11}=\int_0^{1/4}0.021\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_1}{\partial x}dx=0.196,\ K_{12}=\int_0^{1/4}0.021\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial x}dx=-0.224$$

$$K_{13}=\int_0^{1/4}0.021\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_3}{\partial x}dx=0.028,\ K_{21}=\int_0^{1/4}0.021\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_1}{\partial x}dx=-0.224$$

$$K_{22}=\int_0^{1/4}0.021\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_2}{\partial x}dx=0.448,\ K_{23}=\int_0^{1/4}0.021\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_3}{\partial x}dx=-0.224$$

$$K_{31}=\int_0^{1/4}0.021\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_1}{\partial x}dx=0.028,\ K_{32}=\int_0^{1/4}0.021\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_2}{\partial x}dx=-0.224\ \text{and}$$

$$K_{33}=\int_0^{1/4}0.021\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_3}{\partial x}dx=0.196$$

Also

$$f_1=\int_0^{1/4}\psi_1e^{-x}dx=0.042,\ f_2=\int_0^{1/4}\psi_2e^{-x}dx=0.147\ \text{and}\ f_3=\int_0^{1/4}\psi_3e^{-x}dx=0.032$$

The finite element model over an element is given as:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{31} & k_{31} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ Q_3 \end{Bmatrix} \quad (27)$$

i.e.

$$\begin{bmatrix} 0.033 & 0.017 & -0.008 \\ 0.017 & 0.133 & 0.017 \\ -0.008 & 0.017 & 0.033 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{bmatrix} 0.196 & -0.224 & 0.028 \\ -0.224 & 0.448 & -0.224 \\ 0.028 & -0.224 & 0.196 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.042 \\ 0.147 \\ 0.032 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ Q_3 \end{Bmatrix}$$

Now for a 4-Quadratic-element solution, the assembled equation is

$$\begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ K^1_{21} & K^1_{22} & K^1_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ K^1_{31} & K^1_{32} & K^1_{33}K^2_{11} & K^2_{12} & K^2_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & K^2_{21} & K^2_{22} & K^2_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & K^2_{31} & K^2_{32} & K^2_{33}+K^3_{11} & K^3_{12} & K^3_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & K^3_{21} & K^3_{22} & K^3_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & K^3_{31} & K^3_{32} & K^3_{33}K^4_{11} & K^4_{12} & K^4_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & K^4_{21} & K^4_{22} & K^4_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & K^4_{31} & K^4_{32} & K^4_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} +$$

$$\begin{bmatrix} M^1_{11} & M^1_{12} & M^1_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ M^1_{21} & M^1_{22} & M^1_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ M^1_{31} & M^1_{32} & M^1_{33}+M^2_{11} & M^2_{12} & M^2_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & M^2_{21} & M^2_{22} & M^2_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & M^2_{31} & M^2_{32} & M^2_{33}+M^3_{11} & M^3_{12} & M^3_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & K^3_{21} & M^3_{22} & M^3_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & K^3_{31} & M^3_{32} & M^3_{33}+M^4_{11} & M^4_{12} & M^4_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & M^4_{21} & M^4_{22} & M^4_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & M^4_{31} & M^4_{32} & M^4_{33} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \\ \dot{u}_8 \\ \dot{u}_9 \end{Bmatrix} = \begin{Bmatrix} f^1_1 \\ f^1_2 \\ f^1_3+f^2_1 \\ f^2_2 \\ f^2_3+f^3_1 \\ f^3_2 \\ f^3_3+f^4_1 \\ f^4_2 \\ f^4_3 \end{Bmatrix} \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix}$$

i.e.

$$\begin{bmatrix} 0.196 & -0.224 & 0.028 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.224 & 0.448 & -0.224 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.028 & -0.224 & 0.392 & -0.224 & 0.028 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.224 & 0.448 & -0.224 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.028 & -0.224 & 0.392 & -0.224 & 0.028 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.224 & 0.448 & -0.224 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.028 & -0.224 & 0.392 & -0.224 & 0.028 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.224 & 0.448 & -0.224 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.028 & -0.224 & 0.196 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} +$$

$$\begin{bmatrix} 0.033 & 0.017 & -0.008 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.017 & 0.133 & 0.017 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.008 & 0.017 & 0.066 & 0.017 & -0.008 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.017 & 0.133 & 0.017 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.008 & 0.017 & 0.066 & 0.017 & -0.008 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.017 & 0.133 & 0.017 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.008 & 0.017 & 0.066 & 0.017 & -0.008 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.017 & 0.133 & 0.017 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.017 & 0.033 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \\ \dot{u}_8 \\ \dot{u}_9 \end{Bmatrix} = \begin{Bmatrix} 0.042 \\ 0.147 \\ 0.074 \\ 0.147 \\ 0.074 \\ 0.147 \\ 0.074 \\ 0.147 \\ 0.032 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \quad (28)$$

