Numerical Simulation of the Harmonic Oscillator Differential Equation Using Finite Difference Schemes

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ORIGINAL RESEARCH

Abstract— In this study, we have developed a new type of finite difference scheme using dynamically renormalized denominator functions and trigonometric and exponential interpolating functions. These schemes show local stability, convergence, and consistency. The model provides an improved numerical scheme for the Harmonic Oscillator Differential Equation. Additionally, we compared the new model with a previous discrete model for the Harmonic equation and also confirmed the suitability of the new schemes for the numerical simulation of the tested problems.

Keywords— Finite Difference, Interpolant, Harmonic Oscillator, Consistent.

1 Introduction

The concept of a harmonic oscillator is commonly exemplified by a single spring and single mass system, assuming the absence of friction and damping. In reality, such ideal models are rare, as practical applications often involve factors like damping, friction, and external forces. These considerations lead to more complex but realistic harmonic equations, enabling a detailed study of the modelled physical phenomena. Dealing with motion in a resistive medium is a challenging task. We often make simplifying assumptions about the nature of resistance, which is reasonable in many real-life scenarios. Starting with the ideal harmonic oscillator, where there is no resistance, we derive a family of numerical schemes that replicate the behaviour of the appropriate second-order initial value equation expressing specific harmonic oscillator differential equations. We hope that this discrete model will give solutions and curves displaying behaviours close to the exact non-standard schemes proposed by Mickens (1994). Finite difference schemes have become a prominent tool for addressing this equation, as they provide a means to transform the continuous problem into a set of discrete algebraic equations that can be numerically solved (Sprott & Hoover, 2017). The damped harmonic oscillator model, in particular, has been the focus of extensive research, as it represents a more realistic representation of physical systems, where frictional forces are present (Alharthi et al., 2023).

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Several types of damped oscillation have been studied, with the simple harmonic oscillator being a subclass of this broader category.

Researchers have compared various FDM techniques, such as the Euler method, Verlet algorithm, and Gear's predictor-corrector method, to assess their performance in simulating one-dimensional oscillators (Salehi and Granpayeh, 2020). Recent literature has expanded the scope of FDM to tackle complex scenarios, including multi-dimensional oscillators and quantum systems. For example, the Schrödinger equation in polar coordinates has been solved using a finite-difference time-domain (FDTD) method, demonstrating the method's adaptability to different coordinate systems and its ability to handle quantum harmonic oscillators (Jain et al., 2021), while (Darley et al., 2021) studied the impact of lightning electromagnetic pulses (LEMP) on a transmission line using a cross-linked polyethene (XLPE) insulated power cable using finite difference.

These new discrete models are noteworthy due to their close resemblance to earlier schemes for oscillator equations. They have been subjected to numerical experiments, yielding intriguing graphs.

2 MATERIALS AND METHODOLOGY

This method is characterized by replacing the denominator (h) in the first-order discrete derivative with a nonnegative function or a more complex function of the step sizes ($\mu(h)$) that satisfies $\mu(h) \rightarrow h + 0(h^2)$ as $h \rightarrow 0$. This alteration is intended to improve the qualitative behaviour of the numerical solutions. The nonlinear terms in functions are approximated in a nonlocal way, meaning they use a function of several points of the mesh rather than just a single point. This approach is designed to enhance stability and accuracy.

