Mixed convection of couple stress fluid in a vertical channel in the presence of heat generation or heat absorption

S. Narasimha Murthy
Department of Mathematics, College of Science,
Bahir Dar University,
Bahir Dar, Ethiopia.
(Email: Simhamurthy44@gmail.com)

ABSTRACT

The fully developed mixed convection of couple stress fluid in a vertical channel in the presence of heat generation or absorption is analyzed. The two boundaries of the channel are considered as isothermal-isothermal, isoflux-isothermal and isothermal-isoflux for the left and right walls and kept either at equal or at different temperatures. The governing momentum and energy equations are coupled and non linear due to the viscous effects. The velocity field and the temperature field is obtained by perturbation series method which employs a perturbation parameter proportional to the Brinkman number. In addition, closed form expressions for reversal flow conditions at both the left-right channel walls are derived. The results are represented graphically for different values of perturbation parameter and couple stress parameter on velocity and temperature distributions. We observe that for purely viscous fluid the flow reversal was at the hot wall whereas for couple stress fluid there is a flow reversal both at left and right walls. The effect of the perturbation parameter on the flow for couple stress fluid is dominating compare to viscous fluid both on velocity and temperature. The profiles of temperature are significant for couple stress fluid for different values of perturbation parameter whereas the profiles were not sensible for same values for viscous fluid.

Key words: Mixed convection, viscous dissipation, buoyancy force, perturbation series

INTRODUCTION

The study of non-Newtonian fluids has received much attention due to their many practical applications in medical sciences, engineering and technology, such as liquid crystals, fluid film lubrication etc. In the category of non-Newtonian fluids, couple stress fluid has distinct features such as polar effects in addition to possessing large viscosity. The consideration of couple stress in addition to Cauchy stress has led to the recent development of several theories of fluid micro-continua.

Studies on natural convection in a vertical channel with non-Newtonian fluid are relatively sparse compared to the problem with Newtonian fluid. The effect of couple stresses on peristaltic transport has been carried out by (Srivstava, 1986) and (Shehaway and Mecheimer, 1994). However, the natural convection flow and heat transfer in a vertical channel with couple stress fluid has not been studied so far even though the couple stress fluid is one of the simple and interesting models of fluid belonging to the class of non-Newtonian fluids. The theory of (Stokes, 1966) is simplest generalization of the classical theory of fluids, which allows for polar effects such as the presence of a non-Newtonian symmetric stress tensor, couple stresses and body couples. Couple stresses may appear in the flow of liquids that contain additives.

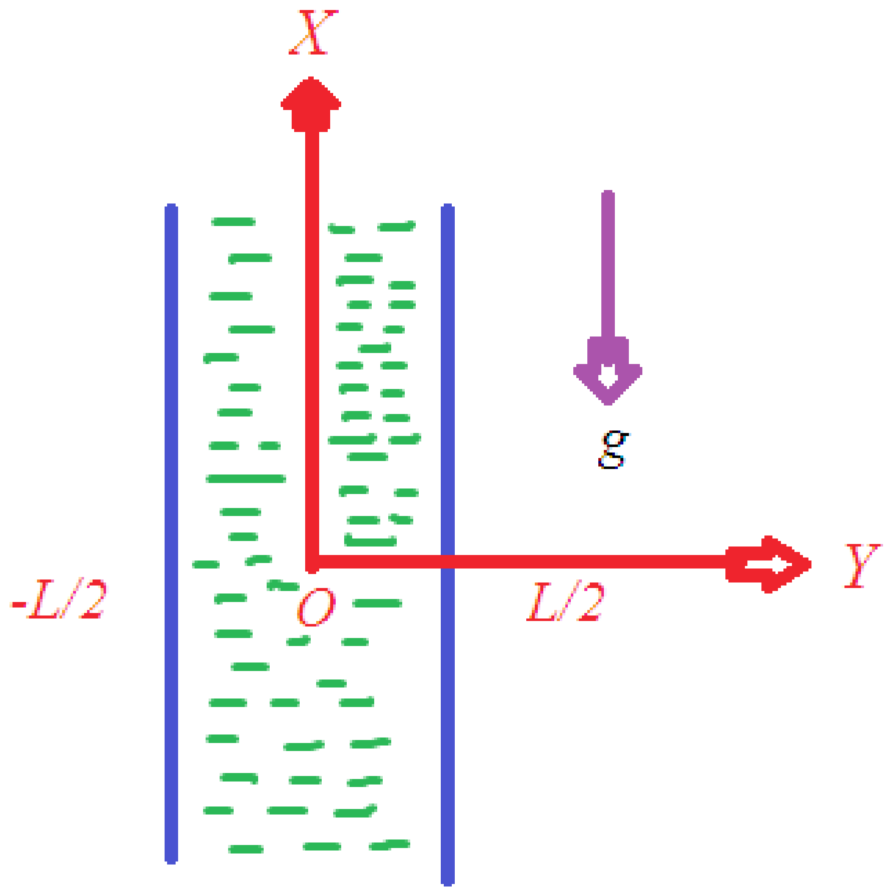
The couple stress fluid model has wide applications in bio-fluids, colloidal fluids and in engineering for pumping fluids such as synthetic lubricants. Studies on natural convection in a vertical channel with non-Newtonian fluid are relatively sparse compared to the problem with Newtonian fluid. Malashetty and Umavathi (1999) analyzed the effects of couple stresses on free convective flow in a vertical channel and Umavathi (2000) analyzed the free convection flow of electrically conducting couple stress fluid in a vertical channel. Umavathi *et al*. (2004) also carried out the convective flow of two immiscible viscous and couple stress fluids through a vertical channel in the absence of viscous dissipation.