Recall boundary conditions

$u(0,t) = 0 \qquad \Rightarrow u_1 = 0$

$u(1,t) = 0 \qquad \Rightarrow u_9 = 0$

Applying boundary conditions and assuming an isolated surface i.e $q_i^c = 0$

The condensed equations become

$$\begin{bmatrix} 0.067 & 0.017 & 0 & 0 & 0 & 0 & 0 \\ 0.017 & 0.067 & 0.017 & -0.008 & 0 & 0 & 0 \\ 0 & 0.017 & 0.067 & 0.017 & 0 & 0 & 0 \\ 0 & -0.008 & 0.017 & 0.067 & 0.017 & -0.008 & 0 \\ 0 & 0 & 0 & 0.017 & 0.067 & 0.017 & 0 \\ 0 & 0 & 0 & -0.008 & 0.017 & 0.067 & 0.017 \\ 0 & 0 & 0 & 0 & 0 & 0.017 & 0.067 \end{bmatrix} \{\dot{u}\} +$$

$$\begin{bmatrix} 0.448 & -0.224 & 0 & 0 & 0 & 0 & 0 \\ -0.224 & 0.392 & -0.224 & 0.028 & 0 & 0 & 0 \\ 0 & -0.224 & 0.448 & -0.224 & 0 & 0 & 0 \\ 0 & 0.028 & -0.224 & 0.392 & -0.224 & 0.028 & 0 \\ 0 & 0 & 0 & -0.224 & 0.448 & -0.224 & 0 \\ 0 & 0 & 0 & 0.028 & -0.224 & 0.392 & -0.224 \\ 0 & 0 & 0 & 0 & 0 & -0.224 & 0.448 \end{bmatrix} \{u\} = \begin{Bmatrix} 0.147 \\ 0.074 \\ 0.147 \\ 0.074 \\ 0.147 \\ 0.074 \\ 0.147 \end{Bmatrix} \quad (29)$$

We would apply the Crank-Nicholson method for finite difference approximation of time schemes. Crank-Nicholson equation states that

$$\left[M + \frac{hK}{2}\right]u(n+1) = \left[M - \frac{hK}{2}\right]u(n) + \frac{h(F(n) + F(n+1))}{2} \quad (30)$$

$$u(n+1) = [S]u(n) + [C] \quad (31)$$

$$S = \left[M + \frac{hK}{2}\right]^{-1}\left[M - \frac{hK}{2}\right] \quad (32)$$

$$C = \left[M + \frac{hK}{2}\right]^{-1}[hF(n)] \quad (33)$$

From equation 29, we have that

$$M=\begin{bmatrix} 0.067 & 0.017 & 0 & 0 & 0 & 0 & 0 \\ 0.017 & 0.067 & 0.017 & -0.008 & 0 & 0 & 0 \\ 0 & 0.017 & 0.067 & 0.017 & 0 & 0 & 0 \\ 0 & -0.008 & 0.017 & 0.067 & 0.017 & -0.008 & 0 \\ 0 & 0 & 0 & 0.017 & 0.067 & 0.017 & 0 \\ 0 & 0 & 0 & -0.008 & 0.017 & 0.067 & 0.017 \\ 0 & 0 & 0 & 0 & 0 & 0.017 & 0.067 \end{bmatrix} \text{ and}$$

$$K=\begin{bmatrix} 0.448 & -0.224 & 0 & 0 & 0 & 0 & 0 \\ -0.224 & 0.392 & -0.224 & 0.028 & 0 & 0 & 0 \\ 0 & -0.224 & 0.448 & -0.224 & 0 & 0 & 0 \\ 0 & 0.028 & -0.224 & 0.392 & -0.224 & 0.028 & 0 \\ 0 & 0 & 0 & -0.224 & 0.448 & -0.224 & 0 \\ 0 & 0 & 0 & 0.028 & -0.224 & 0.392 & -0.224 \\ 0 & 0 & 0 & 0 & 0 & -0.224 & 0.448 \end{bmatrix}$$

