Weak LI-ideals in implicative almost distributive lattices

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ABSTRACT

In the field of many valued logic, lattice valued logic (especially ideals) plays an important role. Nowadays, lattice valued logic is becoming a research area. Researchers introduced weak LI-ideals of lattice implication algebra. Furthermore, other scholars researched LI-ideals of implicative almost distributive lattice. Therefore, the target of this paper was to investigate new development on the extension of LI-ideal theories and properties in implicative almost distributive lattice. So, in this paper, the notion of weak LI-ideals and maximal weak LI- ideals of implicative almost distributive lattice are defined. The properties of weak LI- ideals in implicative almost distributive lattice are studied and several characterizations of weak LI-ideals are given. Relationship between weak LI-ideals and weak filters are explored. Hence, the extension properties of weak LI-ideal of lattice implication algebra to that of weak LI-ideal of implicative almost distributive lattice were shown.

Keywords: Implicative almost distributive lattice; Weak LI-ideals; Maximal weak LI-ideas and weak filters.
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INTRODUCTION

Non-classical logic has become a considerable formal tool for artificial intelligence to deal with uncertainty information and automated reasoning. Many valued logic as a great extension and development of classical logic (Barnes and Mack, 2013) provides an interesting alternative to the classical logic for modeling and reasoning systems. In the field of many valued logic, lattice valued logic plays an important role (Ben-Eliyahu and Dechter, 1996; Barnes and Mack, 2013). Moreover, in order to research many valued logical systems whose propositional value given in lattice in 1993, Xu proposed the notion of lattice implication algebra and investigated many useful properties.

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Ever since, this logical algebra has been extensively investigated by several researchers (Xu, 1992; Jun, 1997; Liu and Xu, 1997; Xu *et al.*, 2001; Bolc and Borowik, 2003; Xu et al., 2003; Jiajun *et al,* 2007; Kolluru and Berhanu Bekele, 2012). Jun *et al.* (1998) defined the concept of LI-ideal in lattice implication algebra and discussed some its properties. Ideals play important role for the general development of these algebras (Swamy and Rao, 1981; Jun *et al.*, 1998, Berhanu Asaye *et al.*, 2019). Berhanu Assaye *et al.* (2018) and Tilahun Mekonnen (2019) introduced implicative almost distributive lattices (IADLs), LI-ideals in IADL and discussed some of their properties. Tilahun Mekonnen (2019) also discussed ILI-ideals in IADL. The aim of this article was to produce extension properties of weak LI-ideal of lattice implication algebra to ideals of IADLs. In this paper, the concept of weak LI-ideal in IADL and maximal weak LI-ideals in IADL were introduced and some of their properties investigated.

METHODOLOGY

Some important definitions, results and methods that will be useful in this study are given below. ADL is an acronym standing for almost distributive lattice.

Definition 2.1 (Swamy and Rao, 1981). An algebra $(L, \vee, \wedge, 0)$ of type $(2,2,0)$ is called an almost distributive lattice (ADL) with 0 if it satisfies the following axioms:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(3) $(x \vee y) \wedge y = y$

(4) $(x \vee y) \wedge x = x$

(5) $x \vee (x \wedge y) = x$

(6) $0 \wedge x = 0$, for all $x, y, z \in L$.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $x, y \in L$, define $x \leq y$ if and only if $x = x \wedge y$ or equivalently $x \vee y = y$, then " $\leq$ " is a partial ordering on L.

Note that here after we use an ADL L to mean an ADL L with 0.

Definition 2.2 (Swamy and Rao, 1981). Let L be an ADL. An element $m \in L$ is called maximal if for any $x \in L$, $m \leq x$ implies $m = x$.

Definition 2.3 (Swamy and Rao, 1981). A non-empty subset I of an ADL L is called an ideal of L, if it satisfies the following:

(1) $x, y \in I$ implies $x \vee y \in I$.

(2) $x \in I$ and $y \in L$ implies that $x \wedge y \in I$. We call I as an ADL ideal of L.

The following important property of the ideals is very useful to develop the algebra of ideals. If I is an ideal of ADL L and $a, b \in L$, then $a \wedge b \in I$ if and only if $b \wedge a \in I$.

Definition 2.4 (Swamy and Rao, 1981). Let L be an ADL. For any $a \in L$, principal filter of L generated by a is $[a) = \{x \vee a : x \in L\}$.