The nonstandard rules from Micken (1994) and their extensions in Anguelov and Lubuma (2003) as well as in Obayomi and Oke (2016) are presented below.

$$y' = \frac{y_{k+1} - y_k}{\mu};\ \mu(h) \to h + 0(h^2)\ as\ h \to 0 \quad (1)$$

$$y' \equiv \frac{(y_{k+1} - \propto y_k)}{\mu};\ \mu(h) \to h + 0(h^2),$$

$$\propto (h) \to 1\ as\ h \to 0 \quad (2)$$

$$\equiv y' = \frac{(y_{k+1} - \propto y_{k-1})}{2\mu};\ \mu(h) \to h + 0(h^2),$$

$$\beta(h) \to 1\ as\ h \to 0 \quad (3)$$

$$y'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{\psi^2};\ \psi(h) \to h^2 + 0(h^n)$$

$$as\ h \to 0\ for\ n > 2 \quad (4)$$

then,

$$y_{k+1} \equiv p y_{k+1} + q y_k;\ \ p + q = 1 \quad (5)$$

$$y_k \equiv \frac{(y_{k+1} + \propto y_k)}{2};\ \propto (h) \to 1\ as\ h \to 0 \quad (6)$$

$$y_{k+1} \equiv \frac{(2y_k + \propto y_{k-1})}{3};\ \propto (h) \to 1 \quad (7)$$

then,

$$\mu = \sin(\beta h),\ \beta \in \mathrm{R} \to h + 0(h^2) \quad (8)$$

$$\mu = \frac{(e^{\gamma h} - 1)}{\gamma},\ \gamma \in \mathrm{R} \to h + 0(h^2) \quad (9)$$

$$\propto = \cos(\beta h),\ \beta \in \mathrm{R} \to 1 \quad (10)$$

$$\left.\begin{array}{c} \psi = 4sin^2\left(\frac{h}{2}\right) \\ h^2\mu \to 0(h^4) \end{array}\right\} \quad (11)$$

2.1 THE HARMONIC OSCILLATOR EQUATION

The equation of the harmonic oscillator describes the model under consideration, as given by

$$y'' + 2\varepsilon y' + y = 0 \quad (12)$$

Where;

x= the distance or displacement of the body involved in the oscillation, and it varies with time.

In his study, Mickens (1994) has proposed a nonstandard strategy for this model.

$$\left\{\frac{y_{k+1} - 2y_k + y_{k-1}}{\psi^2}\right\} = 2\varepsilon\left\{\frac{y_k - \mu y_{k-1}}{\psi}\right\} + \left\{\frac{2(1-\mu)y_k + (x^2 + x^2 - 1)y_{k-1})}{\psi^2}\right\} \quad (13)$$

with $\mu = \cosh$ $\quad \psi = 4\sin^2(\frac{h}{2})$

2.2 NON-LINEAR INTERPOLATION FOR DERIVATION OF THE STANDARD FINITE DIFFERENCE SCHEME

We begin with the general harmonic equation (1), the solutions to the simplified differential equation without the velocity term resemble sine and cosine functions, while the solutions to the simplified differential equation without the acceleration term resemble exponential functions.

$$y(x) = a_0 + a_1 x + e^{\beta x} + a_2 \sin(\propto x^2 + k) \quad (14)$$

The choice of the interpolation function is influenced by the combination of linear displacement, growth, decay at each time t, and sinusoidal motion.

$$\mathrm{y(x)} = a_0 + e^{\beta x} + a_1 x + a_2 \sin(\alpha x^2 + k)$$

$$y' = \beta e^{\beta x} + a_1 + a_2 2 \propto x \cos(\propto x^2 + k) \quad (15)$$

$$y'' = a_2[2 \propto x(-2 \propto x \sin(\propto x^2 + k) + 2 \propto \cos(\propto x^2 + k)] + \beta^2 e^{\beta x}$$

$$y'' = a_2[-(2 \propto x)^2 \sin(\propto x^2 + k) + 2 \propto \cos(\propto x^2 + k)] + \beta^2 e^{\beta x} \quad (16)$$

$$y''' = a_2[-(2 \propto x)^3 \cos(\propto x^2 + k) + 4 \propto x \sin(\propto x^2 + k)] + \beta^3 e^{\beta x} \quad (17)$$