The theoretical investigations on fully developed mixed convection in vertical or inclined ducts are often devoted to a description of the changes on the velocity profiles induced by buoyancy as well as to the determination of the conditions for the onset of flow reversal. Indeed, the flow reversal phenomenon arises when buoyancy forces are so strong that there exists a domain within the duct where the local fluid velocity has a direction opposite to the mean fluid flow. Theoretical investigations have been devoted to the analysis of the interplay between the effect of viscous dissipation and the effect of buoyancy forces. Barletta (1986; 1998; 1999, 1999a, 2002) investigated the heat transfer in vertical channel flow under various flow and boundary conditions like presence of prescribed wall heat flux (Aung and Worku, 1986) , mixed convection with viscous dissipation (Aung and Worku, 1988; Barletta, 1988, Srivastava ,1988), when the boundaries are isothermal-isoflux (Barletta,1999), again with the inclusion of viscous dissipation and fully developed mixed convection with flow reversal in rectangular duct with uniform wall heat flux (Barletta , 2002).
Cheng *et al*. (1990) reported flow reversal and heat transfer of fully developed mixed convection in vertical channels. (Lavine, 1988) studied the fully developed opposing mixed convection between inclined parallel plates. Hamadah and Writz (1991) discussed the laminar fully developed mixed convection in a vertical channel with opposing buoyancy force. In most of the industrial applications we see that the working fluid is non-Newtonian in nature in various applications and much work has not been found in the literature on mixed convection flows. Hence, the present objective is to study the problem of mixed convection couple stress fluid in a vertical channel in the presence of heat generation or heat absorption.

MATHEMATICAL FORMULATION

Consider steady, laminar, fully developed flow in a parallel plate vertical channel.

Cartesian coordinate system is chosen with the transverse coordinate Y and the coordinate in the



direction parallel to the walls is X. The origin of the axes is such that the channel walls are at position Y=-L/2 and Y=L/2. The thermal conductivity, the dynamic viscosity and the thermal expansion coefficient are considered as constant.

The condition of fully developed flow implies that $(\frac{\partial U}{\partial X}) = 0.$

Then, since the velocity field U is solenoid , one obtains $(\frac{\partial V}{\partial Y}) = 0$.

As a consequence, the velocity V is constant in any channel section and is equal to zero at the channel walls, so that V must be vanishing at any position.