It follows therefore that:

$$[S]=\begin{bmatrix} 0.4796 & 0.2829 & -0.0208 & 0.0059 & -0.0017 & 0.0005 & -0.00013 \\ 0.2968 & 0.5022 & 0.3210 & -0.0906 & 0.0263 & -0.0074 & 0.0020 \\ -0.0208 & 0.2887 & 0.4571 & 0.2892 & -0.0227 & 0.0064 & -0.0017 \\ 0.0243 & -0.0906 & 0.3230 & 0.4948 & 0.3230 & -0.0906 & 0.0243 \\ -0.0017 & 0.0064 & -0.0227 & 0.2892 & 0.4571 & 0.2887 & -0.0208 \\ 0.0020 & -0.0074 & 0.0262 & -0.0906 & 0.3210 & 0.5022 & 0.2968 \\ -0.00013 & 0.00048 & -0.0017 & 0.0059 & -0.0208 & 0.2829 & 0.4796 \end{bmatrix}$$

And

$$[C]=\begin{bmatrix} 0.1599 \\ 0.0701 \\ 0.1550 \\ 0.0754 \\ 0.1550 \\ 0.0701 \\ 0.1599 \end{bmatrix}$$

Recall initial condition

$$u(x,0)=U_0(x)$$

Assuming $U(0)=37^{0}C$ human body temperature

Then

$$u_o=\begin{bmatrix} 37 \\ 37 \\ 37 \\ 37 \\ 37 \\ 37 \\ 37 \end{bmatrix}$$

$$\overset{\circ}{u}_1 = \begin{bmatrix} 0.4796 & 0.2829 & -0.0208 & 0.0059 & -0.0017 & 0.0005 & -0.00013 \\ 0.2968 & 0.5022 & 0.3210 & -0.0906 & 0.0263 & -0.0074 & 0.0020 \\ -0.0208 & 0.2887 & 0.4571 & 0.2892 & -0.0227 & 0.0064 & -0.0017 \\ 0.0243 & -0.0906 & 0.3230 & 0.4948 & 0.3230 & -0.0906 & 0.0243 \\ -0.0017 & 0.0064 & -0.0227 & 0.2892 & 0.4571 & 0.2887 & -0.0208 \\ 0.0020 & -0.0074 & 0.0262 & -0.0906 & 0.3210 & 0.5022 & 0.2968 \\ -0.00013 & 0.00048 & -0.0017 & 0.0059 & -0.0208 & 0.2829 & 0.4796 \end{bmatrix} \begin{bmatrix} 37 \\ 37 \\ 37 \\ 37 \\ 37 \\ 37 \\ 37 \end{bmatrix} + \begin{bmatrix} 0.1599 \\ 0.0701 \\ 0.1550 \\ 0.0754 \\ 0.1550 \\ 0.0701 \\ 0.1599 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 27.7687 \\ 38.9678 \\ 37.0149 \\ 37.3773 \\ 37.0149 \\ 38.9278 \\ 27.7687 \end{bmatrix}$$

This iteration is then carried out to get various values of U at various times and nodes and the results are as shown in table 1 below.

4. EXACT SOLUTION

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = \left[x(1-x)\right]^d e^{-mx}$$

$$but\, d = 0, m = 1\, and\, \alpha = 0.021$$

$$\frac{\partial u}{\partial t} - 0.021\frac{\partial^2 u}{\partial x^2} = e^{-x}$$

$$u(x,0) = u(l,0) = 0$$

$$u(x,0) = 37^0 C$$

Rearranging,

$$\frac{\partial u}{\partial t} = 0.021\frac{\partial^2 u}{\partial x^2} + e^{-x}$$

$$Let\ u = U + \psi(x)$$

$$then,\ \frac{\partial u}{\partial t} = \frac{\partial U}{\partial t}$$

$$and\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} + \psi''(x)$$

$$\frac{\partial U}{\partial t} = 0.021\frac{\partial^2 U}{\partial x^2} + \psi''(x) + e^{-x}$$