From (15), (16), (17)

$$a_1 = y' - \beta e^{\propto x} - a_2 2 \propto x \cos(\propto x^2 + k) \quad (18)$$

$$a_2 = \frac{y'' - \beta^2 e^{\beta x}}{[-(2\propto x)^2 \sin(\propto x^2 + k) + 2\propto \cos(\propto x^2 + k)]} \quad (19)$$

The discrete form

$$\mathrm{y(x)} = a_0 + e^{\beta x} + a_1 x + a_2 \sin(\beta x^2 + k)$$

$$\mathrm{y}(x_{n-1}) = a_0 + e^{\beta x_{n-1}} + a_1 x_{n-1} + a_2 \sin(\propto x_{n-1}^2 + k)$$

$$\mathrm{y}(x_n) = a_0 + e^{\beta x_n} + a_1 x_n + a_2 \sin(\propto x_n^2 + k)$$

$$\mathrm{y}(x_{n+1}) = a_0 + e^{\beta x_{n+1}} + a_1 x_{n+1} + a_2 \sin(\propto x_{n+1}^2 + k)$$

$$\mathrm{y}(x_{n+1}) - 2\mathrm{y}(x_n) + \mathrm{y}(x_{n-1}) = (a_o - 2a_o + a_o) + \left(e^{\beta x_{n+1}} - 2e^{\beta x_n} + e^{\beta x_{n-1}}\right) + a_1(x_{n-1} + x_{n+1} - 2x_n) + a_2[(\sin(\propto x_{n-1}^2 + k) + \sin(\propto x_{n+1}^2 + k) - 2\sin(\propto x_n^2 + k)] \quad (20)$$

$$E_n = [(\sin(\propto x_{n-1}^2 + k) + \sin(\propto x_{n+1}^2 + k) - 2\sin(\propto x_n^2 + k)]$$

put $x_{n-1} = a + h(n-1)$, and $x_{n-1} = a + h(n+1)$,

$$y_{n+1} - 2y_n + y_{n-1} \equiv e^{\beta(a+nh)}\left(e^{\beta h} + e^{-\beta h} - 2\right) + a_2[E_n]$$

$$E_n = [(\sin(\propto x_{n-1}^2 + k) + \sin(\propto x_{n+1}^2 + k) - 2\sin(\propto x_n^2 + k)]$$

Let

$$\left.\begin{array}{c} A = \propto (x_n^2 + h^2) + k \\ B = 2 \propto h x_n \\ C = (\propto x_n^2 + k) \end{array}\right\} \quad (21)$$

$$\Rightarrow E_n = [(\sin(A + B) + \sin(A - B) - 2\sin(C)]$$

$$E_n = 2\sin(A)\ \cos(B) - 2\sin(C)]$$

$$E_n = 2\sin(\propto (x_n^2 + h^2) + k)\cos(2 \propto h x_n) - 2\sin(\propto x_n^2 + k)] \quad (22)$$

$$y_{n+1} - 2y_n + y_{n-1} = e^{\beta(a+nh)}\left(e^{\beta h} + e^{-\beta h} - 2\right) + \frac{\{y'' - \beta^2 e^{\beta x}\}[P_n]}{[-(2\propto x)^2 \sin(\propto x^2 + k) + 2\propto \cos(\propto x^2 + k)]} \quad (23)$$

$$= e^{\beta(a+nh)}\left(e^{\beta h} + e^{-\beta h} - 2\right) + \frac{\{y'' - \beta^2 e^{\beta x}\}[2\sin(\propto(x_n^2 + h^2) + k)\cos(2\propto h x_n) - 2\sin(\propto x_n^2 + k)]}{[-(2\propto x)^2 \sin(\propto x^2 + k) + 2\propto \cos(\propto x^2 + k)]} \quad (24)$$

sub ψ for h

$$\equiv \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

$$\equiv \frac{y_{n+1} - 2y_n + y_{n-1}}{\psi^2} \quad (25)$$

using (11), we have a new scheme (New-h)

then

$$\psi = \begin{cases} \sin(h) \\ \dfrac{e^{\gamma h} - 1}{\gamma} \end{cases}$$

we have schemes (New-Sin) and (New-Exp) respectively.