Definition 2.5 (Swamy and Rao, 1981). A non-empty subset F of an ADL L is called a filter of L if it satisfies the following:
(1) $x, y \in F$ implies $x \wedge y \in F$.
(2) $x \in F$ and $y \in L$ implies that $y \vee x \in F$.

Definition 2.6 (Swamy and Rao, 1981). A non-empty subset S of an ADL L is called a sub- ADL of L if
(1) $0 \in S$
(2) $x \wedge y \in S$ and $x \vee y \in S$, for any $x, y \in S$.

Theorem 2.7 (Swamy and Rao, 1981). Let F be filter of an ADL L and $x, y \in L$. Then $x \vee y \in F$ if and only if $y \vee x \in F$.

Lemma 2.8 (Swamy and Rao, 1981). Every ideal (filter) of L is a sub- ADL of ADL L.

Definition 2.9 (Berhanu Asaye *et al*, 2018). Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and maximal element m. Then an algebra $(L, \vee, \wedge, \rightarrow, ', 0, m)$ of type (2, 2, 2, 1, 0, 0) is called implicative almost distributive lattice (IADL) if it satisfies the following conditions:
(1) $x \vee y = (x \rightarrow y) \rightarrow y$
(2) $x \wedge y = [(x \rightarrow y) \rightarrow x']'$
(3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
(4) $m \rightarrow x = x$
(5) $x \rightarrow m = m$
(6) $x \rightarrow y = y' \rightarrow x'$
(7) $0' = m$, for all $x, y, z \in L$.

Now we define the relation "$\leq$" on an IADL L as follows: $x \leq y \Rightarrow x \rightarrow y = m$, for all $x, y \in L$. The relation "$\leq$" on L is a partial ordering. Thus $(L, \leq)$ is a poset.

Theorem 2.10 (Berhanu Asaye *et al*., 2018). In an IADL L, for all $x, y, z \in L$ the following conditions hold:
(1) $[(x \rightarrow y) \rightarrow y] \wedge m = [(y \rightarrow x) \rightarrow x] \wedge m$
(2) $[((x \rightarrow y) \rightarrow x')'] \wedge m = [((y \rightarrow x) \rightarrow y')'] \wedge m$

(3) $x \rightarrow x = m$
(4) $m' = 0$
(5) $(x')' = x$
(6) $x' = x \rightarrow 0$
(7) $0 \rightarrow x = m$
(8) $x \rightarrow y = m = y \rightarrow x$ implies $x = y$.
(9) If $x \rightarrow y = m$ and $y \rightarrow z = m$, then $x \rightarrow z = m$
(10) $x \leq y$ if and only if $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$
(11) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$
(12) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = m$, $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = m$
(13) $(x \rightarrow z) \rightarrow (x \rightarrow y) = (z \rightarrow x) \rightarrow (z \rightarrow y) = (x \wedge z) \rightarrow y$
(14) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$
(15) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$
(16) $x \leq y$ implies $y' \leq x'$
(17) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
(18) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$
(19) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(20) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$

Definition 2.11 (Berhanu Assaye *et al*., 2019). Let L be an IADL.
(1) A subset F of L is called a filter of L if it satisfies:
(F_1) $m \in F$
(F_2) $x \in F$ *and* $x \rightarrow y \in F$ implies $y \in F$, for all $x, y \in L$.
(2) A subset F of L is called implicative filter of L if it satisfies
(F_1) $m \in F$
(I) $x \rightarrow y \in F$ and $x \rightarrow (y \rightarrow z) \in F$ implies $x \rightarrow z \in F$, for all $x, y, z \in L$.

Lemma 2.12 (Berhanu Assaye *et al*., 2019). Let F be a non-empty subset of an IADL L. Then F is a filter of L if and only if it satisfies for all $x, y \in F$ and $z \in L$: $x \leq y \rightarrow z$ implies $z \in F$

Lemma 2.13 (Berhanu Assaye *et al.,* 2019, p. 67). Every filter F of an IADL L has the following property: $x \leq y$ and $x \in F$ implies $y \in F$.

Definition 2.14 (Tilahun Mekonnen, 2019). An implicative almost distributive lattice L is called H-implicative almost distributive lattice, if for any $x, y, z \in L$, $x \vee y \vee ((x \wedge y) \rightarrow z) = m$.

