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Statistical Properties and Applications of a Transmuted Exponential-inverse Exponential Distribution

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Abstract
The Quadratic rank transmutation map was proposed and studied for introducing skewness and flexibility into probability models with a single parameter known as the transmuted parameter. This methodology has received greater attention and application using several classical distributions. However, the Exponential Inverse Exponential distribution has not been transmuted since its introduction. This article used the Quadratic rank transmutation map to add flexibility to the Exponential Inverse Exponential distribution which resulted to a new continuous distribution called "Transmuted Exponential Inverse Exponential distribution". This paper has presented the definition, validation, properties, applications and estimation of unknown parameters of the transmuted Exponential Inverse Exponential distribution using the method of maximum likelihood estimation. The new distribution has been applied to three real life datasets and results provided evidence that it is better than the other existing distributions based on the datasets used. Thus, this new model can be applied fully in modeling real life problems most especially in survival analysis

Keywords: Quadratic Rank Transmutation Map, Transmuted Exponential Inverse Exponential Distribution, Definition, Validity, Properties, Maximum Likelihood Estimation, Applications. Survival Analysis.

1. Introduction

An Exponential distribution which can be used in Poisson processes gives a description of the time between events. Some of its applications have been carried out in life testing experiments. The distribution exhibits memoryless property with a constant failure rate which makes the distribution unsuitable for some real-life problems and hence creating a vital problem in statistical modeling and applications [1].

Despite the applications of Exponential distribution and its attractive properties, its usage has been very limited in modeling real life situations due to the fact that it has a constant failure rate [1]. Another limitation of the exponential distribution is found in its memoryless property, this is because the memoryless assumption is hardly obtained in real life situations. To make up for these limitations, Keller *et al*. [2] proposed a modified version of the Exponential distribution called the Inverse Exponential distribution and it has also been studied in some details [3]. The modified version of the Exponential distribution known as Inverse Exponential distribution was found adequate for modeling datasets with inverted bathtub failure rates [2]. But it also has a limitation which is its inability to efficiently analyze datasets that are highly skewed [4]. This, therefore, creates room for introducing skewness and flexibility into the Inverse Exponential distribution to enable it adequately model heavily skewed datasets.

There are many families of continuous probability distributions useful for adding one or more parameters to a distribution function which makes the resulting distribution more flexible for modeling heavily skewed dataset. Some of these methods or families of distribution among others include the beta generated family (Beta-G) [5], Transmuted family of distributions [6], Gamma-G (type 1) family [7], the Kumaraswamy-G family [8], McDonald-G family [9], Gamma-G (type 2) family [10], Gamma-G (type 3) family [11], Log-gamma-G family [12], Exponentiated T-X family [13], Exponentiated-G (EG) family [14], Weibull-X family [15], Weibull-G family [16], Logistic-G family [17], Gamma-X family [18], a Lomax-G family [19], a new generalized Weibull-G family [20], Beta Marshall-Olkin family of distributions [21], Logistic-X family [22], a new Weibull-G family [23], a Lindley-G family [24], a Gompertz-G family [25]and Odd Lindley-G family [26].

In order to address the problem of memoryless property and constant failure rate of the exponential distribution and to add skewness and flexibility to its modification (Inverse Exponential distribution), many authors have proposed different extensions of the distribution and some of these recent studies include the Exponential Inverse Exponential distribution [27], the Kumaraswamy Inverse Exponential distribution [28], the exponentiated generalized Inverse Exponential distribution [29], a new Lindley-Exponential distribution [30], the Lomax-exponential distribution [31], the transmuted odd generalized exponential-exponential distribution [32], the transmuted exponential distribution [33], transmuted inverse exponential distribution[34], the odd generalized exponential-exponential distribution [35], the transmuted Weibull-exponential distribution [36] and the Weibull-Exponential distribution [37]. This paper focuses on the Exponential Inverse Exponential distribution which has been found to be an improvement over other extensions of the Inverse Exponential distribution [27].

The probability density function (pdf) of the Exponential Inverse Exponential distribution (EIED) according to Oguntunde *et al.* [27] is defined by

$$g(x)=\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \quad (1)$$

The corresponding cumulative distribution function (*cdf*) of Exponential Inverse Exponential distribution (EIED) is given by

$$G(x)=1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \quad (2)$$

where, $x>0,\theta>0,\alpha>0$; α is the shape parameter and θ is a scale parameter.

More about the important mathematical and statistical properties, maximum likelihood estimation of parameters and applications of the Exponential Inverse Exponential distribution showing its efficiency over Inverse Exponential distribution using real life datasets can be found in the literature [27].

The aim of this paper is to introduce a new continuous distribution called the Transmuted Exponential Inverse Exponential distribution (TEIED) using the proposed quadratic rank transmutation map [6]. The whole paper is presented in sections as follows: definition of the new distribution with the proof of its validity and its graphs provided in section 2. Section 3 derived some properties of the new distribution with an estimation of parameters using maximum likelihood estimation (MLE). A comparison of the new model to other existing distributions using three real life datasets was done in section 4 and some useful conclusions are made in section 5

2. The Transmuted Exponential Inverse Exponential distribution (TEIED)

2.1 Definition

The *pdf* and *cdf* of the transmuted Exponential Inverse Exponential distribution (TEIED) are defined using the steps already proposed [6]. According to Shaw and Buckley [6], a random variable X is said to have a transmuted distribution function if its *pdf* and *cdf* are respectively given by;

$$f(x)=g(x)[1+\lambda-2\lambda G(x)] \quad (3)$$

and

$$F(x)=(1+\lambda)G(x)-\lambda[G(x)]^2 \quad (4)$$

where; $x>0$, and $-1\leq\lambda\leq 1$is the transmuted parameter, *G(x)* is the cdf of the continuous distribution to be modified or transmuted while *f(x)* and *g(x)* are the associated *pdf* of *F(x)* and *G(x)* respectively.

Substituting equations (1) and (2) in (3) and (4) and simplifying, the *cdf* and *pdf* of the *TEIED* are obtained as given in equations (5) and (6) respectively:

$$F(x)=(1+\lambda)\left(1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right)-\lambda\left(1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right)^2$$

$$F(x)=1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \quad (5)$$

and

$$f(x)=\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1+\lambda-2\lambda\left(1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right)\right]$$

$$f(x)=\frac{(1-\lambda)\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}+\frac{2\lambda\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$$

(6)

where$x>0,\alpha>0,\theta>0,-1\leq\lambda\leq 1$, α is the shape parameter and θ is the scale parameter while $\lambda$is called the transmuted parameter.

2.2 Validity of the model f(x)

Recall that for any valid continuous probability distribution, the following integral in (7) must hold.

$$\int_{-\infty}^{\infty} f(x)dx=1 \quad (7)$$

Proof

Considering the *pdf* of the transmuted Exponential Inverse Exponential distribution in (6) and substituting this pdf in equation (7) above and simplifying, (7) becomes:

$$\int_0^\infty f(x)dx = \int_0^\infty \left(\frac{(1-\lambda)\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} + \frac{2\lambda\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \right) dx$$

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{(1-\lambda)\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} dx + \int_0^\infty \frac{2\lambda\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} dx \quad (8)$$

Now, from equation (8), let

$y_1 = e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} = e^{-u_1}$, $u_1 = \alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right) = \alpha t$ and $t = \left(\frac{e^{-\frac{\theta}{x_1}}}{1-e^{-\frac{\theta}{x_1}}}\right)$

$y_2 = e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} = e^{-u_2}$, $u_2 = 2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right) = 2\alpha t$ and $t = \left(\frac{e^{-\frac{\theta}{x_2}}}{1-e^{-\frac{\theta}{x_2}}}\right)$

such that

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{(1-\lambda)\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} y_1 dx_1 + \int_0^\infty \frac{2\lambda\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} y_2 dx_2$$

$$\frac{dy_1}{dx_1} = \frac{dy_1}{du_1} \times \frac{du_1}{dt} \times \frac{dt}{dx_1} = -\frac{\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$$

$$\frac{dy_2}{dx_2} = \frac{dy_2}{du_2} \times \frac{du_2}{dt} \times \frac{dt}{dx_2} = -\frac{2\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$$

which implies that

$$dx_1 = -\frac{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2 dy_1}{\alpha\theta e^{-\frac{\theta}{x}} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}} \text{ and } dx_2 = -\frac{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2 dy_2}{2\alpha\theta e^{-\frac{\theta}{x}} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}}.$$

Substituting for dx_1 and dx_2 in (8) and simplifying the resulting expression gives:

$$\int_0^\infty f(x)dx = (1-\lambda)\int_0^\infty -dy_1 + \lambda\int_0^\infty -dy_2 = -(1-\lambda)\int_0^\infty dy_1 - \lambda\int_0^\infty dy_2 \quad (9)$$

Integrating and applying the limit in equation (9) above results in the following:

$$\int_0^\infty f(x)dx = -(1-\lambda)[y_1]_0^\infty - \lambda[y_2]_0^\infty \quad (10)$$

but recall that

$y_1 = e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$ and $y_2 = e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$

