The effect of prey refuge on the dynamics of three species food web system

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ABSTRACT

In this paper, a mathematical model is proposed to study the effect of prey refuge on the dynamics of three species food web system. The food web comprises of a single prey and two competing predators. The two predators predate their prey following Holling type II functional response. In this work we discussed boundedness of the system, existence condition of the equilibrium points and the Jacobean matrix is obtained by linearization techniques. The local stability of the equilibrium points was discussed by using Routh-Hurwitz criteria and the global stability of the equilibrium points by constructing suitable Lyapunov function. Numerical simulation is conducted to support the analytical result. Finally, the effect of prey refuge on the dynamics of one prey two predator was discussed based on the analytical and numerical simulation results. From the numerical simulations, it is found that the dynamical system is persistent for a small value of the refuge constant. However, an increase in the refuge constant leads to the extinction of one of the predator species.

INTRODUCTION

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance (Berryman, 1992). Systems of differential equations have a certain extent, successfully to describe the interactions (relationship) between species. The basic system is the Lotka-Volterra model, which models the interaction between a predator and a prey. Various dynamical relations between predators and their prey in ecology and mathematical ecology have been studied (Kuang and Beretta, 1998). The three species interaction shows very complex dynamical behavior (Gakkhar *et*

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al., 2007; Naji and Balasim, 2007; Lv and Zhao, 2008; Upadhyay and Naji, 2009; Yu and Zhao, 2009; Reddy and Ramacharyulu, 2011; Li *et al*., 2014; Hsu *et al*., 2015; Panday *et al*., 2018; Mishra and Raw, 2019). The co-existence and extinction of interacting species have been of great importance and studied extensively in the past (Dubey and Upadhyay, 2004; Naji and Balasim, 2007; Feng *et al*., 2009; Alebraheem and Abu-Hasan, 2012). Dubey and Upadhyay (2004) have studied the dynamics of one-prey two-predator system with ratio-dependent functional response. Criteria for local stability, instability and global stability of the nonnegative equilibrium points were obtained.

Hsu (1982) proposed and analyzed a model of two predators competing for a single prey. He showed that if the interference coefficient is small, then the winner in purely exploitative system competes its rival successfully and if the interference coefficient is large enough, then the competition outcome depends on the initial population of predator species. Freedman (Freedman and Waltman, 1984) considered three level food webs and two competing predators feeding on a single prey and a single predator feeding on two competing prey species. They obtained criteria for the system to be persistent. Kar (2005) studied on the stability analysis of a prey predator model incorporating a prey refuge.

Jawdat and Yahya (2012) studied two competing predators sharing one prey in homogeneous environment with Holling type-II functional response. The conditions of coexistence and extinction of the predators in the case of non-periodic solution were obtained in terms of efficiency of predator conversion of prey biomass into predator offspring. Edwin (2010) study the dynamics of a system of two prey and one predator in which the predator shows a Holling type II response to one prey that is also harvested, and a ratio-dependent response to the other prey. The major observation from results of numerical simulation is that the predator population density increased significantly when the intrinsic growth rate of both preys increased. This can imply that a high intrinsic growth rate of the prey initially increases their population density, which increases the predator chance of capturing the prey, and so the predator population density increases.

Nature can provide some degree of protection to a given number of prey populations by providing refuges. Such refugia can help in prolonging prey predator interactions by reducing the chance of extinction due to predation (Huang *et al*., 2006; Kar, 2005; Srinivasu and Gayatri, 2005) and damp prey predator oscillations (Collings, 1995). The effects of prey refuges on the population dynamics are very complex in nature, but for modeling purposes, it can be considered as constituted by two components: the first effects, which

affect positively the growth of prey and negatively that of predators, comprise the reduction of prey mortality due to decrease in predation success. The second one may be the trade-offs and by-products of the hiding behavior of prey which could be advantageous or detrimental for all the interacting populations (González-Olivares and Ramos-Jiliberto, 2003).

In the literature studies show that refuges have both stabilizing (Hassell, 2020) and destabilizing effect (McNair, 1986). The traditional ways in which the effect of refuge used by the preys has been incorporated in predator prey models is to consider either a constant number or a constant proportion of the prey population being protected from predation (Smith, 1978). Hassel (Hassell, 2020) notes that in reality refugia fall between these two extremes. It is pointed out that those protecting a proportion of the prey population appearing to be more common (Collings, 1995). However, the refuges, which protect a constant number of preys, have a stronger stabilizing effect on population dynamics than the refuges, which protect a constant proportion of prey (González-Olivares and Ramos-Jiliberto, 2003; Krivan, 1998). For more biological background and results on the effects of a prey refuge, one could refer to several other reports (McNair, 1986; Sih, 1987; Collings, 1995; Krivan, 1998; Kar, 2005; Ko and Ryu, 2006).