$$Now,\ let\ \psi''(x) + e^{-x} = 0.$$

The boundary conditions becomes

$$u(0,t) = U(0,t) + \psi(0) = 0$$

$\therefore U(0,t) = -\psi(0)$

Also, $U(1,t) + \psi(1) = 0$

$\therefore U(1,t) = -\psi(1)$

Let $\psi(0) = \psi(1) = 0$

$\therefore \psi'' + e^{-x} = 0$

$\Rightarrow \psi'' = -e^{-x}$

$\psi'(x) = e^{-x} + C_1$

$\therefore \psi(x) = -e^{-x} + C_1 x + C_2$

applying boundary conditions,

then $C_1 = -0.632 \quad C_2 = 1$

$\therefore \psi(x) = 1 - e^{-x} - 0.632x$

$\Rightarrow 37^0 = U(x,0) + 1 - e^{-x} - 0.632x$

$U(x,0) = 36 + e^{-x} + 0.632x$

Now the governing equation becomes,

$$\frac{\partial U}{\partial t} = 0.021\frac{\partial^2 U}{\partial x^2}$$

with

$U(0,t) = U(l,0) = 0$

$U(x,0) = 36 + e^{-x} + 0.632x$

Let $U = XT \quad \frac{\partial U}{\partial t} = XT'$ *and* $\frac{\partial^2 U}{\partial x^2} = X''T$

$\therefore XT' = 0.021X''T$

separating variables, we have

$$\frac{T'}{0.021T} = \frac{X''}{X} = -p^2$$

$\Rightarrow T' = -0.021p^2 T.$

or $T' + 0.021p^2 T = 0$

$\therefore$ *the solution is*

$T = Ae^{-0.021p^2 t}$

Also, $X'' + p^2 X = 0$

$\Rightarrow X = B\cos px + C\sin px$

but $X(0) = 0$ *hence* $B = 0$

$\therefore X = C\sin px$ *and* $T = Ae^{-0.021p^2 t}$

since $U = XT. \quad \therefore U = De^{-0.021p^2 t}\sin(px)$

$$Also,\ U(l,t)=0 \quad \therefore 0 = De^{-0.021p^2t}\sin(px)$$

$$i.e. \sin p = 0$$

$$or\ p = n\pi \quad n = 1,2,3,...$$

$$\because U = D\sin(n\pi x)\cdot e^{-0.021n^2\pi^2 t}$$

Applying initial condition

$$i.e. U(x,0) = 36 + e^{-x} + 0.632x$$

$$36 + e^{-x} + 0.632x = D\sin(n\pi x)$$

$$\therefore D = 2\int_0^1 (36 + e^{-x} + 0.632x)\sin(n\pi x)dx$$

Hence,

$$U = \sum_{n=1}^{\infty}\left[2\int_0^1 (36 + e^{-x} + 0.632x)\sin(n\pi x)dx\right]\sin(n\pi x)\cdot e^{-0.021n^2\pi^2 t}$$

$$recall,\ u(x,t) = U(x,t) + \psi(x)$$

$$u(x,t) = (1 - e^{-x} - 0.632x) + \sum_{n=1}^{\infty}\left[2\int_0^1 (36 + e^{-x} + 0.632x)\sin(n\pi x)dx\right]\sin(n\pi x)\cdot e^{-0.021n^2\pi^2 t}$$

5. RESULTS

The results obtained considering a four-Quadratic element mesh (9-nodes) and sixteen-linear element mesh ([illegible]-nodes) are shown in table 1 below and compared with the exact solution obtained by the Fourier series technique.