Hence, substituting for y_1 and y_2 in equation (10) and simplifying will result in the following:

$$\int_0^\infty f(x)dx = -(1-\lambda)[y_1]_0^\infty - \lambda[y_2]_0^\infty = -(1-\lambda)\left[e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]_0^\infty - \lambda\left[e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]_0^\infty$$

$$\int_0^\infty f(x)dx = -(1-\lambda)\left[e^{-\alpha\left(\frac{e^{-\frac{\theta}{\infty}}}{1-e^{-\frac{\theta}{\infty}}}\right)} - e^{-\alpha\left(\frac{e^{-\frac{\theta}{0}}}{1-e^{-\frac{\theta}{0}}}\right)}\right] - \lambda\left[e^{-2\alpha\left(\frac{e^{-\frac{\theta}{\infty}}}{1-e^{-\frac{\theta}{\infty}}}\right)} - e^{-2\alpha\left(\frac{e^{-\frac{\theta}{0}}}{1-e^{-\frac{\theta}{0}}}\right)}\right]$$

$$\int_0^\infty f(x)dx = -(1-\lambda)\left[e^{-\alpha\left(\frac{1}{1-1}\right)} - e^{-\alpha\left(\frac{0}{1-0}\right)}\right] - \lambda\left[e^{-2\alpha\left(\frac{1}{1-1}\right)} - e^{-2\alpha\left(\frac{0}{1-0}\right)}\right]$$

$$\int_0^\infty f(x)dx = -(1-\lambda)[0-1] - \lambda[0-1] = 1$$

Therefore,

$$\int_0^\infty f(x)dx = 1$$

which proves that equation (6) is a probability density function.

2.3 Graphical Description of the Pdf and Cdf of TEIED

The *pdf* and *cdf* of the TEIED using some parameter values are displayed in Figures 2.1 and 2.2 as follows:

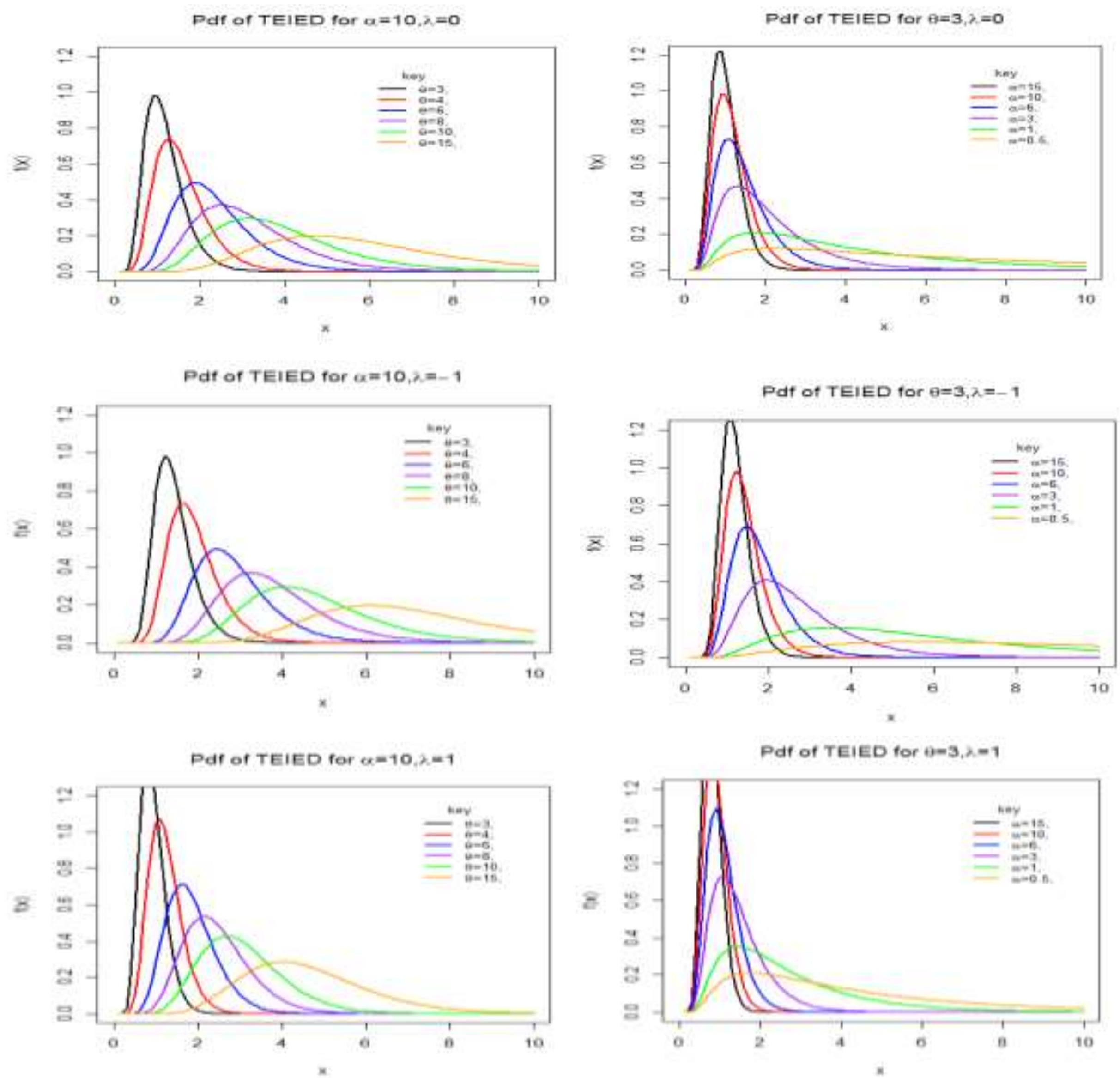

Figure 2.1: *PDF* of the *TEIED* for Different Values of the Parameters.

Figure 2.1 indicates that the *TEIED* distribution is positively skewed and takes various shapes depending on the parameter values. The values of the parameters were chosen and varied against each other to detect the effects in the plots with respect to the variations.

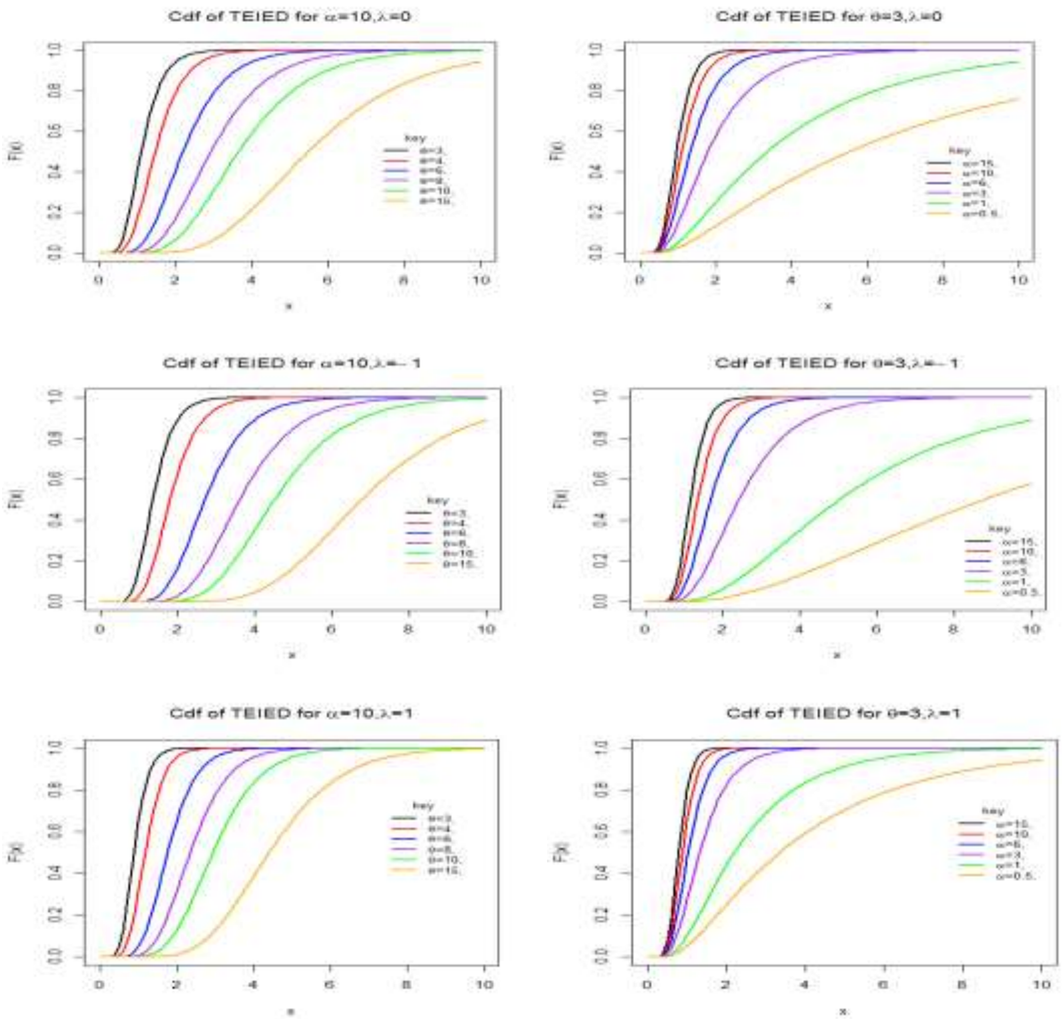

Figure 2.2: *CDF* of the *TEIED* for Different Values of the Parameters.