In this paper, we intended to improve Jawdat Alebraheem and Yahya Abu-Hasan (2012) study on one prey and two-predator system by incorporating prey refuge and taking the classical Holling type II numerical response on the two predators to study the co-existence, boundedness and stability of the system and see the effect of prey refuge on the dynamics of the proposed system.

The organization of the paper is as follows. In section two, the mathematical model is presented. Section three is devoted to mathematical analysis: existence and boundedness of the solutions, existence and stability of the unique steady state of the system. In section four, numerical simulations are shown. At last, conclusion is presented in section five.

Model formulation

Jawdat Alebraheem and Yahya Abu-Hasan (2012) considered the case of an environment where there is a prey and two predators. A two-predator one prey model was studied with a prey population $X(t)$ and predator populations $Y(t)$ and $Z(t)$. The predators prey on the prey species based on the Holling type-II functional response. Portion of the prey species is supposed to take refuge. The prey species are assumed to grow logistically. Interspecific competition among

the predator species was also included in the model. The model has had three non-linear autonomous ordinary differential equations describing how the population densities of the three species would vary with time.

The Model equation is written as

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - F_1(X)Y - F_2(X)Z$$
$$\frac{dY}{dT} = e_1F_1(X)Y - \bar{\beta}_1YZ - \delta_1Y$$
$$\frac{dZ}{dT} = e_2F_2(X)Z - \bar{\beta}_2YZ - \delta_2Z$$

for $X(0) = X_0 \geq 0, Y(0) = Y_0 \geq 0, Z(0) = Z_0 \geq 0$, $F_i(X) = \frac{a_iX}{b_i+X}$, $i = 1,2$.

It is assumed that prey grows logistically in the absence of predators such that r is the per capita intrinsic growth rate of prey and with carrying capacity K. $\bar{\beta}_1$ and $\bar{\beta}_2$ measure the inter-specific competition of the predator Z on predator Y and vice-versa; b_1 and b_2 are the saturation value of the functional response of the predator Y and predator Z, respectively. And a_1 and a_2 measure efficiency of the searching and the capture of predators Y and Z, respectively; e_1 and e_2 represent efficiency of converting consumed prey into predator offspring. δ_1 and δ_2 are the death rate of predators Y and Z, respectively. The functions $F_1(X)$ and $F_2(X)$ represent the Holling type-II functional responses. The constant m represents prey refuge constant.

In this work, we intended to modify the model (1) by incorporating a refuge protecting mX of the prey. This leaves $(1-m)X$ of the prey available to the predators. Thus, the modified system becomes:

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - F_1(X)Y - F_2(X)Z$$
$$\frac{dY}{dT} = e_1F_1(X)Y - \bar{\beta}_1YZ - \delta_1Y$$
$$\frac{dZ}{dT} = e_2F_2(X)Z - \bar{\beta}_2YZ - \delta_2Z$$

for

$X(0) = X_0 \geq 0, Y(0) = Y_0 \geq 0, Z(0) = Z_0 \geq 0, F_i(X) = \frac{a_i(1-m)X}{b_i+(1-m)X}$, $i = 1,2$,

where all the parameters in the model are assumed to be positive.

The following non-dimensional state variables and parameters are chosen.

$x = \frac{X}{K}, y = \frac{Y}{K}, z = \frac{Z}{K}, t = rT, \alpha_1 = \frac{b_1}{K}, \alpha_2 = \frac{b_2}{K}, \beta_1 = \frac{\bar{\beta}_1K}{r}, \beta_2 = \frac{\bar{\beta}_2K}{r}, c_1 = \frac{a_1}{r},$
$c_2 = \frac{a_2}{r}, d_1 = \frac{\delta_1}{r}, d_2 = \frac{\delta_2}{r}.$

The model equation (2) takes the following non-dimensional form

$$\frac{dx}{dt} = x(1-x) - \frac{c_1(1-m)xy}{\alpha_1 + (1-m)x} - \frac{c_2(1-m)xz}{\alpha_2 + (1-m)x} = f(x,y,z)$$

$$\frac{dy}{dt} = \frac{e_1 c_1(1-m)xy}{\alpha_1 + (1-m)x} - \beta_1 yz - d_1 y = g(x,y,z)$$

$$\frac{dz}{dt} = \frac{e_2 c_2(1-m)xz}{\alpha_2 + (1-m)x} - \beta_2 yz - d_2 z = h(x,y,z)$$

for $x(0) = x_0 \geq 0, y(0) = y_0 \geq 0, z(0) = z_0 \geq 0$.

MATHEMATICAL ANALYSIS

Positivity and boundedness of the model

In the theory of the dynamics of food web system, the positivity and boundedness of the system implies that the system is biologically well behaved. One can easily prove that the function $f = (f, g, h)$ of the system (3) is Locally Lipschitz on the region $\Omega = \{(x, y, z): x \geq 0, y \geq 0, z \geq 0\}$. Fundamental theorem of existence and uniqueness assures existence and uniqueness of solution of the system (3) with the given initial condition. The uniqueness and existence theorem ensures that any trajectory starting from the first octant remains in it, that is, no trajectory will cross the coordinate axes.