Theorem 2.15 (Tilahun Mekonnen, 2019, P.67). Let L be H-IADL. Then for any $x, y, z \in L$, $x \to (y \to z) = (x \to y) \to (x \to z)$.
Theorem 2.16 (Xu, 1993) Let L be IADL, then for any $x, y, z \in L$,
$\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big) = x \vee y \vee ((x \wedge y) \to z)$.

Definition 2.17 (Tilahun Mekonnen, 2019, P.99). Let L be an IADL. An LI-ideal A of L is a non-empty subset of L such that
(1) $0 \in A$,
(2) $y \in A$ and $(x \to y)' \in A$ implies $x \in A$, for $x, y \in L$.

Theorem 2.18 (Tilahun Mekonnen, 2019, P.99; Xu *et al.*, 2003) Let L be an IADL. Every LI-ideal of L is an ADL ideal of L.

Theorem 2.19 (Tilahun Mekonnen, 2019, P.100). In H-IADL, every ADL ideal is an LI-ideal.

Theorem 2.20 (Tilahun Mekonnen, 2019, P.98). Let A be LI-ideal of IADL L. If $x \in A$ and $y \leq x$ for some $y \in L$, then $y \in A$.

Theorem 2.21 (Tilahun Mekonnen, 2019, P.97). If A is a non-empty subset of an IADL L, then $< A > = \{ x \in L | \, a_n' \to (\cdots (a_1' \to x') \cdots) = m$ for some $a_1, \cdots, a_n \in A\}$ is smallest LI-ideal containing A. In particular $<\{a\}> = <a> = \{x \in L : n(a') \to x' = m\}$ for any $a \in L$.

Definition 20.22 (Tilahun Mekonnen, 2019, P.130). A non-empty subset I of an implicative almost distributive lattice L is said to be an implicative LI-ideals (briefly, ILI-ideals) of L if it satisfies:
(1) $0 \in I$ and
(2) $(((x \to y)' \to y)' \to z)' \in I$ and $z \in I$ implies $(x \to y)' \in I$ for all $x, y, z \in L$.

Theorem 2.23 (Tilahun Mekonnen, 2019, P.131). In IADL L, any ILI-ideal of L is an LI-ideal of L. Conversely, in H-IADL, any LI-ideal is an ILI-ideal.

RESULTS AND DISCUSSION

In this work, weak LI-ideal in implicative almost distributive lattice was introduced and some of its properties investigated. Characterizations of WLI-ideals was discussed. Also, maximal weak LI-ideal in IADL was introduced and some of its properties investigated.

Definition 3.1. Let L be an implicative almost distributive lattice (IADL). A subset A of L is called a weak LI-ideals (briefly, WLI-ideal) of L if it satisfies the following condition: For all $x, y \in L$; $(x \to y)' \in A$ implies $((x \to y)' \to y)' \in A$, where $\to$ is a binary operation.

Example 3.2. Let $A = \{0, m\}$ be a sub set of IADL L. Claim: A is WLI-ideal of L. Now if $x = 0, y = m$, then $(0 \to m)' \in A$ implies $((0 \to m)' \to m)' = 0 \in A$; if $x = m, y = 0$, then $(m \to 0)' = m \in A$ implies $((m \to 0)' \to 0)' = m \in A$. Hence, A is WLI-ideal of L.

Example 3.3. Let $B = \{0\}$ be a sub set of IADL L. Then $(0 \to 0)' = 0 \in B$ implies $((0 \to 0)' \to 0)' = 0 \in B$. Hence, B is WLI-ideal of L (by Definition 3.1).

Theorem 3.4. Let L be an IADL, $A \subseteq L$ is an LI-ideal of L. Then A is a WLI-ideal of L. Proof. Suppose that A is an LI-ideal of IADL L and let $x, y \in L$ such that $(x \to y)' \in A$. Then

$$(((x \to y)' \to y)' \to (x \to y)')' = ((x \to y) \to ((x \to y)' \to y))' = ((x \to y) \to (y' \to (x \to y)))' = (y' \to ((x \to y) \to (x \to y)))' = 0 \in A$$

, i.e., $(((x \to y)' \to y)' \to (x \to y)')' \in A$. Since A is an LI-ideal of L and $(x \to y)' \in A$, then $((x \to y)' \to y)' \in A$ (by Definition2.17. Thus $(x \to y)' \in A$ implies $((x \to y)' \to y)' \in A$. Hence, A is a WLI-ideal of L.

Theorem 3.5. Let L be an IADL. Every ILI-ideal of L is a WLI-ideal of L.