3 QUANTITATIVE PROPERTIES OF THE NEW SCHEME

Theorem 2 (Henrici, 1962)

In the defined region, let the incremental function of the one-step scheme above be continuous and jointly a function of its arguments.

$x \in [a,b]$ *and* $y \in (-\infty, \infty), 0 \le h \le h_0; h_0 > 0$

and let there exist a constant L such that

$\zeta(x_n, y_n, h) - \zeta(x_n, y_n^*, h) \le L|y_n - y_n^*|$

for all (x_n, y_n, h) *and* (x_n, y_n^*, h) *in the defined region.*

Then

$(x_n, y_n; 0) = (x_n, y_n^*)$ *is a necessary condition for the convergence of the new derived scheme.*

Theorem 2 (Fatunla, 1998)

Let $y_n = y(x_n)$ *and* $p_n = p(x_n)$ *denote two different numerical solutions of the differential equation with the initial condition specified as*

$y_0 = y(x_0) = \vartheta$ *and* $q_0 = q(x_0) = \vartheta^*$

$\Rightarrow |\vartheta - \vartheta^*| < \xi \quad \xi > 0$

we have

$y_{n+1} = y_n + h\zeta(x_n, y_n; h)$

$q_{n+1} = q_n + h\zeta(x_n, q_n; h)$

$|y_{n+1} - q_{n+1}| \le k|\vartheta - \vartheta^*|$ *is the schemes' stability and convergence necessary and sufficient condition.*

3.1 PROOF OF CONVERGENCE

Let

$$\mathrm{E_n} = 2\sin(\propto (\mathrm{x_n^2} + \mathrm{h^2}) + \mathrm{k}\ )\cos(2 \propto \mathrm{hx_n}) - 2\sin(\propto \mathrm{x_n^2} + \mathrm{k})] \quad (26)$$

$y' = f_n$, $y'' = f_n'$ and $y''' = f_n''$

$$\mathrm{P_n} = [-(2 \propto \mathrm{x_n})^2 \sin(\propto \mathrm{x_n^2} + \mathrm{k}) + 2 \propto \cos(\propto \mathrm{x_n^2} + \mathrm{k})]$$

$$\mathrm{Q_n} = \beta^2 e^{\beta \mathrm{x_n}}$$

$$\mathrm{R_n} = e^{\beta(a+nh)}\left(e^{\beta h} + e^{-\beta h} - 2\right)$$

$$y_{n+1} = 2y_n + y_{n-1} + \mathrm{R_n} + \left[\frac{(f_n' - Q_n)E_n}{P_n}\right] \quad (27)$$

$$2y_n - y_{n-1} \cong y_n \text{ (small h)}$$

$$y_{n+1} = y_n + \mathrm{R_n} + \left[\frac{(f_n' - Q_n)E_n}{P_n}\right]$$

$$y_{n+1} = y_n + \mathrm{R_n} + \left[\frac{(f_n' - Q_n)E_n}{P_n}\right] \quad (28)$$

$$y_{n+1} = y_n + \left[\mathrm{R_n} - \frac{Q_n E_n}{P_n}\right] + \left[\frac{E_n}{P_n}\right] f_n' \quad (29)$$

with incremental function

$$\zeta(x_n, y_n; h) = \left[\mathrm{R_n} - \frac{Q_n E_n}{P_n}\right] + \left[\frac{E_n}{P_n}\right] f_n' \quad (30)$$