Theorem 1: All the solutions of system (3) which start in R_+^3 are uniformly bounded.

Proof. Let $x(t)$, $y(t)$ and $z(t)$ be any solutions of the system (3) with positive initial condition.

Let $w = x + \frac{1}{e_1}y + \frac{1}{e_2}z$.

Then, we have $\frac{dw}{dt} = \frac{dx}{dt} + \frac{1}{e_1}\frac{dy}{dt} + \frac{1}{e_2}\frac{dz}{dt}$. From equation (3), we have

$$\frac{dw}{dt} = x(1-x) - \frac{c_1(1-m)xy}{\alpha_1+(1-m)x} - \frac{c_2(1-m)xz}{\alpha_2+(1-m)x} + \frac{c_1(1-m)xy}{\alpha_1+(1-m)x} - \frac{\beta_1 yz}{e_1} - \frac{d_1 y}{e_1} + \frac{c_2(1-m)xz}{\alpha_2+(1-m)x}$$

$$- \frac{\beta_2 yz}{e_2} - \frac{d_2 z}{e_2},$$

$$\leq -x - \frac{d_1 y}{e_1} - \frac{d_2 z}{e_2} - (x-1)^2 + 1,$$

$$\leq -\theta w + 1,$$

where $\theta = \min(1, d_1, d_2)$.

Thus, we have $\frac{dw}{dt} + \theta w \leq 1$.

Applying the theory of differential inequality, we obtain

$$0 < w(x, y, z) \leq \frac{1 - e^{-\theta t}}{\theta} + w(x(0), y(0), z(0))e^{-\theta t},$$

which upon letting $t \to \infty$ yields $w(x, y, z) \le \frac{1}{\theta}$. Therefore, the solutions of the system are bounded for $\eta > 0$ and $w(x, y, z) < \frac{1}{\theta} + \eta$.

Hence, all the solutions of (3) that initiate in $\Re_+^3$ are confined in the region $S = \{(x, y, z) \in \Re_+^3 : w(x, y, z) < \frac{1}{\theta} + \eta\}$ for any $\eta > 0$.

Existence of equilibrium points of the system

In this section, conditions for the existence of the equilibrium points of the system (3) are established. One can see that the system (3) has five biologically feasible equilibrium points:

1. The trivial equilibrium point: $E_0(0,0,0)$,
2. The axial equilibrium points: $E_1(1,0,0)$, $E_2(x^*, 0, z^*)$, $E_3(\bar{x}, \bar{y}, 0)$,

where $x^* = \dfrac{d_2 \alpha_2}{(e_2 c_2 - d_2)(1-m)}$, $z^* = \dfrac{(\alpha_2 + (1-m)x^*)(1-x^*)}{c_2(1-m)}$,

$\bar{x} = \dfrac{d_1 \alpha_1}{(e_1 c_1 - d_1)(1-m)}$, $\bar{y} = \dfrac{(1-\bar{x})(\alpha_1 + (1-m)\bar{x})}{c_1(1-m)}$.

The equilibrium point $E_2(x^*, 0, z^*)$ exists when

$$c_2 e_2 > d_2 \quad \text{and} \quad m < 1 - \frac{d_2 \alpha_2}{c_2 e_2 - d_2}$$

The equilibrium point $E_3(\bar{x}, \bar{y}, 0)$ exists when

$$c_1 e_1 > d_1 \quad \text{and} \quad m < 1 - \frac{d_1 \alpha_1}{c_1 e_1 - d_1}$$

3. The co-existence equilibrium point $E_4(\tilde{x}, \tilde{y}, \tilde{z})$, where

$$\tilde{y} = \frac{(e_2 c_2 - d_2)(1-m)\tilde{x} - d_2 \alpha_2}{\beta_2(\alpha_2 + (1-m)\tilde{x})}, \quad \tilde{z} = \frac{(e_1 c_1 - d_1)(1-m)\tilde{x} - d_1 \alpha_1}{\beta_1(\alpha_1 + (1-m)\tilde{x})},$$

and $\tilde{x}$ is the positive solution of the cubic polynomial

$$A_3 \tilde{x}^3 + A_2 \tilde{x}^2 + A_1 \tilde{x} + A_0 = 0,$$

with

$A_3 = (1-m)^2 \beta_1 \beta_2, A_2 = (1-m)(\alpha_1 + \alpha_2 - (1-m))\beta_1 \beta_2,$
$A_1 = \beta_1(1-m)(c_1(e_2 c_2 - d_2)(1-m) - \beta_2(\alpha_1 + \alpha_2)) + \beta_1 \beta_2 \alpha_1 \alpha_2$
$+\beta_2 c_2 (1-m)^2 (e_1 c_1 - d_1),$
$A_0 = (1-m)(c_1 d_2 \alpha_2 \beta_1 + c_2 d_1 \alpha_1 \beta_2) + \alpha_1 \alpha_2 \beta_1 \beta_2.$