Table 1: Comparison of finite element solutions with the exact solution

Time t (s)	Exact solution			4-quadratic elements			16-linear elements		
	x = 0.25	x = 0.50	x = 0.75	x = 0.25	x = 0.50	x = 0.75	x = 0.25	x = 0.50	x = 0.75
0.1	36.997	37.001	36.997	38.928	39.377	38.928	37.086	37.095	37.086
0.2	35.767	37.003	35.767	37.095	36.776	37.095	37.042	37 190	37.042
0.2	36.045	37.003	36.045	35.394	37.199	35.394	36.471	37.284	36 471
0.4	35.017	36.997	35.017	33.713	37.445	33.713	35.251	37.376	35.251
0.5	33.882	36.965	33.882	32.225	37.515	32.225	34.190	37.453	34.190
0.6	32.744	36.887	32.744	30.909	37.387	30.909	33.058	37.490	33.058
0.7	31.652	36.747	31.652	29.448	37.100	29.448	32.047	37.457	32.047
0.8	30.624	36.538	30.624	28.711	36.687	28.711	31.085	37.342	31.085
0.9	29.665	36.263	29.665	27.777	36.182	27.777	20.211	37.154	20.211
1.0	28.772	35.925	28.772	26.926	35.610	26.926	29.400	36.897	29.400
2.0	22.298	30.772	22.298	20.819	28.960	20.819	23.666	32.481	23.666
3.0	17.961	25.273	17.961	16.676	23.251	16.676	19.880	27.671	19.880
4.0	14.584	20.595	14.584	13.577	18.887	13.577	16.993	23.583	16.993
5.0	11.862	16.760	11.862	11.234	15.568	11.234	14.559	20.238	14.559
10.0	4.249	5.996	4.249	5.755	7.259	5.755	7.936	10.873	7.936

The solutions at the position $x = 0.25$ is further compared in the graphical representations below

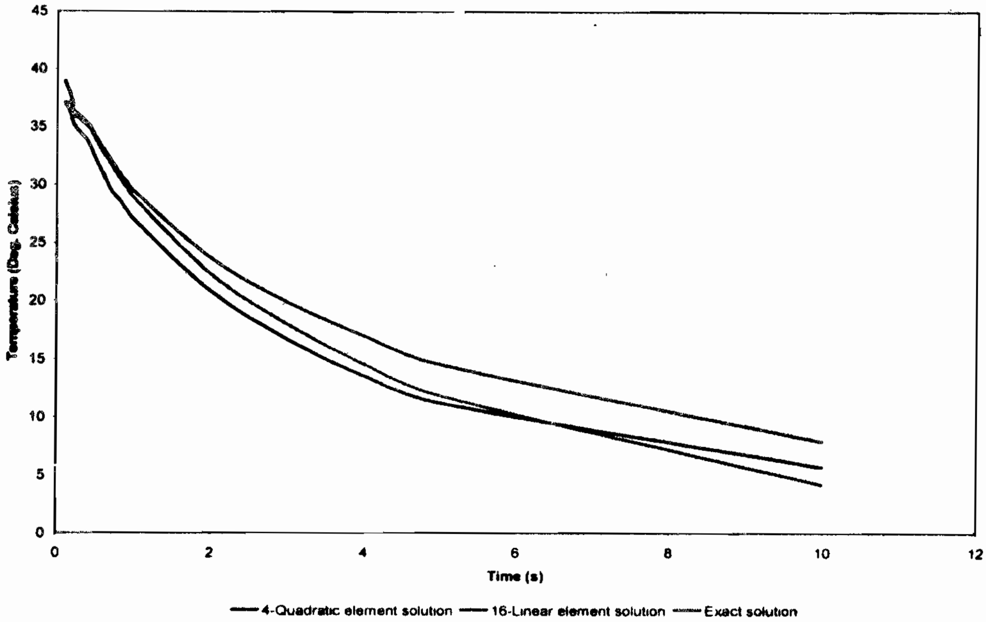

Figure 1: Temperature variation with Time at position $x = 0.25$

6. DISCUSSION

The results displayed in Table 1 show that the numerical values obtained by the finite element-finite difference method were very close to the exact values, which are also displayed in Table 1. Furthermore, Figure 1, which is a graphical representation of the time rate of variation of temperature, emphasizes the close proximity between the finite element-finite difference solution and the exact solution. A close examination of the results in Table 1 indicates that the temperature values were on a decline along an exponential trajectory (path), this is due to the type of heating source employed (i.e. $f(x,t,u,) = [x(1-x)]^d e^{-mx}$) which decays exponentially with time.

Therefore cancerous cells in a biological tissue can be destroyed with the application of heat without damaging other useful cells, as the temperature in every other part is known and controllable. It should be noted however that the thermal conductivity of the tissue is assumed to be constant.

Hence, the finite element-finite difference method is a highly accurate numerical tool for predicting the temperature variation with time at any point on a body whose ends are isolated.

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