The oberbeck-Boussinesq approximation is assumed to hold good for the evaluation of the gravitational body force, which is typical in this type of buoyancy driven flows in which the density will depend on temperature according to the equation of state

$$\rho = \rho_0\left(1-\beta\left(T-T_0\right)\right). \tag{1}$$

The momentum balance equation for couple stress fluid is Stokes (1966).

$$g\beta(T-T_0)-\frac{1}{\rho_0}\frac{\partial P}{\partial X}+\frac{\mu}{\rho_0}\frac{d^2U}{dY^2}-\frac{\eta}{\rho_0}\frac{d^4U}{dY^4}=0 \tag{2}$$

The Y-momentum balance equation can be expressed as

$$\frac{\partial P}{\partial Y}=0 \tag{3}$$

where $P = p + \rho_0 g X$ is the difference between the pressure and the hydrostatic pressure. The temperature is T_1, at the left wall $Y = -L/2$ and the temperature is T_2, at the right wall $Y = L/2$ with $T_2 \geq T_1$. These conditions are compatible with equation (2) only when $\partial P/\partial X$ is independent of X. Hence, there exists a constant A such that,

$$\frac{\partial P}{\partial X}=A \tag{4}$$

Solving the equations (2) and (4), we obtain

$$\frac{\partial T}{\partial X}=0, \tag{5}$$

which implies that the temperature also depends on Y. By considering the effects of viscous dissipation, along with heat generation or absorption, the energy balance equation relevant to the present situation is

$$\alpha\frac{d^2T}{dY^2}+\frac{\mu}{\rho_0 C_p}\left(\frac{dU}{dY}\right)^2 \pm \frac{Q(T-T_0)}{\rho_0 C_p}=0 \tag{6}$$

Simplifying the equations (2) and (6) allow one to obtain a non-linear differential equation for U, in the form

$$\frac{d^6U}{dY^6}=\left(\frac{\mu}{\eta}\mp\frac{Q}{K}\right)\frac{d^4U}{dY^4}\pm\frac{\mu Q}{K\eta}\frac{d^2U}{dY^2}-\frac{\mu\rho_0\beta g}{K\eta}\left(\frac{dU}{dY}\right)^2\mp\frac{QA}{K\mu} \tag{7}$$

The boundary conditions on U are both the no slip conditions

$$U=\frac{d^2U}{dY^2}=0 \text{ at } Y=\pm\frac{L}{2} \tag{8}$$

$$\frac{d^4U}{dY^4} = -\frac{A}{\eta} + \frac{\beta g \rho_0 \left(T_1 - T_0\right)}{\eta} \text{ at } Y = -\frac{L}{2}$$

$$\frac{d^4U}{dY^4} = -\frac{A}{\eta} + \frac{\beta g \rho_0 \left(T_2 - T_0\right)}{\eta} \qquad \text{at } Y = \frac{L}{2} \tag{9}$$

The following quantities are employed for writing equations (6) to (9) in the dimensionless form:

$$u = \frac{U}{U_0};\quad \theta = \frac{T - T_0}{\Delta T};\quad y = \frac{Y}{D};\quad Gr = \frac{g\beta \Delta T D^3}{\nu^2};\ \lambda = \frac{Gr}{Re};$$

$$R_T = \frac{T_2 - T_1}{\Delta T}; Re = \frac{U_0 D}{\nu}; Pr = \frac{\nu}{\alpha}; Br = \frac{\mu U_0^2}{K\Delta T}; \phi = \frac{QD^2}{K} \tag{10}$$

k= $\frac{\eta}{\mu D^2}$; $a^2 = \frac{1}{k}$, where D=2L, is the hydraulic parameter.

The reference velocity U_0 and the reference temperature T_0 are,

$$U_0 = -\frac{AD^2}{48\mu};\qquad T_0 = \frac{T_1 + T_2}{2} \tag{11}$$

The temperature difference ΔT is given by

$\Delta T = T_2 - T_1$ if $T_1 < T_2$ or by

$$\Delta T = \frac{\nu^2}{Gr D^2} \text{ if } T_1 = T_2 \tag{12}$$

The dimensionless form of equations (6) to (9) are as follows,

$$\frac{d^2\theta}{dy^2} = -Br\left(\frac{du}{dy}\right)^2 \mp \phi\theta \tag{13}$$

$$\frac{d^6u}{dy^6} = \left(a^2 \mp \phi\right)\frac{d^4u}{dy^4} \pm a^2\phi\frac{d^2u}{dy^2} - \lambda Bra^2\left(\frac{du}{dy}\right)^2 \pm 48\phi a^2 \tag{14}$$

$$u = \frac{d^2u}{dy^2} = 0 \text{ at } y = \pm\frac{1}{4} \tag{15}$$

$$\frac{d^4u}{dy^4} = 48a^2 - \frac{R_T\lambda a^2}{2} \text{ at } y=-1/4,$$

$$\frac{d^4u}{dy^4} = 48a^2 + \frac{R_T\lambda a^2}{2} \text{ at } y= 1/4. \tag{16}$$