Therefore, sufficient conditions for the existence of the positive equilibrium point in the interior of the first octant are easily obtained as follows:

$$0 < \tilde{x} < \min\left\{\frac{\alpha_1 d_1}{e_1 c_1 - d_1}, \frac{\alpha_2 d_2}{e_2 c_2 - d_2}\right\}, \quad m < 1 - \max\left\{1, \frac{\beta_2}{c_1(e_2 c_2 - d_2)}\right\}(\alpha_1 + \alpha_2)$$

Local stability of the equilibrium points

The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and checking the sign of the real part of the

eigenvalues of the Jacobean matrix evaluated at each equilibrium point. For stability of the equilibrium points, the real parts of the eigenvalues of the Jacobian matrix must be negative.

The Jacobian matrix is

$$J(E_i) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

Where $a_{11} = 1 - 2x - \dfrac{c_1 a_1 (1-m) y}{(a_1 + (1-m)x)^2} - \dfrac{c_2 a_2 (1-m) z}{(a_2 + (1-m)x)^2}$, $a_{12} = -\dfrac{c_1 (1-m) x}{a_1 + (1-m) x}$,

$a_{13} = -\dfrac{c_2 (1-m) x}{a_2 + (1-m) x}$, $a_{21} = \dfrac{e_1 c_1 a_1 (1-m) y}{(a_1 + (1-m)x)^2}$, $a_{22} = \dfrac{e_1 c_1 (1-m) x}{a_1 + (1-m) x} - \beta_1 z - d_1$,

$a_{23} = -\beta_1 y$, $a_{31} = \dfrac{e_2 c_2 a_2 (1-m) z}{(a_2 + (1-m)x)^2}$, $a_{32} = -\beta_2$, $a_{33} = \dfrac{e_2 c_2 (1-m) x}{a_2 + (1-m) x} - \beta_2 y - d_2$.

One can see that the trivial equilibrium point $E_0 (0,0,0)$ is always unstable.

Theorem 2: The axial equilibrium point E_1 is stable provided

$$m > 1 - min\left\{ \frac{d_1 a_1}{e_1 c_1 - d_1}, \frac{d_2 a_2}{e_2 c_2 - d_2} \right\}.$$

Proof. The characteristic equation at $E_1(1,0,0)$ is

$(-1 - \lambda)(\chi_1 - \lambda)(\chi_2 - \lambda) = 0$, where

$\chi_1 = \dfrac{e_1 c_1 (1-m)}{a_1 + (1-m)} - d_1$, $\chi_2 = \dfrac{e_2 c_2 (1-m)}{a_2 + (1-m)} - d_2$.

Thus, all the eigenvalues are negative if $m > 1 - min\left\{ \dfrac{d_1 a_1}{e_1 c_1 - d_1}, \dfrac{d_2 a_2}{e_2 c_2 - d_2} \right\}$.

Hence, the equilibrium point $E_1(1,0,0)$ is locally asymptotically stable.

Theorem 3: The axial equilibrium point $E_2 (x^*, 0, z^*)$ is stable provided

$c_1 > \dfrac{(c_2 d_1 a_1 - a_2 \beta_1)(c_2 e_2 - d_2) + c_2 d_1 d_2 a_2}{c_2 d_2 e_1 a_2}$, $m > 1 - \dfrac{2 c_2 e_2 a_2}{(c_2 e_2 - d_2)(c_2 e_2 - d_2 + 1)}$.

Proof. The characteristic equation at E_2 is $(\Omega_2 - \lambda)(\lambda^2 + \Omega_1 \lambda + \Omega_0) = 0$,

where

$$\Omega_2 = -\frac{1}{c_2 e_2 - d_2}(c_1 - \chi_1) - (1 - x^*)((1 - m)x^* + a_1) - \beta_1 (1-m)x^*,$$

$$\Omega_1 = \left(\frac{2}{(1-m)x^* + a_2} \right) ((c_2 e_2 - d_2)(c_2 e_2 - d_2 + 1)\chi_2 + d_2 a_2),$$

$$\Omega_0 = \frac{(1-m)(c_2 e_2 - d_2)(1 - x^*)x^*}{(1-m)x^* + a_2},$$

$\chi_1 = \dfrac{(c_2 d_1 a_1 - a_2 \beta_1)(c_2 e_2 - d_2) + c_2 d_1 d_2 a_2}{c_2 d_2 e_1 a_2}$, $\chi_2 = m - \left(1 - \dfrac{2 c_2 e_2 a_2}{(c_2 e_2 - d_2)(c_2 e_2 - d_2 + 1)} \right)$.

The eigenvalues are negative if and only if $\Omega_2 < 0, \Omega_1 > 0$ and $\Omega_0 > 0$. Since $\Omega_0 > 0$, one can easily see that $\Omega_2 < 0$ and $\Omega_1 > 0$ if condition (10)

holds. Thus, $E_2(x^*, 0, z^*)$ is locally asymptotically stable.