Also, from the above plots of the cdf in Figure 2.2, it is clear that the cdf equals to one (1) when x approaches infinity and equals zero when x tends to zero as normally expected.

3. Mathematical and Statistical Properties of TEIED

In this section, some properties of the TEIED distribution are defined and studied as follows:

3.1 Asymptotic behavior

Here, we investigate the asymptotic properties of the TEIED, that is, the limit of the *PDF* and *CDF* of the TEIED as $x$approaches infinity ($x \to \infty$) and as x tends to zero ($x \to 0$). This is demonstrated as follows

Lemma 1: The limit of the PDF of the TEIED as $x$approaches infinity ($x \to \infty$) is equal to zero (0) and the limit of the PDF of the TEIED as x tends to zero ($x \to 0$) is equal to zero.

Proof

(i) The limit of the f(x) of the TEIED as x approaches infinity ($x \to \infty$)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left\{\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right\}$$

(11)

Recall that $e^{-\frac{\theta}{x}}$ is the CDF of the Inverse Exponential distribution and its limit as $x$approaches infinity, $x \to \infty$ is equal to one (1), therefore simplifying the equation above gives:

$$\lim_{x\to\infty} f(x) = \frac{\alpha\theta}{x^2}\frac{1}{[1-1]^2} e^{-\alpha\left(\frac{1}{1-1}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{1}{1-1}\right)}\right]$$

$$\lim_{x\to\infty} f(x) = \frac{\alpha\theta}{x^2}\frac{1}{[0]^2} e^{-\alpha\left(\frac{1}{0}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{1}{0}\right)}\right] = \frac{\alpha\theta}{x^2}\frac{1}{[0]^2}(0)\left[1-\lambda+2\lambda(0)\right]$$

$$\lim_{x\to\infty} f(x) = \frac{\alpha\theta}{x^2}\frac{1}{[0]^2}(0)\left[1-\lambda+2\lambda(0)\right]=0$$

(12)

(ii) The limit of f(x) of the TEIED as $x$tends to zero (0), $x \to 0$

$$\lim_{x\to0} f(x) = \lim_{x\to0}\left\{\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right\}$$

(13)

Recall that $e^{-\frac{\theta}{x}}$ is the CDF of the Inverse Exponential distribution and its limit as x tends to zero ($x \to 0$) is equal to zero, therefore simplifying the equation above gives:

$$\lim_{x\to0} f(x) = \frac{\alpha\theta}{x^2}\frac{(0)}{[1-0]^2}e^{-\alpha\left(\frac{0}{1-0}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{0}{1-0}\right)}\right]$$

$$\lim_{x\to0} f(x) = \frac{\alpha\theta}{x^2}\frac{(0)}{[1-0]^2}e^{-\alpha\left(\frac{0}{1-0}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{0}{1-0}\right)}\right]=\frac{\alpha\theta}{x^2}\frac{(0)}{1}(1)\left[1-\lambda+2\lambda(1)\right]$$

$$\lim_{x\to0} f(x) = \frac{\alpha\theta}{x^2}(0)(1)\left[1-\lambda+2\lambda(1)\right]=0$$

(14)

Lemma 2: The limit of F(x) of the TEIED as $x$approaches infinity ($x \to \infty$) is equal to one (1) and the limit of F(x) of the TEIED as $x$tends to zero (0) ($x \to 0$) is equal to zero (0).

Proof

The limit of F(x) of the TEIED as$x$approaches infinity ($x \to \infty$)

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left\{1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right\}$$

(15)

Recall that $e^{-\frac{\theta}{x}}$ is the CDF of the Inverse Exponential distribution and its limit as $x$approaches infinity, $x \to \infty$ is equal to one (1), therefore simplifying the equation above gives:

$$\lim_{x\to\infty} F(x) = 1-(1-\lambda)e^{-\alpha\left(\frac{1}{1-1}\right)}-\lambda e^{-2\alpha\left(\frac{1}{1-1}\right)}$$

$$\lim_{x\to\infty} F(x) = 1-(1-\lambda)e^{-\alpha\left(\frac{1}{0}\right)}-\lambda e^{-2\alpha\left(\frac{1}{0}\right)}=1-(1-\lambda)(0)-\lambda(0)$$

$$\lim_{x\to\infty}(1-\lambda)(0)(0)$$

(16)

(i) The limit of F(x) of the TEIED as $x$tends to zero ($x \to 0$)

$$\lim_{x\to0} F(x) = \lim_{x\to0}\left\{1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right\}$$

(17)

Recall that $e^{-\frac{\theta}{x}}$ is the CDF of the Inverse Exponential distribution and its limit of the CDF, F(x) of the TEIED as $x$tends to zero (0), $x \to 0$ is equal to zero (0), therefore simplifying the equation above gives:

$$\lim_{x\to0} F(x) = 1-(1-\lambda)e^{-\alpha\left(\frac{0}{1-0}\right)}-\lambda e^{-2\alpha\left(\frac{0}{1-0}\right)}$$

$$\lim_{x\to0} F(x) = 1-(1-\lambda)e^{-\alpha\left(\frac{0}{1}\right)}-\lambda e^{-2\alpha\left(\frac{0}{1}\right)}=1-(1-\lambda)(1)-\lambda(1)$$

$$\lim_{x\to0}(1-\lambda)(1)(1) \qquad (18)$$

This demonstration above affirms that the distribution i.e., the TEIED has at least one mode or it is a unimodal distribution and that it is a valid probability distribution.

3.2 Moments

Let X denote a continuous random variable, the n^{th} ordinary moment or moment about the origin of a random variable X is $\mu'_n = E\left(X^n\right)=\int_0^\infty x^n f(x)dx$ given by:

(19)

Where *f(x)* the *pdf* of the TEIED is as given in equation (6) as:

$$f(x)=\frac{(1-\lambda)\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}+\frac{2\lambda\alpha\theta e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}$$

(20)

Before substitution in (19), we perform the expansion and simplification of the *pdf* as follows:

First, by expanding the exponential term in (20) using power series, we obtain:

$$e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}=\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)^k$$

(21)

$$e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}=\exp\left\{-2\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}=\sum_{k=0}^{\infty}\frac{(-2)^k\alpha^k}{k!}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)^k$$

(22)

Making use of the result in (21) and (22) above, equation (23) becomes:

$$f(x) = (1-\lambda)\alpha\theta\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\frac{e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)^k + 2\lambda\alpha\theta\sum_{k=0}^{\infty}\frac{(-2)^k\alpha^k}{k!}\frac{e^{-\frac{\theta}{x}}}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)^k$$

$$f(x) = (1-\lambda)\alpha\theta\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}x^{-2}e^{-\frac{\theta}{x}(k+1)}\left[1-e^{-\frac{\theta}{x}}\right]^{-(k+2)} + 2\lambda\alpha\theta\sum_{k=0}^{\infty}\frac{(-2)^k\alpha^k}{k!}x^{-2}e^{-\frac{\theta}{x}(k+1)}\left[1-e^{-\frac{\theta}{x}}\right]^{-(k+2)}$$

$$f(x) = \alpha\theta\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}((-1)^k(1-\lambda) + 2\lambda(-2)^k)\,x^{-2}e^{-\frac{\theta}{x}(k+1)}\left[1-e^{-\frac{\theta}{x}}\right]^{-(k+2)} \quad (24)$$

Also, using the generalized binomial theorem, we can write the last term from the above result in equation (24) as:

$$\left[1-e^{-\frac{\theta}{x}}\right]^{-(k+2)} = \sum_{l=0}^{\infty}\frac{\Gamma(l+k+2)}{l!\Gamma(k+2)}e^{-\frac{\theta}{x}l} \quad (25)$$

Making use of the result in (25) above in equation (24) and simplifying, we obtain:

$$f(x) = \alpha\theta\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k\Gamma(l+k+2)}{k!l!\Gamma(k+2)}((-1)^k(1-\lambda) + 2\lambda(-2)^k)\,x^{-2}e^{-\frac{\theta}{x}(k+l+1)} \quad (26)$$

Hence, the pdf in equation (26) can also be written in its simple form as follows:

$$f(x) = \eta_{k,l}x^{-2}e^{-\frac{\theta}{x}(k+l+1)} \quad (27)$$

where

$$\eta_{k,l} = \alpha\theta\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k\Gamma(l+k+2)}{k!l!\Gamma(k+2)}((-1)^k(1-\lambda) + 2\lambda(-2)^k)$$

Now, using the simplified form of the pdf of the TEIED in equation (27), the n^th ordinary moment of the TEIED is derived as follows:

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \int_0^{\infty}x^n\left(\eta_{k,l}x^{-2}e^{-\frac{\theta}{x}(k+l+1)}\right)dx$$