$\zeta(x_n, y_n, h) = M + N f_n'$ [M is fixed (n<∞)]

$$\zeta(x_n, y_n; h) - \zeta(x_n, y_n^*; h) = M[f'(x_n, y_n; h) - f'(x_n, y_n^*; h)] \quad (31)$$

$$= M[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= M\left[\frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n - y_n^*)\right]$$

$$\mathrm{L} = \mathrm{SUP}_{(x_n, y_n) \in D} \frac{\partial f'(x_n, \bar{y})}{\partial y_n}$$

then

$$\zeta(x_n, y_n; h) - \zeta(x_n, y_n^*; h) = M[L(y_n - y_n^*)] \quad (32)$$

Let $A = ML$

$\zeta(x_n, y_n, h) - \zeta(x_n, y_n^*, h) \le A|y_n - y_n^*|$

3.2 SCHEME CONSISTENCY

$$y_{n+1} = y_n + \left[\mathrm{R_n} - \frac{Q_n E_n}{P_n}\right] + \left[\frac{E_n}{P_n}\right] f_n' \quad (33)$$

$$y_{n+1} = y_n + M + N f_n'$$

If $h = 0, \mathrm{R_n} = 0, \mathrm{E_n} = 0$ & $M = N = 0$

$$\Rightarrow \zeta(x_n, y_n; 0) \equiv 0 \quad (34)$$

3.3 SCHEME STABILITY

$y_{n+1} = y_n + M + N f_n'(x_n, y_n)$

let $G_{n+1} = G_n + M + N f_n'(x_n, G_n)$

$$y_{n+1} - G_{n+1} = y_n - G_n + \{M - M\} + N[f_n'(x_n, y_n) - f_n'(x_n, G_n)] \quad (35)$$

$$\mathrm{L} = \mathrm{SUP}_{(x_n, y_n) \in D} \frac{\partial f'(x_n, G_n)}{\partial E_n}$$

$y_{n+1} - G_{n+1} = y_n - G_n + ML(y_n - G_n)$

$|y_{n+1} - G_{n+1}| = |y_n - G_n| + [ML]|(y_n - E_n)|$

$|y_{n+1} - G_{n+1}| \le \mathrm{N}\, Q|y_n - S_n| \quad (Q = ML)$

$$|y_{n+1} - G_{n+1}| \le K|\vartheta - \vartheta^*| \quad (36)$$

3.4 APPLICATION TO THE HARMONIC OSCILLATOR

Using (1), we have

$$\mathrm{y}'' = -2\varepsilon \mathrm{y}' - \mathrm{y} \quad (37)$$

$$\mathrm{y}''' = -2\varepsilon \mathrm{y}'' - \mathrm{y}' \quad (38)$$

From the Nonstandard theory;

$$\mathrm{y}' = \frac{y_{n+1} - y_n}{\mu}$$

$$f_n' = -2\varepsilon\left(\frac{y_{n+1} - y_n}{\mu}\right) - y_n$$

$$f_n' = -2\varepsilon\left(\frac{y_{n+1}}{\mu}\right) + 2\varepsilon\left(\frac{y_n}{\mu}\right) - y_n$$

from (25)

$$y_{n+1} = y_n + M + N f_n'(x_n, y_n)$$

$$y_{n+1} = y_n + \left[\mathrm{R_n} - \frac{\mathrm{Q_n E_n}}{\mathrm{P_n}}\right] - \left\{\frac{\mathrm{E_n}}{\mathrm{P_n}}\right\}\left\{\left(\frac{2\varepsilon}{\mu}\right)\right\} y_{n+1} + \left\{\left(\frac{2\varepsilon}{\mu}\right) - 1\right\}\left\{\frac{\mathrm{E_n}}{\mathrm{P_n}}\right\} y_n \quad (39)$$

$$\left\{\frac{\mu \mathrm{P_n} + 2\varepsilon \mathrm{E_n}}{\mu \mathrm{P_n}}\right\} y_{n+1} = \left\{\mathrm{R_n} - \frac{\mathrm{Q_n E_n}}{\mathrm{P_n}}\right\} + \left\{1 + \frac{2\varepsilon \mathrm{E_n}}{\mu \mathrm{P_n}} - \frac{\mathrm{E_n}}{\mathrm{P_n}}\right\} y_n$$