Theorem 4: The axial equilibrium point $E_3(\bar{x}, \bar{y}, 0)$ is locally asymptotically stable provided

$$c_2 > \frac{a_2(c_1d_2 - \beta_2)(c_1e_1 - d_1) + c_1d_1d_2\alpha_1}{c_1d_1e_2\alpha_1}, \quad m > 1 - \frac{2c_1e_1\alpha_1}{(c_1e_1 - d_1)(c_1e_1 - d_1 + 1)}.$$

Proof. The characteristic equation at E_3 is $(\gamma_2 - \lambda)(\lambda^2 + \gamma_1\lambda + \gamma_0) = 0$,

where $\gamma_2 = -\frac{1}{c_1e_1 - d_1}(c_2 - V_1) - (1 - \bar{x})((1 - m)\bar{x} + \alpha_2) - \beta_2(1 - m)\bar{x}$,

$\gamma_1 = \left(\frac{2}{(1-m)\bar{x} + \alpha_1}\right)((c_1e_1 - d_1)(c_1e_1 - d_1 + 1)V_2 + d_1\alpha_1)$,

$\gamma_0 = \frac{(1-m)(c_1e_1 - d_1)(1 - \bar{x})\bar{x}}{(1-m)\bar{x} + \alpha_1}$,

$V_1 = \frac{a_2(c_1d_2 - \beta_2)(c_1e_1 - d_1) + c_1d_1d_2\alpha_1}{c_1d_1e_2\alpha_1}, V_2 = m - \left(1 - \frac{2c_1e_1\alpha_1}{(c_1e_1 - d_1)(c_1e_1 - d_1 + 1)}\right)$.

The eigenvalues are negative if and only if $\gamma_2 < 0, \gamma_1 > 0$ and $\gamma_0 > 0$. Since $\gamma_0 > 0$, one can easily see that $\gamma_2 < 0$ and $\gamma_1 > 0$ if condition (11) holds. Thus, $E_3(\bar{x}, \bar{y}, 0)$ is locally asymptotically stable.

Theorem 5: Suppose that the positive equilibrium point $E_4(\tilde{x}, \tilde{y}, \tilde{z})$ exists in the interior of the positive octant. Then $E_4(\tilde{x}, \tilde{y}, \tilde{z})$ is not stable.

Proof. The entries of the Jacobian matrix evaluated at the positive equilibrium point E_4 are

$$a_{11} = \left(-1 + \frac{c_1(1-m)^2\tilde{y}}{(\alpha_1 + (1-m)\tilde{x})^2} + \frac{c_2(1-m)^2\tilde{z}}{(\alpha_2 + (1-m)\tilde{x})^2}\right)\tilde{x}, \quad a_{22} = 0, a_{33} = 0,$$

$a_{12} = -\frac{c_1(1-m)\tilde{x}}{\alpha_1 + (1-m)\tilde{x}} < 0, \ a_{13} = -\frac{c_2(1-m)\tilde{x}}{\alpha_2 + (1-m)\tilde{x}} < 0, a_{21} = \frac{e_1c_1\alpha_1(1-m)\tilde{y}}{(\alpha_1 + (1-m)\tilde{x})^2} > 0,$

$a_{23} = -\beta_1\tilde{y} < 0, a_{31} = \frac{e_2c_2\alpha_2(1-m)\tilde{z}}{(\alpha_2 + (1-m)\tilde{x})^2} > 0, a_{32} = -\beta_2\tilde{z} < 0.$

The characteristics equation of the Jacobian matrix around E_4 is:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0,$$

Where $A = -a_{11}, B = -a_{23}a_{32} - a_{12}a_{21} - a_{13}a_{31}$,
$C = a_{11}a_{23}a_{32} - a_{21}a_{32}a_{13} - a_{12}a_{23}a_{31}$. According to Routh-Hurwitz criteria, $E_4(x^*, y^*, z^*)$ is local asymptotic stability if and only if A, C and $AB - C$ are all positive.

Let $a_{11} > 0$, then it can be seen that $A < 0$. If $a_{11} < 0$, then from the signs of Jacobean matrix elements $a_{ij}, (i, j = 1,2,3)$ one can conclude that $C < 0$. Therefore, the Routh-Hurwitz criteria is not satisfied and the positive equilibrium point E_4 is not stable.

Numerical simulation

The dynamical behavior of the system (3) about the equilibrium points has been seen in the previous sections and obtained the analytical results. Now in this section, we have performed some numerical simulations by using MatLab to observe and describe the effect of prey refuge on the dynamics of the system (3) with a set of reasonable parameter values given in table (1). The effect of prey refuge on the dynamics of the system (3) is numerically investigated by keeping all parameters in table (1) constant except the prey refuge m.

Table 1. Parameter values of the system (3).