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \eta_{k,l}\int_0^{\infty}x^{n-2}e^{-\frac{\theta}{x}(k+l+1)}dx$$

(28)

Making use of integration by substitution method in equation (28), we perform the following operations:

Let $u = \frac{\theta}{x}(k+l+1) \Rightarrow x = \frac{\theta(k+l+1)}{u} = u^{-1}[\theta(k+l+1)]$

$\frac{du}{dx} = -\frac{\theta}{x^2}(k+l+1) \Rightarrow dx = -\frac{x^2du}{\theta(k+l+1)}$

Substituting for x, u and dx in equation (28) and simplifying; we have:

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \eta_{k,l}\int_0^{\infty}x^{n-2}e^{-u}\left(-\frac{x^2du}{\theta(k+l+1)}\right)$$

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \left(-\frac{1}{\theta(k+l+1)}\right)\eta_{k,l}\int_0^{\infty}x^n e^{-u}du$$

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \left(-\frac{1}{\theta(k+l+1)}\right)\eta_{k,l}\int_0^{\infty}u^{-n}\left[\theta(k+l+1)\right]^n e^{-u}du$$

$$\mu_n' = E\left(X^n\right) = \int_0^{\infty}x^n f(x)dx = \eta_{k,l}(-1)\left(\theta(k+l+1)\right)^{n-1}\int_0^{\infty}u^{-n}e^{-u}du$$

$$\mu_n' = E\left(X^n\right) = \eta_{k,l}(-1)\left(\theta(k+l+1)\right)^{n-1}\int_0^{\infty}u^{1-n-1}e^{-u}du$$

(29)

Hence, recall that $\int_0^{\infty}t^{n-1}e^{-t}dt = \Gamma(n)$ and that $\int_0^{\infty}t^n e^{-t}dt = \int_0^{\infty}t^{n+1-1}e^{-t}dt = \Gamma(n+1)$

Thus, we obtain the n^{th} ordinary moment of X for the TEIED as follows:

$$\mu_n' = E\left(X^n\right) = \eta_{k,l}(-1)\left(\theta(k+l+1)\right)^{n-1}\Gamma(1-n)$$

(30)

Again, recall that $\eta_{k,l}$ is a constant and making use of its value as defined above, we can write the expression for the nth ordinary moment of TEIED as:

$$\mu_n' = E\left(X^n\right) = \alpha\theta\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)\alpha^k\Gamma(l+k+2)\left((-1)^k(1-\lambda)+2\lambda(-2)^k\right)}{k!l!\Gamma(k+2)\left(\theta(k+l+1)\right)^{-(n-1)}\left(\Gamma(1-n)\right)^{-1}}$$

(31)

The mean (μ_1'), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be calculated from the ordinary and non-central moments using some well-known relationships such as:

$\mu_1' = E(X), Var(X) = \sigma^2 = \mu_2' - \{\mu_1'\}^2$,

$CV = \left\{\frac{\sigma^2}{(\mu_1')^2}\right\}^{\frac{1}{2}} CS = E\left(\frac{x-\mu_1'}{\sigma}\right)^3 = \frac{\mu_3}{(\sigma)^3}$ and

$CK = E\left(\frac{x-\mu_1'}{\sigma}\right)^4 = \frac{\mu_4}{(\sigma)^4}$

3.3 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \quad (32)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r \quad (33)$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r=0}^{\infty} \frac{t^r}{r!} [\mu_r']$$

Using the result in equation (33) and simplifying the integral in (32) therefore we have:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\alpha\theta \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)\alpha^k \Gamma(l+k+2)\left((-1)^k(1-\lambda)+2\lambda(-2)^k\right)}{k!l!\Gamma(k+2)\left(\theta(k+l+1)\right)^{-(r-1)}\left(\Gamma(1-r)\right)^{-1}}\right]$$

(34)

3.4 Characteristics Function

The characteristics function of a random variable X is defined by:

$$\varphi_x(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x)dx \quad (35)$$

Again, applying power series expansion and simplifying equation (35), we obtained the characteristics function of X as:

$$\varphi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\alpha\theta \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)\alpha^k \Gamma(l+k+2)\left((-1)^k(1-\lambda)+2\lambda(-2)^k\right)}{k!l!\Gamma(k+2)\left(\theta(k+l+1)\right)^{-(r-1)}\left(\Gamma(1-r)\right)^{-1}}\right]$$

(36)

3.5 Quantile Function

Hyndman and Fan [38] defined the quantile function for any distribution in the form $Q(u) = X_q = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$

Taking *F(x)* to be the cdf of the TEIED and inverting it as above will give us the quantile function as follows:

$$F(x) = 1 - (1-\lambda) e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} - \lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} = u$$

(37)

Simplifying equation (37) above and solving for X presents the quantile function of TEID as:

$$Q(u) = X_q = \theta\left\{log\left[\left(-\frac{log(1-u)}{3\alpha}\right)^{-1} + 1\right]\right\}^{-1} \quad (38)$$

This function is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question.

3.6 Skewness and Kurtosis

This paper presents the quantile-based measures of skewness and kurtosis due to non-existence of the classical measures in some cases.

According to to Kenney and Keeping [39], the Bowley's measure of skewness based on quartiles is given by:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad (39)$$

Also, the Moors kurtosis based on octiles proposed [40] and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \quad (40)$$

Where$Q(.)$ is obtainable with the help of equation (38).

3.7 Reliability analysis of the TEIED.

This section presents the derivation and study of the survival (or reliability) function and the hazard (or failure) rate function. Also, the cumulative hazard function, the reverse hazard function and the odds function are obtained for the TEIED.

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \quad (41)$$

Applying the *cdf* of the TEIED in (41), the survival function for the TEIED is obtained as:

$$S(x) = 1 - \left\{1 - (1-\lambda) e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} - \lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right\}$$

$$S(x) = e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1 - \lambda + \lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]$$

(42)

The following are plots for the survival function of the TEIED using different parameter values as shown in Figure 3.1, the selection of parameter values was done based on the range of values of the parameters and in such a way as to detect the changes in the plots or shapes with respect to the variation in the parameter values.

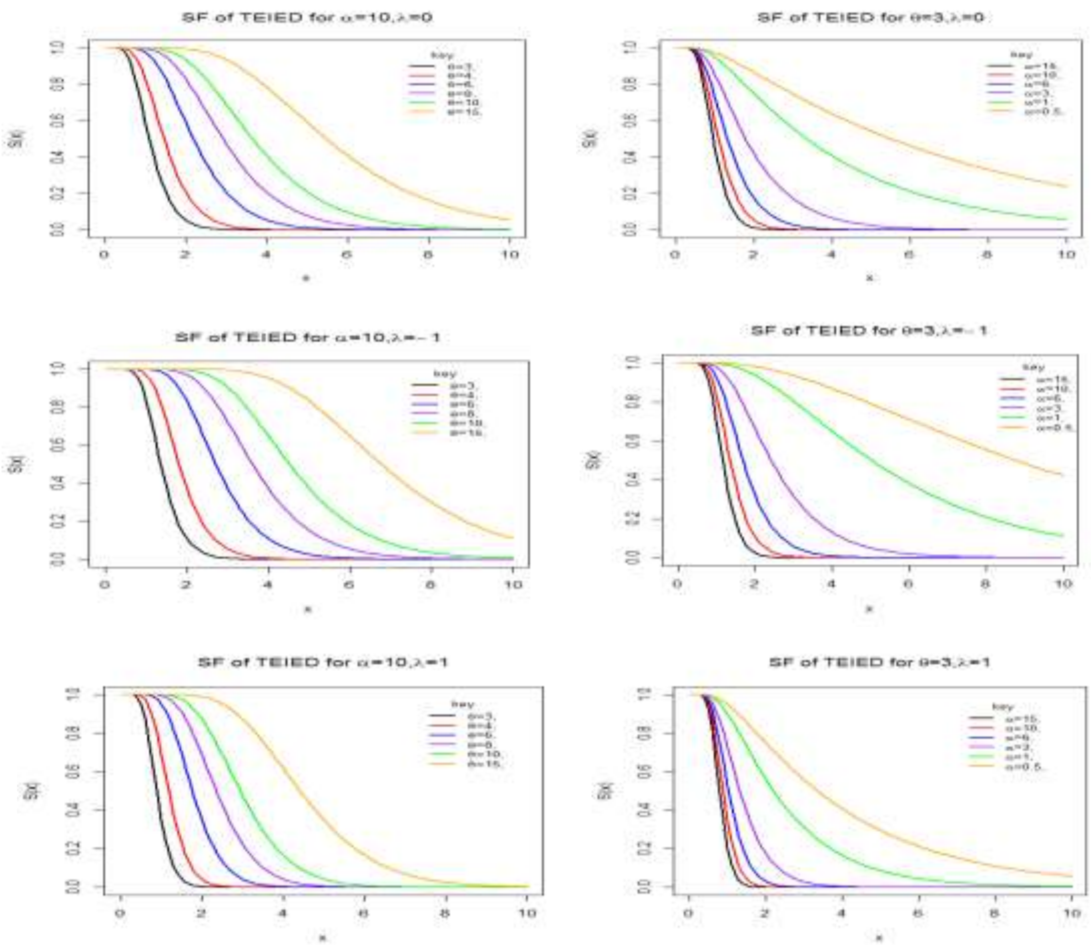

Figure 3.1: Survival Function of the TEIED at Different Parameter Values.