Proof. Suppose that A is an ILI-ideal of an IADL L, and then A is an LI-ideal of L by Theorem 2.23. Hence, A is the WLI-ideal of L by Theorem 3.4.

Definition 3.6. Let L be an IADL. A sub set A of L is a weak filter of L if it satisfies following condition:
$x \to y \in A$ implies $x \to (x \to y) \in A$ for all $x, y \in L$.

Theorem 3.7. If A is a non-empty subset of IADL L and $A' = \{x' : x \in A\}$, then A' is a WLI-ideal of L if and only if A is a weak filter of L.

Proof. Let L be an IADL. Assume that for any $x, y \in L$, A is a weak filter of L, i.e., $x \to y \in A$ implies $x \to (x \to y) \in A$ holds. Then $(x \to y)' \in A'$ implies $(x \to (x \to y))' \in A'$ by definition of A'. That is, $(y' \to x')' \in A'$, implies $((y' \to x')' \to x')' \in A'$ (see Definition 2.9). Thus, A' is a WLI-ideal of L (by Definition 3.1).
Conversely, let A' is a WLI-ideal of L and $x, y \in L$. Then $(x \to y)' \in A'$ implies $((x \to y)' \to y)' \in A'$. This implies $x \to y = y' \to x' \in A$ implies $((x \to y)' \to y)')' = (x \to y)' \to y = (y' \to (y' \to x')) \in A$. In addition, I get $y' \to x' \in A$ implies $(y' \to (y' \to x')) \in A$ (by Definition 2.9, $x \to y = y' \to x'$). Hence, A is weak filter of L by Definition 3.6.

Theorem 3.8. Let L be an H-IADL. Then any ADL ideal I of L is a WLI-ideal of L.

Proof. In an H-IADL L, any ADL ideal A of L is an LI-ideal of L (see Theorem 2.19). By Theorem 3.4, an LI-ideal A of L is a Weak LI-ideal of L. Thus, any ADL ideal A of L is a WLI-ideal of L.

Corollary 3.9. Let L be H-IADL. Then LI-ideal $\{0\}$ of L is WLI-ideal.

Theorem 3.10. Let L be H-IADL. If $A(t) = \{x \in L : (x \to t)' = 0\}$ for all elements t of L, then $A(t)$ is WLI-ideal of L.

Proof. Let $x, y \in L$ for an H-IADL L. Suppose $(x \to y)' \in A(t)$, then $((x \to y)' \to t)' = 0 \Rightarrow ((x \to y) \vee t)' = 0$, i.e., $(x \to y)' \wedge t' = 0$, since $(((x \to y)' \to y)' \to t)' = (((x \to y) \vee y)' \to t)' = (((x \to y) \vee y) \vee t)' = ((x \to y)' \wedge y') \wedge t' = y' \wedge ((x \to y)' \wedge t') = y' \wedge 0 = 0$
, we have $((x \to y)' \to y)' \in A(t)$. Therefore, $A(t)$ is a WLI-ideal of L.

Theorem 3.11. Let L be an H-IADL. If A is an LI-ideal of L, then $A_t = \{x \in L : (x \to t)' \in A\}$ is a WLI-ideal for any $t \in L$.

Proof. Assume that $(x \to y)' \in A_t$ for any x, y in H-IADL L, then $((x \to y)' \to t)' \in A$. Since

$((((x \to y)' \to y)' \to t)' \to ((x \to y)' \to t)')' = (((x \to y)' \to t) \to (((x \to y)' \to y)' \to t))' = (((x \to y)' \to y)' \to (((x \to y)' \to t) \to t))' = (((x \to y)' \to y)' \to ((x \to y)' \vee t))' = ((((x \to y)' \to y)' \to (x \to y)') \vee (((x \to y)' \to y)' \to t))' = (((x \to y) \to ((x \to y)' \to y)) \vee (t' \to ((x \to y)' \to y)))' = ((y' \to m) \vee (y' \to (t' \to (x \to y)))') = (m \vee (y' \to (t' \to (x \to y))))' = 0 \wedge (((x \to y)' \to t)' \to y)' = 0 \in A$

This implies $((((x \to y)' \to y)' \to t)' \to ((x \to y)' \to t)')' \in A$. As A is an LI-ideal of L and $((x \to y)' \to t)' \in A$, we get $(((x \to y)' \to y)' \to t)' \in A$. This implies $((x \to y)' \to y)' \in A_t$ by definition of A_t. Hence, A_t is a WLI-ideal of L by Definition 3.1.