Parameter	Value	Source
c_1	0.82	(Edwin, 2010)
c_2	0.86	(Edwin, 2010)
e_1	0.085	Estimated
e_2	0.086	Estimated
α_1	0.46	Estimated
α_2	0.48	Estimated
β_1	0.01	(Edwin, 2010)
β_2	0.01	(Edwin, 2010)
d_1	0.02	Estimated
d_2	0.02	Estimated
m	0.2	Estimated

The effect of prey refuge m to the equilibrium point $E_2(x^*, 0, z^*)$

The equilibrium point $E_2(x^*, 0, z^*)$ exists for $m < 0.82209$ and the rest of the parametric values are given in Table 1. The system (3) is locally asymptotically stable around the equilibrium point $E_2(x^*, 0, z^*)$ for $0.164181 < m < 0.82209$ and unstable otherwise. The existence condition (5) and the stability condition (10), (C.F. Theorem 3), are satisfied. The parametric plot of the solution of the system (3) in Figure 1 (top) shows a limit cycle about the equilibrium point $E_2(0.19337996, 0, 0.67073185)$ with parameter values as in Table 1 except $m = 0.08$ in the $x - z$ plane. The time series solution of the system (3) in Figure 1 (bottom) shows that the system oscillates around the equilibrium point E_2 in the given time interval with parameter value as in Table 1. Thus, for the given set of parametric values as in Table 1 except for $m = 0.08$, the equilibrium point E_2 is unstable.

Figure 2 (top and bottom) represent the phase portrait and time series solution of the system (3) at the equilibrium point $E_2\,(0.22598870,0,0.74340068)$ for the set of parametric values as in Table 1, respectively. From Figure 2 (top and bottom), we can observe that the equilibrium point E_2 is locally asymptotically stable for the parametric values as in Table 1.
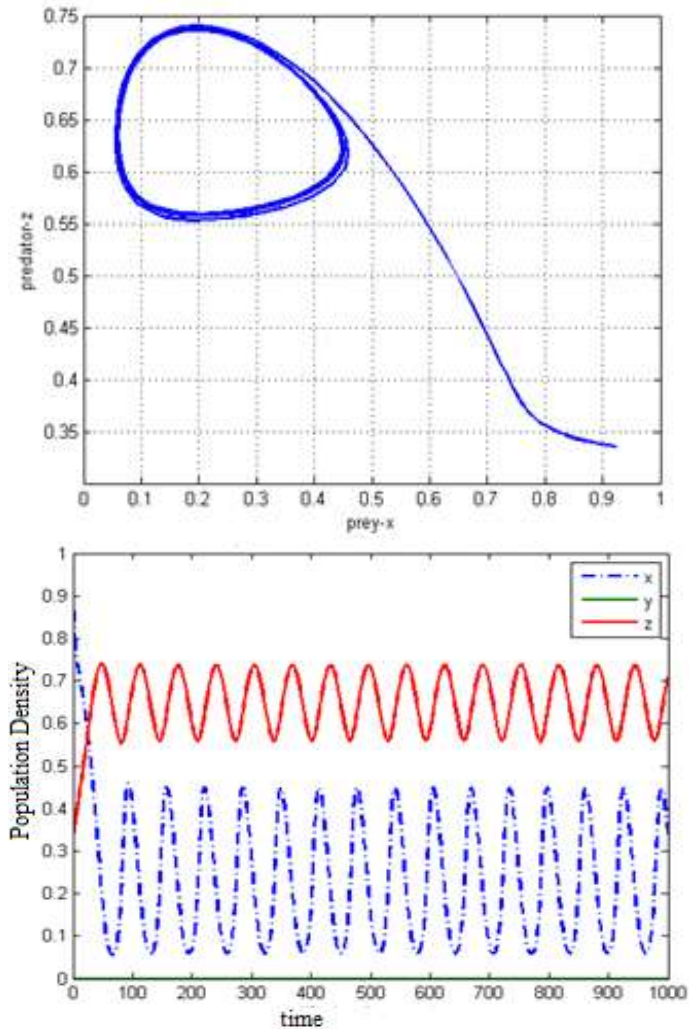