The plots in Figure 3.1 shows that the probability of survival equals one (1) at initial time or early age and it decreases as time increases and equals zero (0) as time approaches infinity.

Hazard function are plots probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)} \tag{43}$$

Meanwhile, the expression for the hazard rate of the TEIED is given by:

$$h(x) = \frac{\alpha\theta e^{-\frac{\theta}{x}}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2\left[1-\lambda+\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]}$$

(44)

where $\alpha, \theta > 0$ and $-1 \le \lambda \le 1$.

The following is a plot of the hazard function for arbitrary parameter values in Figure 3.2, the selection of parameter values was done based on the range of values of the plots or shapes with respect to the variation in the parameter values.

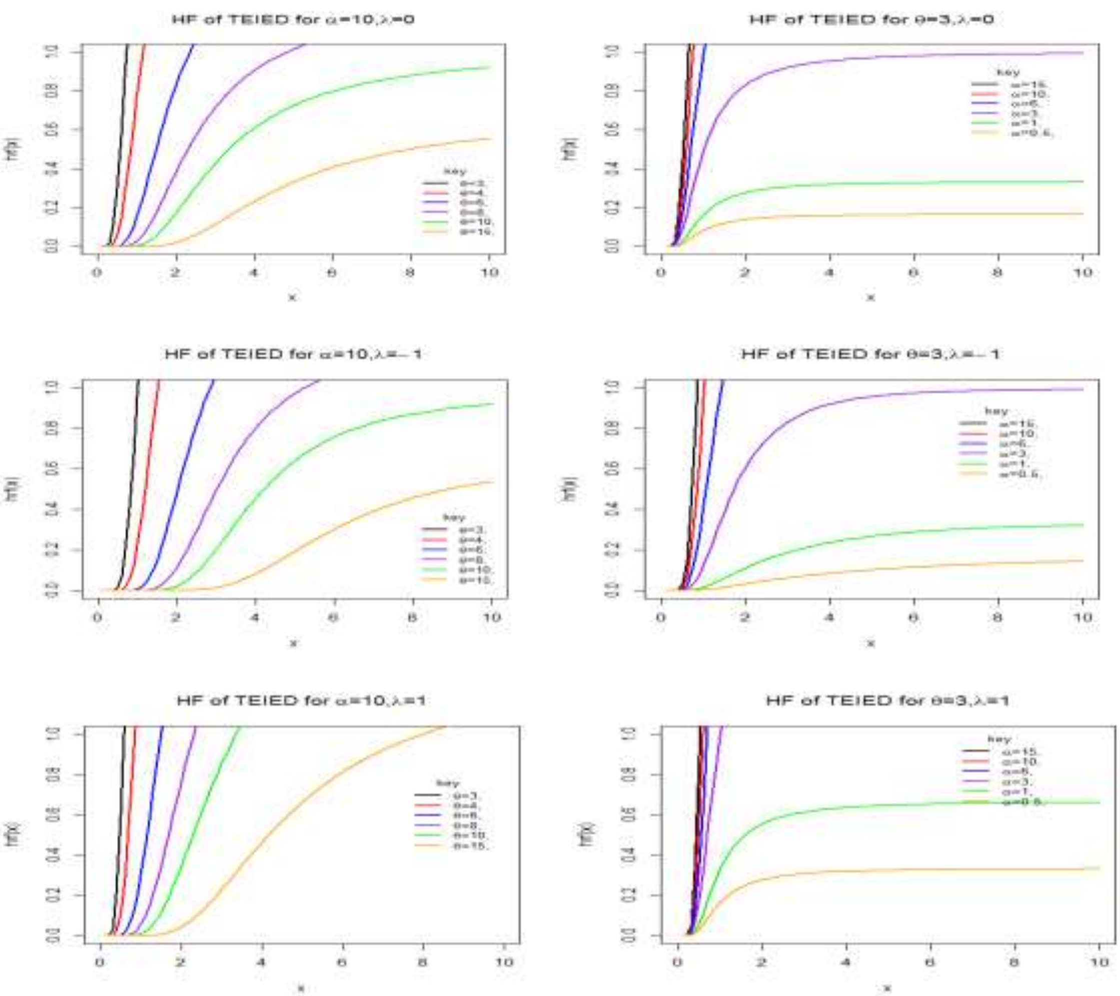

Figure 3.2: Hazard function of the TEIED.

The Figure above revealed that the TEIED has increasing failure rate which implies that the probability of failure for any random variable following a TEIED increases as time increases, that is, probability of failure or death increases with age.

The cumulative hazard function is defined as:

$$H(x) = \int_0^x h(t)\,dt = \int_0^x \frac{f(t)}{1-F(t)}\,dt = -\ln S(x) \quad (45)$$

Considering the cdf of the TEIED in (5), the cumulative hazard function for the TEIED is derived as:

$$H(x) = -\ln\left\{ e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right\}$$

(46)

The reversed hazard function of a distribution is obtained by dividing the pdf by the cdf in. It is mathematically defined as:

$$Rh(x) = \frac{f(x)}{F(x)} \quad (47)$$

Making use of the pdf and cdf of the TEIED in (47), the reverse hazard function for the TEIED is obtained as:

$$Rh(x) = \frac{\alpha\theta e^{-\frac{\theta}{x}} e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1+\lambda-2\lambda\left(1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right)\right]}{x^2\left[1-e^{-\frac{\theta}{x}}\right]^2\left[1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]}$$

(48)

Also, the odds function of a distribution is obtained by dividing the cdf by the reliability (survival) function. That is:

$$O(x) = \frac{F(x)}{1-F(x)} = \frac{F(x)}{S(x)} \quad (49)$$

Using the cdf of the TEIED in (49), the odds function for the TEIED is obtained as:

$$O(x)=\frac{1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}}{(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}}$$

(50)

where $x > 0, \alpha, \theta > 0$ and $-1 \leq \lambda \leq 1$.

3.8 Entropy Measurement

Entropy is a function used to quantify the uncertainty, disorderliness or randomness in a system or a probability distribution. In this section, we present the most frequently used measure of entropy known as Renyi entropy [41]. The Renyi entropy of a random variable *X* which represents a variation of the uncertainty is defined as:

$$I_\delta(X) = \frac{1}{1-\delta} log \int_{-\infty}^{\infty} f^\delta(x)dx \quad (51)$$

for $\delta > 0$ and $\delta \neq 1$.

Now, using the pdf of the TEIED in equation (51) we get:

$$I_\delta(X) = \frac{1}{1-\delta} log \left[\int_{x=0}^{\infty} (\eta_{k,l})^\delta x^{-2\delta} e^{-\frac{\theta}{x}(k+l+1)\delta} dx\right] \quad (52)$$

where

$f^\delta(x) = (\eta_{k,l})^\delta x^{-2\delta} e^{-\frac{\theta}{x}(k+l+1)\delta}$ and

$\eta_{k,l} = \alpha\theta \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\alpha^k \Gamma(l+k+2)}{k!l!\Gamma(k+2)} ((-1)^k(1-\lambda) + 2\lambda(-2)^k)$

Therefore, solving the integral above, the Renyi entropy of *X*~TEIED after simplification reduces to:

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\left\{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\theta\alpha^{k+1}\left((-1)^k(1-\lambda)+2\lambda(-2)^k\right)}{\left(\Gamma(l+k+2)\right)^{-1}k!l!\Gamma(k+2)}\right\}^\delta\frac{(-1)\Gamma(2\delta-1)}{\left(\theta(k+l+1)\delta\right)^{2\delta-1}}\right]$$

(53)

3.9 Distribution of order Statistics

Suppose $X_1, X_2, \ldots\ldots, X_n$ is a random sample from the TEIED and let $X_{1:n}, X_{2:n}, \ldots\ldots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The *pdf*, $f_{i:n}(x)$ of the *i*^th^ order statistic can be obtained by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}f(x)F(x)^{k+i-1} \quad (54)$$

Using (5) and (6), the *pdf* of the *i*^th^ order statistics $X_{i:n}$, can be expressed from (54) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right]$$
$$\times\left[1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]^{i+k-1}$$

(55)

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the TEIED are respectively given by:

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right]$$
$$\times\left[1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]^{k}$$

(56)

and

$$f_{n:n}(x)=n\left[\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]\right]$$
$$\times\left[1-(1-\lambda)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}-\lambda e^{-2\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)}\right]^{n-1}$$

(57)

3.10 Estimation of Unknown Parameters of the TEIED Via MLE

In this section, the estimation of the parameters of the TEIED is done by using the method of maximum likelihood estimation (MLE). Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the TEIED with unknown parameters α, θ and λ defined previously.