Theorem 3.12. Let L be IADL and $\{A_i: i \in J\}$ be the set of WLI-ideal of L where a nonempty set J is the index set. Then $\cup_{i\in J} A_i$ and $\cap_{i\in J} A_i$ are WLI-ideals of L.

Proof. Let $\{A_i: i\in J\}$ be the set of WLI-ideals of IADL L where J (non-empty set) is index set. Assume $(x \to y)' \in \cup_{i\in J} A_i$, then there exist $i \in J$ such that $(x\to y)' \in A_i$. Since A_i is a WLI-ideal of L which implies that $((x \to y)' \to y)' \in A_i$ for some $i \in J$. Hence, we get $((x \to y)' \to y)' \in \cup_{i\in J} A_i$. By Definition 3.1, $\cup_{i\in J} A_i$ is a WLI-ideal of L. Suppose that $(x \to y) \cap_{i\in J} A_i$ for any $x, y \in L$, then $(x \to y)' \in A_i$ for any $i \in J$. Since A_i is a WLI-ideal of L, we have $((x\to y)' \to y)' \in A_i$ for any $i \in J$. Thus, $((x\to y)' \to y)' \in \cap_{i\in J} A_i$. Therefore, $\cap_{i\in J} A_i$ is a WLI-ideal of L by Definition 3.1.

Definition 3.13. Let L be an IADL. A WLI-ideal A of L is called maximal WLI-ideal of L if it is not equal to L and it is maximal elements of the set of all WLI-ideals with respect to set inclusion.

Remark 3.14. Suppose $A \subseteq B$, By definition $< B > \subseteq < A >$. From Theorem 3.12, the intersection of WLI-ideals of L is WLI-ideal of L. Suppose $A \subseteq L$ and $A \neq L$ then the maximal WLI-ideal containing A is called the WLI-ideal generated by A and denoted by $< A >$.

Now for any $a \in L$,

$L_a^1 = \{((x_1 \to y_1)' \to y_1)': x_1, y_1 \in L, (x_1 \to y_1)' = a\}$

$L_a^2 = \{((x_2 \to y_2)' \to y_2)': x_2, y_2 \in L, (x_2 \to y_2)' \in L_a^1\}$

$L_a^3 = \{((x_3 \to y_3)' \to y_3)': x_3, y_3 \in L, (x_3 \to y_3)' \in L_a^2\}$

$\vdots \qquad\qquad \vdots$

$L_a^n = \{((x_n \to y_n)' \to y_n)': x_n, y_n \in L, (x_n \to y_n)' \in L_a^{n-1}\}.$

It can be easily verified that
$((x_i \to y_i)' \to y_i)' = ((x_i \to y_i)' \to y_i)' \to 0)'$; $((x_i \to y_i)' \to y_i)' \leq (x_i \to y_i)'$ for each i =1, 2, 3,⋯.
Hence, $L_a^n \subseteq L_a^{n-1} \subseteq \cdots \subseteq L_a^3 \subseteq L_a^2 \subseteq L_a^1$ and denoted by $T_a = \cap_1^\infty L_a^i$
We also observe that
(1) L_a^i for each $i = 1,2,3,\cdots$ is an LI-ideal and hence weak LI-ideal by Theorem 3.4. In this condition we see that L_a^1 is maximal weak LI-ideal of L as $L_a^1 \neq L$ and $L_a^n \subseteq L_a^{n-1} \subseteq \cdots \subseteq L_a^3 \subseteq L_a^2 \subseteq L_a^1 \subseteq L$.
(2) $L_a^1 = \{(a \to y_1)' : x_1, y_1 \in L, (x_1 \to y_1)' = a\}$
$L_a^2 = \{((a \to y_1)' \to y_2)' : x_2, y_2 \in L, (x_2 \to y_2)' = (a \to y_1)' \in L_a^1\}\}$
$L_a^3 = \{(((a \to y_1)' \to y_2)' \to y_3)' : x_3, y_3 \in L, (x_3 \to y_3)' = ((a \to y_1)' \to y_2)' \in L_a^2$ }
⋮ ⋮
$L_a^n = \{(\cdots((a \to y_1)' \to y_2)' \cdots \to y_n)' : x_n, y_n \in L, (x_n \to y_n)' \in L_a^{n-1}\}\}$. By considering $y_i = 0$ for every $i = 1,2,3\cdots$, we get $a \in L_a^i$ and hence $a \in T_a = \cap_1^\infty L_a^i$.