Figure 1. The parametric plot (top) and time series solution (bottom) of the system (3) about the equilibrium point E_2 with parametric values as in Table 1 except $(m = 0.08)$.
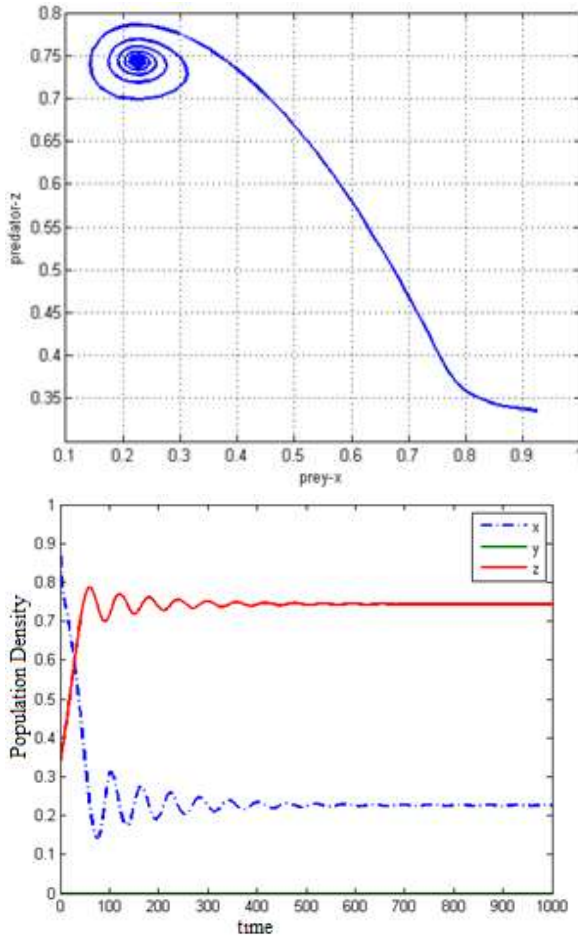
Figure 2. (Top) The phase portrait and (Bottom) time series of the system (3) around the equilibrium point E_2 with parametric values as in table 1.

The effect of prey refuge m to the equilibrium point $E_3(\bar{x}, \bar{y}, 0)$

The equilibrium point $E_3(\bar{x}, \bar{y}, 0)$ exists for $m < 0.814889$ and the rest of the parametric values are given as in Table 1. The system (3) is locally asymptotically stable around the equilibrium point $E_3(\bar{x}, \bar{y}, 0)$ for $0.169779 < m < 0.814889$ and unstable otherwise. The existence condition (6) and the stability condition (11), (C.F. Theorem 4), are satisfied.

The phase portrait in Figure 3 (top) shows the limit cycle around the equilibrium point $E_3 (0.21777725, 0.72398886, 0)$ with the parametric value in Table 1 except $m = 0.15$. The time series solution in Figure 3 (bottom) shows an oscillation with high amplitude around the equilibrium point E_3 with the parametric value in Table 1 except $m = 0.15$. Figure 3 (top and bottom) both signify that E_3 is unstable equilibrium point and the behavior of the system (3) is oscillatory.
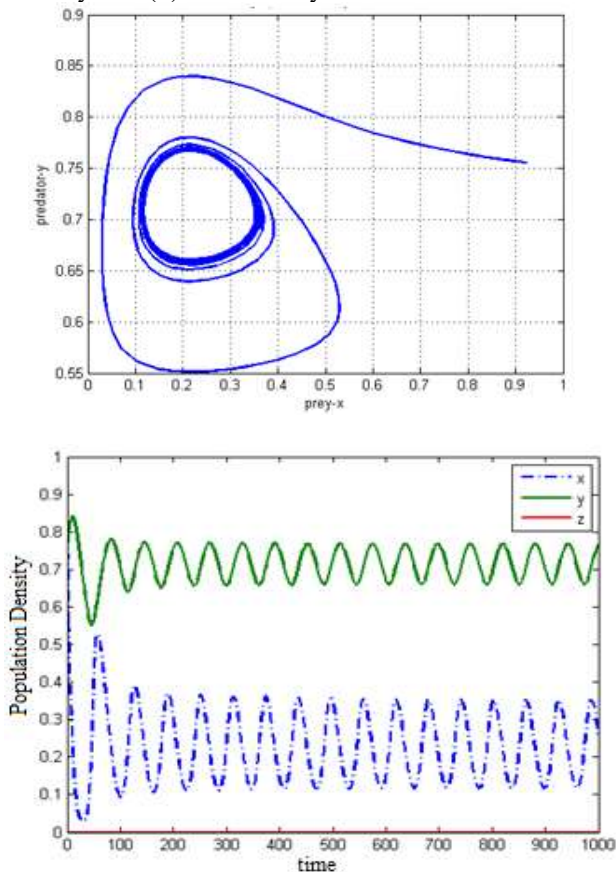

Figure 3. The phase portrait (top) and time series solution (bottom) of the system (3) around the equilibrium point E_3 with parametric values as in Table 1 except $(m = 0.15)$.