The likelihood function of the TEIED using the pdf in equation (6) is given by;

$$L\left(\underline{X}/\alpha,\theta,\lambda\right)=\left(\alpha\theta\right)^{n}\prod_{i=1}^{n}\left(x_i^{-2}\left[1-e^{-\frac{\theta}{x_i}}\right]^{-2}e^{-\frac{\theta}{x_i}}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}\right)\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}\right] \quad (58)$$

Let the natural logarithm of the likelihood function be, $l(\eta) = log\, L\, (\underline{X}|\alpha,\theta,\lambda)$, therefore, taking the natural logarithm of the function above gives:

$$l(\eta)=n\log\alpha+n\log\theta-2\sum_{i=1}^{n}\log x_i-2\sum_{i=1}^{n}\log\left(1-e^{-\frac{\theta}{x_i}}\right)-\theta\sum_{i=1}^{n}x^{-1}{}_{i}-\alpha\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}\right]$$

(59)

Differentiating $l(\eta)$ partially with respect to α, θ and λ respectively gives the following results;

$$\frac{\partial l(\eta)}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)-2\lambda\sum_{i=1}^{n}\left\{\frac{\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}}{1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}}\right\}$$

(60)

$$\frac{\partial l(\eta)}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x^{-1}{}_{i}+2\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{x_i}}}{x_i\left(1-e^{-\frac{\theta}{x_i}}\right)}\right\}+\alpha\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{x_i}}}{x_i\left[1-e^{-\frac{\theta}{x_i}}\right]^{2}}\right\}-2\alpha\lambda\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{x_i}}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}}{1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}\left(x_i\left[1-e^{-\frac{\theta}{x_i}}\right]^{2}\right)}\right\}$$

(61)

$$\frac{\partial l(\eta)}{\partial\lambda}=\sum_{i=1}^{n}\left\{\frac{2e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}-1}{1-\lambda+2\lambda e^{-\alpha\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}}\right\}$$

(62)

Making equation (60), (61) and (62) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α, θ and λ. However, these solutions could not be obtained manually except numerically with the aid of suitable statistical software R using adequacy model package.

4. Applications to Three Real Life Datasets

This section presents the three real life datasets, their descriptive statistics, graphical summary and applications. The section compares the fits of the transmuted Exponential Inverse Exponential distribution (TEIED), Exponential Inverse Exponential distribution (EIED), Inverse Exponential distribution (IED) and Exponential distribution (ED) using three real life datasets (dataset A, dataset B and dataset C).

To compare the above listed distributions, we have considered some model selection criteria which include the value of the log-likelihood function evaluated at the maximum likelihood estimates (ℓ), Akaike Information Criterion, *AIC*, Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in [42]. Some of these statistics are computed with the following formulas:

$AIC = -2\ell + 2k$, $BIC = -2\ell + k\,log(n)$, $CAIC = -2\ell + \frac{2kn}{(n-k-1)}$ and $HQIC = -2\ell + 2k\,log[log(n)]$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, *k* is the number of model parameters and *n* is the sample size. Meanwhile, when taking our decisions, we consider any model with the lowest values for these statistics to be a best model that fit the dataset. The required computations are carried out using the R package “Adequacy Model” which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Tables 4.1 (for dataset A), 4.5 (for dataset B) and 4.9 (for dataset C) list the Maximum Likelihood Estimates of the model parameters whereas the statistics AIC, CAIC, BIC, HQIC, A*,W* and K-S for the fitted TEIED, EIED, IED and ED models are given in Tables 4.2, 4.3 & 4.4 for dataset A, 4.6, 4.7 & 4.8 for dataset B and 4.10, 4.11 & 4.12 for dataset C.

Dataset A: The dataset represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by [43]. They are the Regiment 4.3, Study M.: 10, 33, 44,56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116,120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176,183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327,342, 347, 361, 402, 432, 458, 555.

Table 4.1: Descriptive Statistics for dataset A

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Dataset A	72	10.0	108.0	149.5	224.0	176.8	555.0	10705.1	1.34128	1.98852

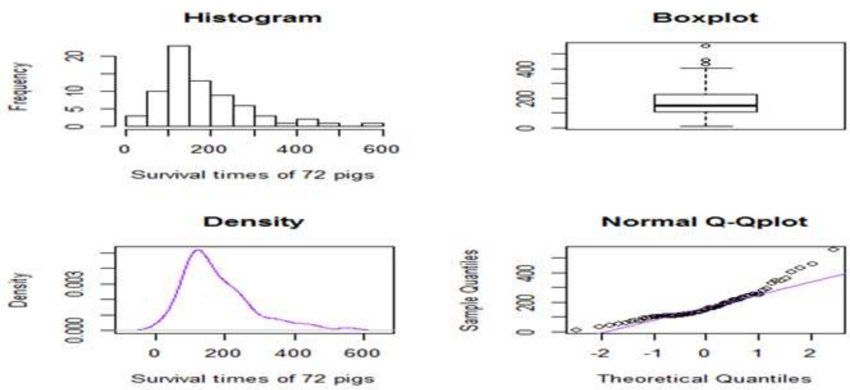

Figure 4.1: A Graphical Summary of Dataset A

Based on the descriptive statistics in Table 4.1 and the histogram, box plot, density and normal Q-Q plot generally known as graphical summary shown in Figure 4.1 above, it is seen that dataset A is skewed to the right or positively skewed.

Table 4.2: Maximum Likelihood Parameter Estimates for dataset A

Distribution	θˆ	αˆ	λˆ
TEIED	9.74623485	0.09072364	-0.99742240
EIED	8.81638559	0.04973562	-
IED	9.879388	-	-
ED	0.01326009	-	

The values in Table 4.2 are estimates of the parameters of the fitted distributions for dataset A and the empty spaces in the cells of the table is as a result of the fact that some distributions have less than three parameters unlike the proposed TEIED.

Table 4.3: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset A

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
TEIED	429.7125	865.4249	865.7779	872.2549	868.144	1st
EIED	442.9481	889.8962	890.0702	894.4496	891.7089	2nd
IED	561.9758	1125.952	1126.009	1128.228	1126.858	4th
ED	480.0832	962.1664	962.2236	964.4431	963.0728	3rd

Table 4.4: The A^*, W^*, K-S statistic and P-values for dataset A.

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
TEIED	0.585328	0.09079332	0.19861	0.006825	1st
EIED	0.6156711	0.09710336	0.27903	2.704e-05	2nd
IED	1.671922	0.2436275	0.80234	2.2e-16	4th
ED	0.6373625	0.1056187	0.59364	2.2e-16	3rd

Based on the results in Table 4.3 and Table 4.4 above, it can be seen that the TEIED has minimum values of AIC, CAIC, BIC, HQIC, A^*, W^* and K-S statistic for dataset A among all the fitted models. This means that the TEIED fits the dataset better compared to the other three fitted distributions.

The following figure displayed the histogram and estimated densities and cdfs of the fitted models to dataset A

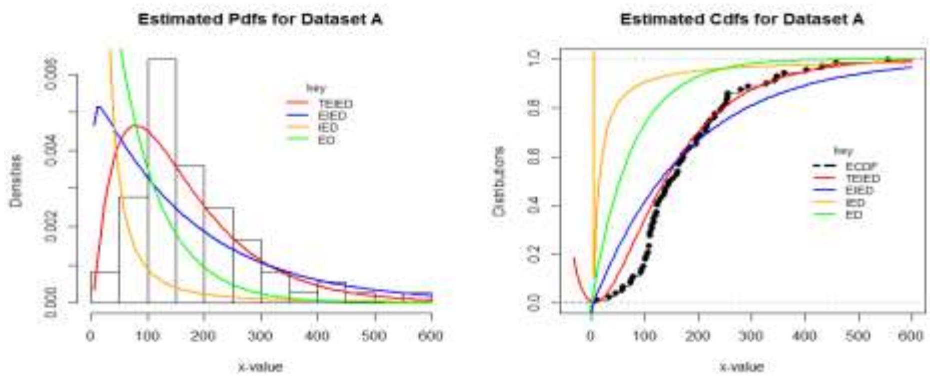

Figure 4.2: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset A.

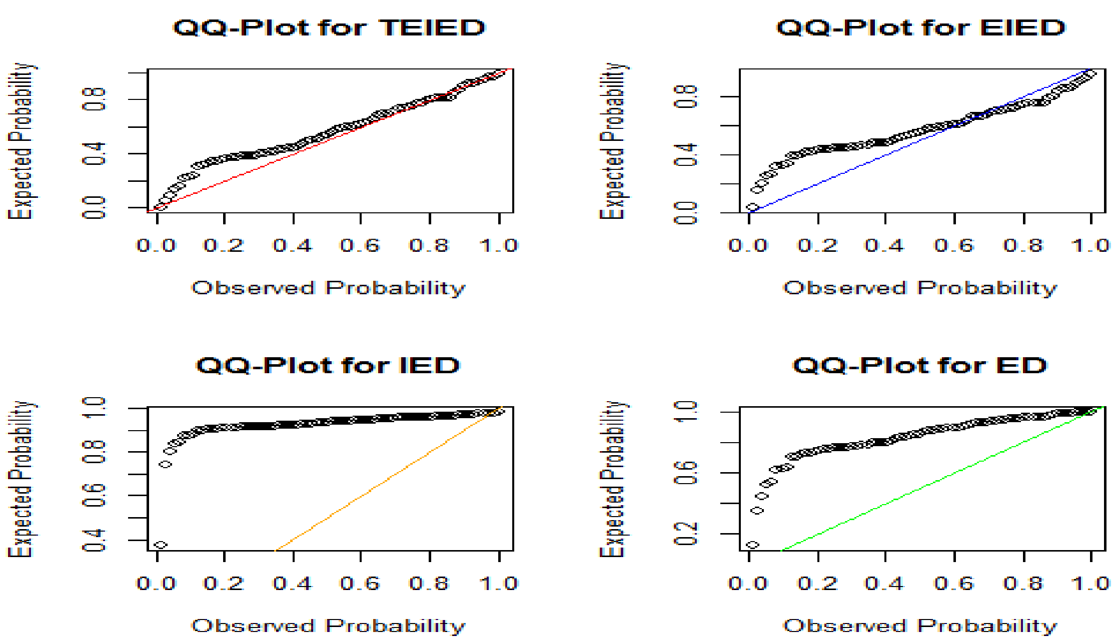

Figure 4.3: Probability Plots for the Fit of the TEIED, EIED, IED & ED Based on Dataset A.