$$y_{n+1} = \left\{\frac{\mu \mathrm{P_n}}{\mu \mathrm{P_n} + 2\varepsilon \mathrm{E_n}}\right\}\left\{\mathrm{R_n} - \frac{\mathrm{Q_n E_n}}{\mathrm{P_n}}\right\} + \left\{\frac{\mu \mathrm{P_n}}{\mu \mathrm{P_n} + 2\varepsilon \mathrm{E_n}}\right\}\left\{1 + \frac{2\varepsilon \mathrm{E_n}}{\mu \mathrm{P_n}} - \frac{\mathrm{E_n}}{\mathrm{P_n}}\right\} y_n \quad (40)$$

$$\mathrm{E_n} = 2\sin(\propto (\mathrm{x_n^2} + \mathrm{h^2}) + \mathrm{k}\ )\cos(2 \propto \mathrm{hx_n}) - 2\sin(\propto \mathrm{x_n^2} + \mathrm{k})]$$

$$\mathrm{P_n} = [-(2 \propto \mathrm{x_n})^2 \sin(\propto \mathrm{x_n^2} + \mathrm{k}) + 2 \propto \cos(\propto \mathrm{x_n^2} + \mathrm{k})]$$

$$\mathrm{Q_n} = \beta^2 e^{\beta x_n}$$

$$\mathrm{R_n} = e^{\beta(a+nh)}\left(e^{\beta h} + e^{-\beta h} - 2\right)$$

4 RESULTS OF NUMERICAL EXPERIMENT

The method is used to solve harmonic equations (Mickens, 1994), and the schemes are tested using different step sizes. The results of the numerical simulation are presented in the graphs. NTSD-Ex represents the exact solution as derived from (Mickens, 1994). NTSD-Exp is the solution with the step size replaced by an exponential function, and NTSD-Sin is the solution with the step size replaced by a sine function. Similarly, we have NewSchExp, which is the new derived scheme using exponential functions as step-size, and NewSchSin, which is the new derived scheme employing a sine function as step-size.

Problem I

Graph for all schemes for h=0.001, $\varepsilon = 0.0001$, simulation parameters $r = 0.528, \gamma = 0.66\ ,\beta = 1, \alpha = -0.65$.

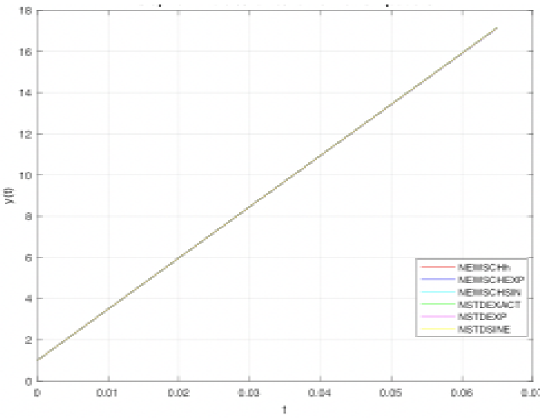

Fig. 1: Graph showing the solutions from all new developed schemes.

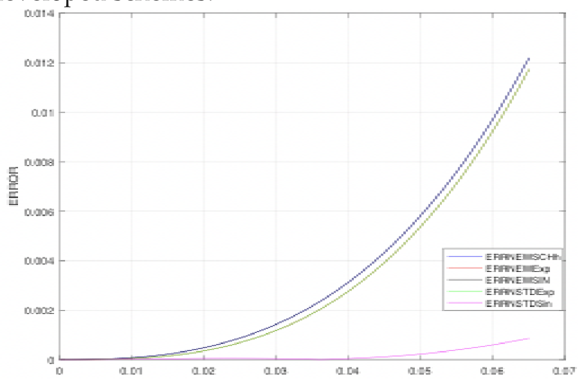

Fig. 2: Graph showing the error deviation of new developed schemes.