Temperature field can also be obtained while substituting equations (10) and (11) in momentum equation (2) one obtains,

$$\theta = -\frac{1}{\lambda}\left(8 + \frac{d^2u}{dy^2} - \frac{1}{a^2}\frac{d^4u}{dy^4} \right) \quad . \tag{17}$$

Equation (14) is highly nonlinear through viscous dissipation term. If the viscous dissipation is negligible so that $B = 0$, the dimensionless temperature θ and dimensionless velocity u are uncoupled. In this case, the solution of equation (14) by applying the boundary conditions (15) and (16) becomes

$$u = \frac{3}{2} - \frac{2R_T\lambda}{\phi}y - 24\ y^2 - \frac{8}{a^2}\left(1 - \frac{Coshay}{Cosha/4}\right) + \frac{\lambda\ R_T}{2(a^2+\phi)}\left(\frac{a^2 Sin\sqrt{\phi}y}{\phi\, Sin\sqrt{\phi}/4} + \frac{Sinhay}{Sinha/4}\right) \tag{18}$$

for the case of heat generation and

$$u = \frac{3}{2} + \frac{2R_T\lambda}{\phi}y - 24\ y^2 - \frac{8}{a^2}\left(1 - \frac{Coshay}{Cosha/4}\right) + \frac{\lambda\ R_T}{2(a^2-\phi)}\left(\frac{a^2 Sinh\sqrt{\phi}y}{\phi\, Sinh\sqrt{\phi}/4} + \frac{Sinhay}{Sinha/4}\right) \tag{19}$$

for the case of heat absorption.

The corresponding temperature field for these two cases can be obtained by substituting the expressions (18) and (19) in equation (17).

$$\theta = \frac{R_T}{2}\frac{Sin\sqrt{\phi}y}{Sin\sqrt{\phi}/4} \tag{20}$$

$$\theta = \frac{R_T}{2}\frac{Sinh\sqrt{\phi}y}{Sinh\sqrt{\phi}/4} \quad . \tag{21}$$

In the absence of couple stress parameter i.e. $a = 0$ the velocity field becomes,

$$u = \frac{3}{2} - 24\ y^2 - \frac{2\lambda\ R_T}{\phi}\left(y - \frac{Sin\sqrt{\phi}y}{4Sin\sqrt{\phi}/4}\right) \tag{22}$$

for the case of heat generation and

$$u = \frac{3}{2} - 24\ y^2 + \frac{2\lambda\ R_T}{\phi}\left(y - \frac{Sinh\sqrt{\phi}y}{4Sinh\sqrt{\phi}/4}\right) \tag{23}$$

for the case of heat absorption, and the temperature field is similar to the above expressions (20) and (21).

In the absence of couple stress parameter a and the heat generation or absorption coefficient ϕ, the velocity and temperature fields reduces to,

$$u = \left(\frac{R_T\ \lambda}{3}y + 24\right)\left(\frac{1}{6} - y^2\right) \tag{24}$$

$$\theta = 2R_T\ y \tag{25}$$

which corresponds to the velocity and temperature fields determined by (Aung and Worku, 1986).

In the case of asymmetric heating, when buoyancy forces are dominated i.e.,

when $\lambda \to \pm\infty$, equations (18) and (19) for the cases of heat generation or absorption gives

$$\frac{u}{\lambda} = \frac{2}{\phi}y + \frac{1}{2(a^2+\phi)}\left(\frac{Sinhay}{Sinha/4} - \frac{a^2}{\phi}\frac{Sin\sqrt{\phi}\ y}{4Sin\sqrt{\phi}/4}\right) \tag{26}$$

$$\frac{u}{\lambda} = \frac{2}{\phi} y + \frac{1}{2(a^2 - \phi)} \left(\frac{Sinhay}{Sinha/4} - \frac{a^2}{\phi} \frac{Sinh\sqrt{\phi}\, y}{4Sinh\sqrt{\phi}/4} \right) \tag{27}$$

In the absence of couple stress parameter the above equations become,

$$\frac{u}{\lambda} = \frac{2}{\phi} \left(y - \frac{Sin\sqrt{\phi} y}{4Sin\sqrt{\phi}/4} \right) \tag{28}$$

$$\frac{u}{\lambda} = \frac{2}{\phi} \left(y - \frac{Sinh\sqrt{\phi} y}{4Sinh\sqrt{\phi}/4} \right) \tag{29}$$

In the absence of source and sink, the above equations for clear viscous fluid reduces to

$$\frac{u}{\lambda} = \frac{y}{3} \left(\frac{1}{16} - y^2 \right) \tag{30}$$

which is Batchelor's velocity field for free convection (Batchelor, 1954).