Dataset B. This data represents the survival times of a group of patients suffering from head and neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT) ([44], [45],[46]). The observations are as follows:12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776. The summary is given as follows:

Table 4.5: Descriptive Statistics for dataset B.

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	44	12.20	67.21	128.5	219.0	223.48	1776.00	93286.4	3.38382	13.5596

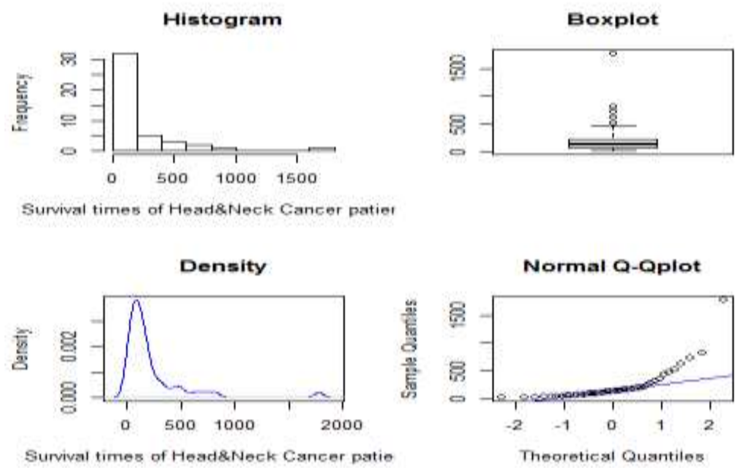

Figure 4.4: A graphical summary of dataset B

Using the descriptive statistics in Table 4.5 and the graphical summary in Figures 4.4 above, we observed that the second data (dataset B) is also skewed to the right or positively skewed just like the first data (dataset A).

Table 4.6: Maximum Likelihood Parameter Estimates for Dataset B

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
TEIED	9.22948531	0.02576929	0.79001208
EIED	6.33690078	0.02542512	-
IED	9.879388	-	-
ED	0.016902	-	-

The values in Table 4.2 are estimates of the parameters of the fitted distributions for dataset B and the empty spaces in the cells of the Table is as a result of the fact that some distributions have less than three parameters unlike the proposed TEIED.

Table 4.7: The Statistics ℓ, AIC, CAIC, BIC and HQIC for Dataset B

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
TEIED	279.9114	565.8228	566.4228	571.1754	567.8078	1st
EIED	281.7682	567.5365	567.8292	571.1049	568.8598	2nd
IED	331.4209	664.8418	664.9371	666.626	665.5035	3rd
ED	345.7314	693.4629	693.5581	695.247	694.1245	4th

Table 4.8: The A^*, W^*, K-S statistic and P-values for dataset B

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
TEIED	0.5742914	0.09887078	0.12346	0.4762	1st
EIED	0.7891442	0.1363498	0.19041	0.07171	2nd
IED	0.2349885	0.03740618	0.65501	2.2e-16	3rd
ED	1.218726	0.2113589	0.44325	2.145e-08	4th

Also, it has been observed from Tables 4.7 and 4.8 that the TEIED has the lowest values of AIC, CAIC, BIC, HQIC, A^*, W^* and K-S statistic for dataset B compared to the other fitted distributions. This shows that the TEIED fits the dataset better than the other three fitted models.

The following Figure displayed the histogram and estimated densities and cdfs of the fitted models to dataset B.

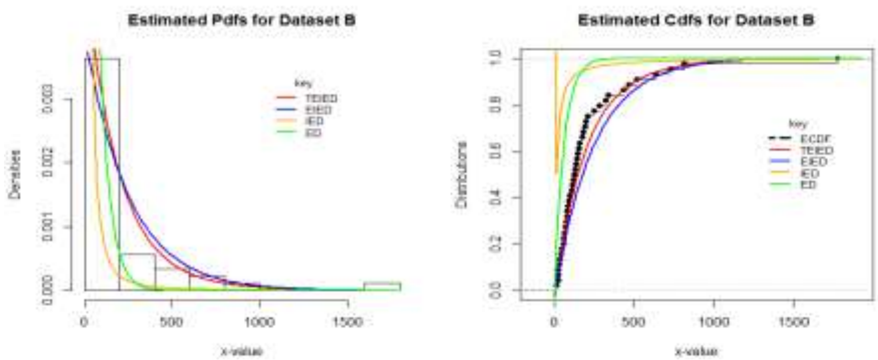

Figure 4.5: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset B.

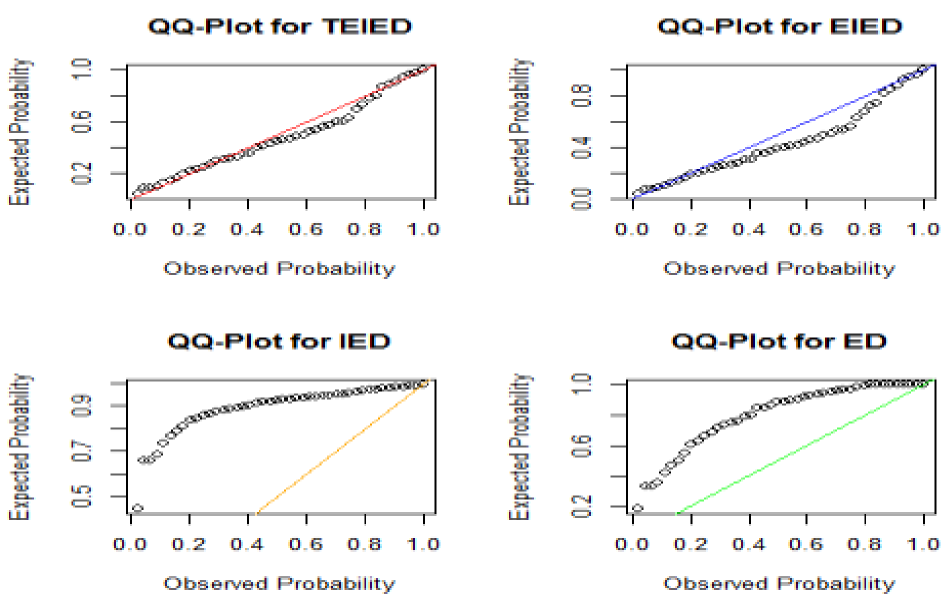

Figure 4.6: Probability Plots for the Fit of the TEIED, EIED, IED & ED Based on Dataset B

Dataset C: Actuarial Science (Mortality Deaths) data

This third dataset (dataset C) represents 280 observations on the age of death (in years) of retired women with temporary disabilities. This dataset has been studied [47]. It is important for the Mexican Institute of Social Security (IMSS) to study the distributional behavior of the mortality of retired people on disability because it enables the calculation of long- and short-term financial estimation, such as the assessment of the reserve required to pay the minimum pensions.