Problem II

Graphs of all schemes for h=0.01 and $\varepsilon = 0.0001$ simulation parameters $r = 0.528, \gamma = -1.5\ ,\beta = -1.45, \alpha = -0.65$

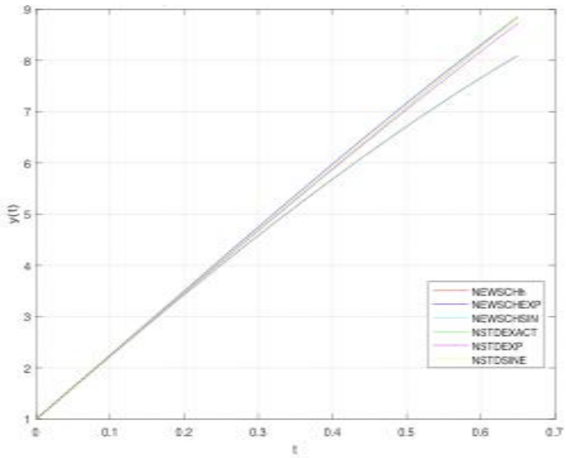

Fig. 3: Graph showing the solutions from all new developed schemes for problem II.

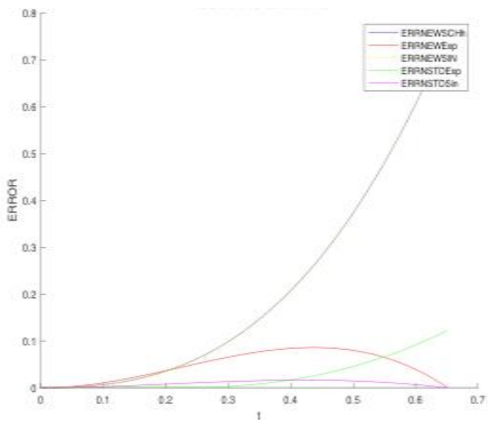

Fig. 4: Graph showing the error deviation of new schemes compared to (Mickens 1994) for problem II.

Problem III

Graphs for all schemes for h=0.01 and $\varepsilon = 0.0001$ simulation parameters $r = 1, = -0.01\ ,\beta = 0.5, \alpha = 0.75$

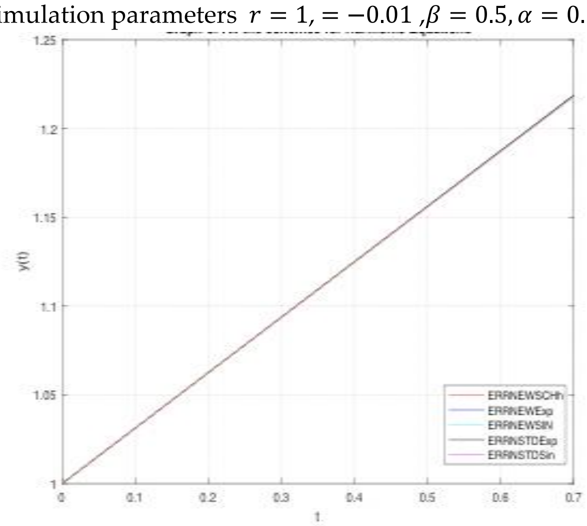

Fig. 5: Graph showing the solutions for all new developed schemes for problem III.