The phase portrait in Figure 4 (top) and the time series solution (bottom) show that the equilibrium point $E_3 (0.23138833, 0.75585303, 0)$ is locally

asymptotically stable for the parameter values in Table 1. Thus, in the absence of the predator-z the system (3) converges to the equilibrium point E_3 $(0.23138833, 0.75585303, 0)$ for the parametric value as in Table 1.
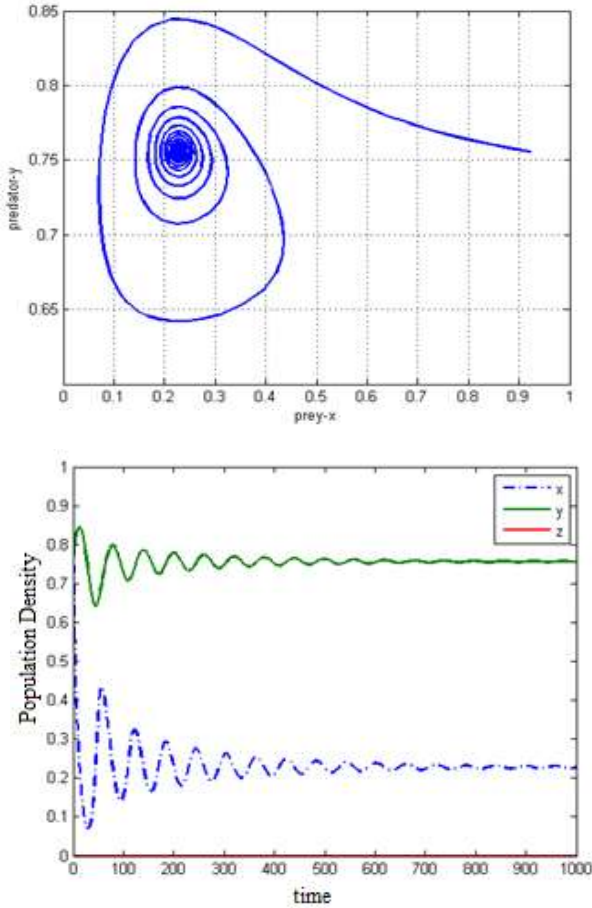

Figure 4. The phase portrait (top) and time series solution (bottom) of the system (3) around the equilibrium point E_3 with parametric values as in Table 1.

The effect of prey refuge to the co-existence equilibrium point $E_4(\tilde{x}, \tilde{y}, \tilde{z})$

The positive equilibrium point E_4 exists for *m < 0.8122699* and the rest of the parametric values are given as in Table 1. The time series solution in Figure 5 shows that for the given set of parametric values as in Table 1 except *(m=0.07)*, the prey and the predators oscillate around the positive equilibrium point $E_4(0.24389215, 0.37338956, 0.30181707)$. It is evident to see that all the three species persist for *m < 0.164181*.
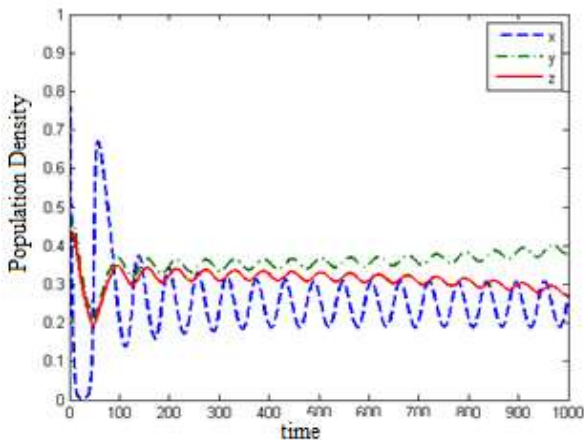

Figure 5. The time series solution of the system (3) around the positive equilibrium point E_4 with parametric values as in table 1 except *(m=0.07)*.

CONCLUSION

In this paper, a mathematical model is proposed and analyzed to study the effect of prey refuge on the dynamics of a three species food web system consisting of two predators competing for a single prey. Both predators show a Holling type II functional response to the prey. All the biologically feasible equilibrium points were calculated. The boundedness and positivity of the solutions of the system (3) are obtained. Conditions for the local stability of each equilibrium points, except the positive equilibrium point, are given with the help of the Routh-Hurwiz criteria.

Numerical simulations are performed to support and verify our analytical findings. In achieving this, we have concentrated on the equilibrium points,

namely E_2, E_3 and E_4. These observations depend on the parameter values in Table 1 except the prey refuge constant m, where m is taken as a control parameter. The refuge constant is found to play a crucial role in stabilizing the dynamics of the system (3). Huang *et al*. (2006) stated that the non-zero equilibrium point could change from unstable state surrounded by a stable limit cycle to globally asymptotically stable state as increasing the prey in the refuge. Our numerical simulation shows also that an increase in prey refuge will lead to an extinction of one of the predators (see Figures 1 to 4).

From the numerical simulations we found that the system (3) is not persistent for $0.164181 < m < 0.8122699$. This is due to the fact that the system (3) has a stable boundary equilibrium points, E_2 and $E3$, under this range. The principle of competitive exclusion holds in such cases. However, it is observed, c.f. Figure 5, that the system (3) exhibits persistence for $m < 0.164181$, as the system (3) shows oscillatory behavior around the boundary equilibrium points E_2 and $E3$. Therefore, one of the predators manages to survive at a very low level leading to persistence. Generally, as we have seen from the numerical simulation, as the number of preys in the refuge increases, the dynamics of a three species system changes the stability from limit cycle to a stable equilibrium point.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest in the publication of this paper.

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