Theorem 3.15. Let L be IADL. Then T_a is a WLI-ideal of L for any a ∈ L.
Proof. Let L be IADL and T_a be defined as above. Suppose that $(x \to y)' \in T_a$ for any $a \in L$ and for any $x, y \in L$, then $(x \to y)' \in L_a^i$ for each i =1,2,3,⋯, implies $((x \to y)' \to y)' \in L_a^{i+1}$ by definition of L_a^i above. Thus $(x \to y)' \in T_a$ implies $((x \to y)' \to y)' \in T_a$. Therefore, T_a is a WLI-ideal of L by Definition 3.1.

Theorem 3.16. Let L be IADL, $x, y, a \in L$, then $x \in T_a$ if and only if there exists $k \in \{1,2,3,\cdots\}$, such that $x_k, x_{k-1}, \cdots, x_2, x_1 \in L$ and $y_k, y_{k-1}, \cdots, y_2, y_1 \in L$, if it satisfies the following conditions:
(1) $(x_1 \to y_1)' = a$;
(2) $(x_i \to y_i)' \in L_a^{i-1}$ and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})'$; for $i = 2,3,4,\cdots$
(3) $((x_k \to y_k)' \to y_k)' = x$.

Proof. Assume conditions (1) to (3) hold. Clearly, $x \in T_a$. Conversely, let $x \in T_a$, then $x \in L_a^i$ for each $i \in \{1,2,3,\cdots\}$. In particular take $i = k$, i.e., $\exists x_k, y_k \in L$ such that $x = ((x_k \to y_k)' \to y_k)'$. Thus, $(x_k \to y_k)' \in L_a^{k-1}$. Since there exist $x_{k-1}, y_{k-1} \in L$ such that $(x_k \to y_k)' = ((x_{k-1} \to y_{k-1})' \to y_{k-1})'$ for $(x_k \to y_k)' \in L_a^{k-1}$ and so we get $(x_{k-1} \to y_{k-1})' \in L_a^{k-2}$. Similarly there exist $x_{k-2}, y_{k-2} \in L$ such that $(x_{k-1} \to y_{k-1})' = ((x_{k-2} \to y_{k-2})' \to y_{k-2})' \in L_a^{k-2}$and so we get $(x_{k-2} \to y_{k-2})' \in L_a^{k-3}$. It follows that we can get sequences x_k, $x_{k-1}, \cdots, x_2, x_1 \in L$ and $y_k, y_{k-1}, \cdots, y_2, y_1 \in L$ such that the three conditions hold.

Theorem 3.17. Let L be IADL. Then $T_a = < a >$ for any $a \in L$.

Proof. From Remark 3.14, we know that $a \in T_a$, then $a \in \cap_1^{\infty} L_a^i$. This implies $a \in L_a^i$ for every i. Since $a \in < a >$ for every $a \in L_a^i$, then $a \in \cap_{a \in L_a^i} < a >$. Thus, $< a > \subseteq T_a$. On the other hand, let $a \in T_a$, then there exist $k \in N^+$ such that $x_k, x_{k-1}, \cdots, x_2, x_1 \in L$ and $y_k, y_{k-1}, \cdots, y_2, y_1 \in L$ satisfying the following conditions:

(1) $(x_1 \to y_1)' = a$;

(2) $(x_i \to y_i)' \in L_a^{i-1}$ and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})'$ $(i = 2, 3, \cdots)$;

(3) $((x_k \to y_k)' \to y_k)' = x$ (by Theorem 3.16). Moreover, we have $(x_i \to y_i)' \in < a >$ $(i = 1, 2, 3, \cdots, k)$, i.e., $T_a \subseteq < a >$. hence, the result.

CONCLUSION

In order to research the many-valued logical systems whose propositional value is given in a lattice. Xu (1993) initiated the concept of lattice implication algebra. Furthermore, Berhanu Assaye *et al.* (2018) proposed implicative almost distributive lattices. Hence for the development of many-valued logical systems, clarifying the structure of this algebra is needed. To investigate the structure of a logical system, the ideals with special properties play an important role. In this article, the concept of WLI-ideals in IADL was introduced and related properties investigated. Hence, the search for the properties of ideals of IADL will advance the research of logical systems with propositional value. Furthermore, this work will enhance the development of logical algebra and initiates further research work.

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