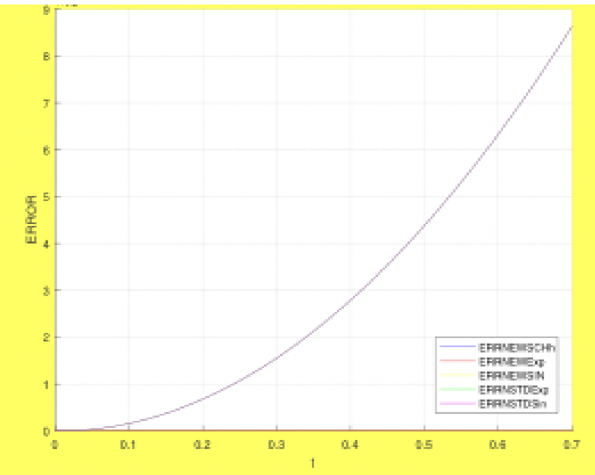

Fig. 6: Graph showing the error deviation of new developed schemes for problem III.

Problem IV

Graphs of all schemes for h=0.0001, $\varepsilon = 0.0001$ simulation parameters $r = 1, \gamma = -0.01, \beta = 0.5, \alpha = -0.75$.

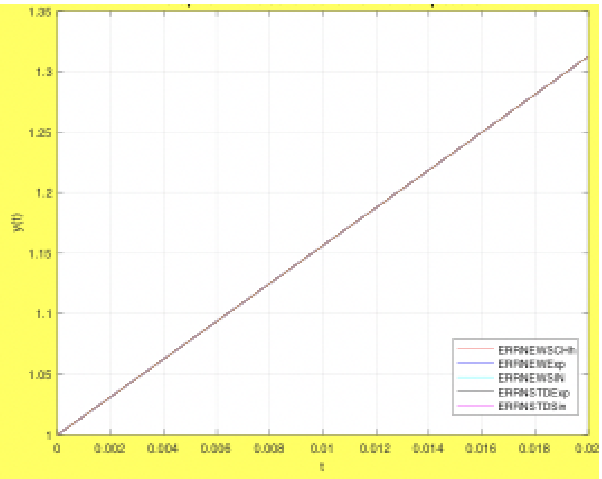

Fig. 7: Graph showing the solutions from all new schemes and the exact solution for problem IV.

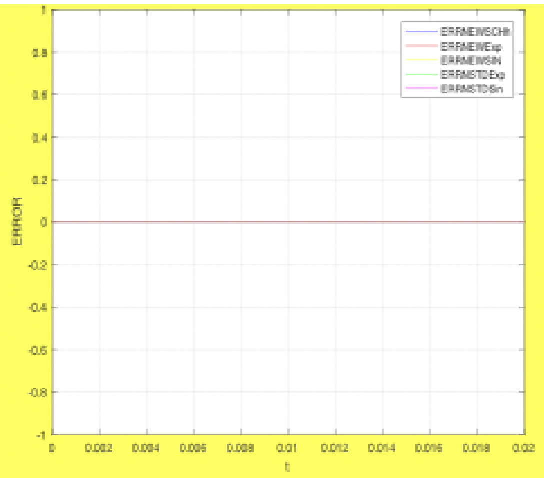

Fig. 8: Graph showing the error deviation of new developed schemes for problem IV.

5 CONCLUSION

In our comparative study, we evaluated (Mickens, 1994) scheme alongside new schemes using various step sizes. Our observations indicated the monotonicity and total monotonicity of solutions, as well as their monotonous dependence on initial values. As the step size, h decreases for any finite number of iterations, we noted total congruency. The new developed discrete hybrid nonstandard models, NewSchEx and NewSchSin, were found to produce solutions for the harmonic oscillator equation and have been analytically proven to be consistent, stable and convergent. Notably, our numerical experiments demonstrated that the absolute error of deviation for the hybrid schemes becomes zero as $h \longrightarrow 0$. We reaffirm the efficacy of Mickens' non-standard modeling rules for discrete modeling of dynamical systems, emphasizing the powerful impact of carefully chosen interpolation functions and the potential for improved results when combined with nonstandard modeling. The harmonic oscillator model can simulate the behavior of springs and pendulums, which are essential components in machinery and vehicles, help design systems with desired oscillation properties, and analyze vibration issues.

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