The data corresponding to lifetimes (in years) of retired women with temporary disabilities who died during 2004 and which are incorporated in the Mexican insurance public system are: 22,24, 25(2), 27, 28, 29(4), 30, 31(6), 32(7), 33(3), 34(6), 35(4), 36(11), 37(5), 38(3), 39(6), 40(14),41(12), 42(6), 43(5), 44(7), 45(10), 46(6), 47(5), 48(11), 49(8), 50(8), 51(8), 52(14), 53(10), 54(13), 55(11), 56(10), 57(15), 58(11), 59(9), 60(7), 61(2), 62, 63, 64(4), 65(2), 66(3), 71, 74, 75, 79, 86. Its summary is given as follow

Table 4.9: Descriptive Statistics for Dataset C.

parameters	n	Min.	Q_1	Median	Q_3	Mean	Max	Var	Skewness	Kurtosis
Values	280	22.00	40.00	49.00	55.25	47.79	86.00	108.63	0.06703	0.0524

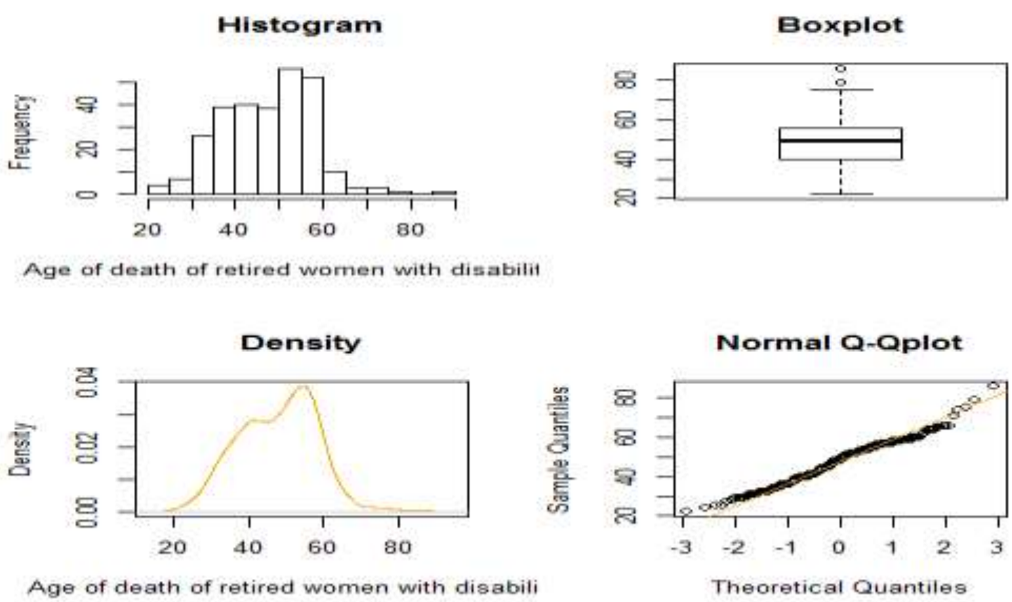

Figure 4.7: A graphical summary of dataset C.

Considering the values in Table 4.9 and the histogram, box plot, density and normal Q-Q plot in Figure 4.7, it is clear to conclude that the third data (dataset C) is normally distributed or approximately normal.

Table 4.10: Maximum Likelihood Parameter Estimates for dataset C

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
TEIED	9.8006389	0.3469193	-0.9773374
EIED	9.3101940	0.2158965	-
IED	9.879388	-	-
ED	0.02092807	-	-

The values in Table 4.10 are estimates of the parameters of the fitted distributions for dataset C and the empty spaces in the cells of the table is as a result of the fact that some distributions have less than three parameters unlike the proposed TEIED.

Table 4.11: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset C

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
TEIED	1247.671	2501.343	2501.43	2512.247	2505.717	1st
EIED	1336.062	2676.124	2676.167	2683.394	2679.04	2nd
IED	1570.938	3143.877	3143.891	3147.511	3145.335	4th
ED	1362.683	2727.367	2727.381	2731.002	2728.825	3rd

Table 4.12: The A^*, W^*, K-S statistic and P-values for dataset C

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
TEIED	2.940039	0.5217251	0.33412	2.2e-16	1st
EIED	2.829021	0.5020088	0.42078	2.2e-16	2nd
IED	4.266137	0.7503556	0.68987	2.2e-16	4th
ED	2.688549	0.4772946	0.43803	2.2e-16	3rd

From Tables 4.11 and 4.12 above, it is observed that the TEIED has minimum values of AIC, CAIC, BIC, HQIC, A^*, W^* and K-S statistic for dataset C among all the fitted models. This is an indication that the TEIED fits the dataset better than the other three fitted models.

The following figure displayed the histogram and estimated densities and cdfs of the fitted models to dataset C.

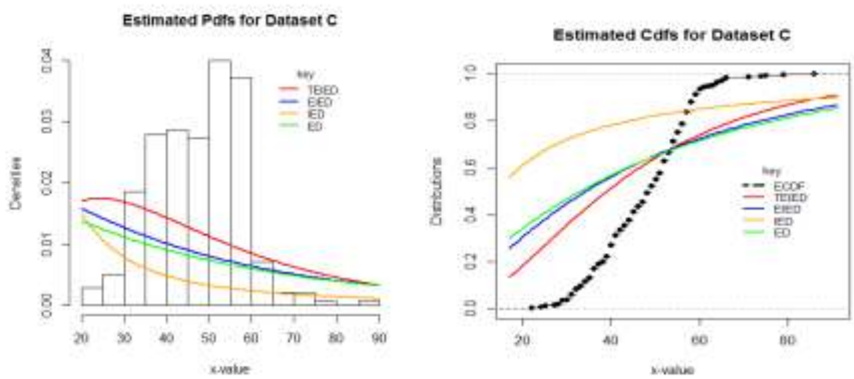

Figure 4.8: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset C.

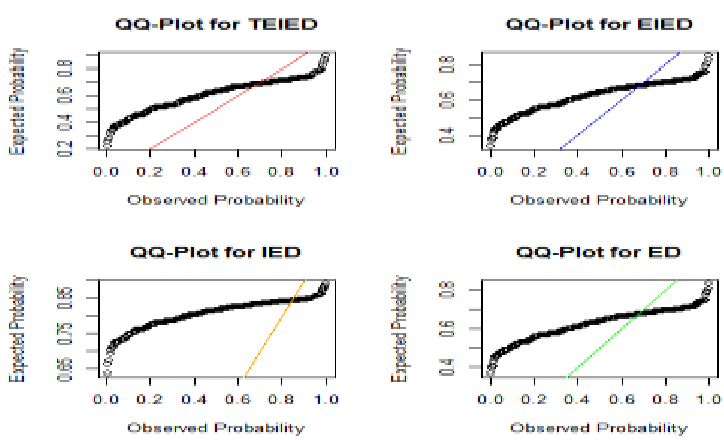

Figure 4.9: Probability Plots for The Fit of the TEIED, EIED, IED & ED Based on Dataset C.

Deciding on the best distribution based on the statistics in Tables 4.3 and 4.4 and 4.7 and 4.8, we conclude that the TEIED distribution has the smallest or minimum values of AIC, CAIC, BIC, HQIC, A^*, W^* and K-S statistic for dataset A and dataset B among all fitted models. Based on these criteria, it is clear that TEIED produces the overall best fit and therefore is selected as the most adequate model for explaining the two data sets as considered in this paper. The histogram of the data, fitted densities and estimated cumulative distribution functions displayed in Figures 4.4 and 4.5 for dataset A and dataset B respectively also confirm that the TEIED performs better than the EIED, IED and the conventional ED. Similarly, the Q-Q plots displayed in Figures 4.3 and 4.6 for dataset A and dataset B respectively also provide evidences that the proposed distribution (TEIED) is more flexible than the other three distributions (EIED, IED and ED) as already shown previously in Tables 4.2 and 4.3 as well as 4.6 and 4.7 for dataset A and dataset B respectively.

Again, deciding on the best distribution based on the statistics in Tables 4.11 and 4.12, we conclude that even though the TEIED distribution has the minimum values of AIC, CAIC, BIC, HQIC, A^*, W^* and K-S statistic for dataset C among all fitted models, it is not a suitable model for the data because the dataset is normally distributed while the distribution is a skewed model. The histogram of the data, fitted densities and estimated cumulative distribution functions displayed in Figure 4.8 for dataset C also confirmed that none of the fitted distributions is good for dataset C because it is a normally distributed data and therefore not suitable for skewed models. Similarly, the Q-Q plots displayed in Figure 4.9 for dataset C also provide evidences that the proposed distribution (TEIED) and the other three fitted distributions (EIED, IED and ED) are not good for dataset C as already demonstrated previously in Tables 4.11 and 4.12 as well as Figure 4.9 all for dataset C.

These results above also prove the fact that Quadratic Rank Transmutation Map [6] has additional advantage to the Exponential Inverse Exponential distribution by increasing its skewness and flexibility in modeling real life data and therefore, we agree and conclude that the quadratic rank transmutation map proposed [6] is very useful in increasing the flexibility of continuous probability distributions as seen in the previous studies [32-34, 36, 48-49].

5. Summary and Conclusion

This article proposed a new three-parameter generalization of the Inverse Exponential distribution named "Transmuted Exponential Inverse Exponential distribution". A very good reason for generalizing a classical distribution is that the generalization makes it more flexible for analyzing real life data. In this article, some mathematical and statistical properties of the transmuted Exponential Inverse Exponential distribution are derived and studied. These properties include validity and limiting behavior of the new model, moments, moment generating function, characteristics function, quantile function, coefficient of skewness and kurtosis, survival function, odds function, hazard function, cumulative hazard function, reversed hazard function and Renyi entropy. The paper also obtained the density function for the distribution of minimum and maximum order statistics. It also estimated the unknown parameters of the distribution by method of maximum likelihood estimation. The new distribution has been applied to three real life datasets and the results provided much evidence that the transmuted Exponential Inverse Exponential distribution is better than the Exponential Inverse Exponential distribution, Inverse Exponential distribution and the conventional Exponential distribution based on the datasets used and therefore it is our great hope that this new model will be applied fully in modeling real life problems most especially in survival analysis.

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