When buoyancy forces are negligible and viscous dissipation is relevant, i.e. $\lambda = 0$, so that a purely forced convection occurs. For this condition, the solutions of velocity and temperature are obtained from equations (13) and (14) as,

$$u = \frac{3}{2} - 24\, y^2 - \frac{48}{a^2} \left(1 - \frac{Coshay}{Cosha/4} \right) \tag{31}$$

$$\theta = C_1 Cos\sqrt{\phi} y + C_2 Sin\sqrt{\phi} y + l_1 Cosh2ay + l_2 Coshay + l_3 Sinhay + l_4 y^2 + l_5 \tag{32}$$

where,

$$l_1 = -\frac{1152B}{a^2(4a^2 + \phi)Cosh^2 P_1};\ l_2 = -\frac{9216B}{(a^2 + \phi)^2 CoshP_1};\ l_3 = \frac{4608B}{a(a^2 + \phi)CoshP_1}$$

$$-\frac{2304B}{\phi};\quad l_5 = \frac{4608B}{\phi^2} + \frac{2304B}{2a^2\phi Cosh^2 P_1};\qquad C_2 = \frac{K}{2Si}$$

$$-\frac{1}{CosP_2}\left(l_1 CoshP_3 + l_2 CoshP_1 + \frac{l_3}{4} SinhP_1 + \frac{l_4}{16} \right.$$

for the case of heat generation and

$$\theta = C_1 Cosh\sqrt{\phi} y + C_2 Sinh\sqrt{\phi} y + l_1 Cosh2ay + l_2 Coshay + l_3 Sinhay + l_4 y^2 + l_5 \tag{33}$$

where,

$$l_1 = -\frac{1152B}{a^2(4a^2 - \phi)Cosh^2 P_1};\ l_2 = -\frac{9216B}{(a^2 - \phi)^2 CoshP_1};\ l_3 = \frac{4608B}{a(a^2 - \phi)CoshP_1}$$

$$l_4 = \frac{2304B}{\phi};\ l_5 = \frac{4608\,B}{\phi^2} - \frac{1152B}{a^2\phi\, Cosh^2 P_1};\ C_2 =$$

$$C_1 = -\frac{1}{CoshP_2}\left(l_1 CoshP_3 + l_2 CoshP_1 + \frac{l_3}{4} SinhP_1 + \frac{l_4}{16} + l_5 \right)$$

for the case of heat absorption.

Solutions of equations (13) and (14) for clear viscous fluid in the absence of buoyancy force, viscous dissipation and source and sink leads to the Hagen-Poiseuille velocity profile

$$u = 24\left(\frac{1}{16} - y^2\right) \tag{34}$$

and temperature profile is given by

$$\theta = -192\ B\ \ y^4 + 2R_T y + \frac{3B}{4} \tag{35}$$

which agree with the results obtained by (Cheng and Wu, 1976) in the case of forced convection with asymmetric heating.

Solutions

The solution of nonlinear differential equation (13) can be simplified by employing a perturbation technique

$$\varepsilon = B \quad \lambda = R \quad P \ \frac{\beta g D}{C_p} \tag{36}$$

The temperature field is obtained in terms of velocity from the equation(17). The solution of equation (14) using (36) is.

$$u(y) = u_0(y) + \varepsilon\ u_1(y) + \varepsilon^2\ u_2\ (y) + \cdots\cdots\cdots = \sum_{n=0}^{\infty} \varepsilon^n\ u_n(y) \tag{37}$$

The second and higher order terms of ε give a correction to $u_{0,}\ \theta_0$ accounting for the viscous dissipation effect.

Isothermal-isothermal $(T_1 - T_2)$ walls

In this case both the walls are maintained at different temperatures.

Substituting equation (37) in equation (14) and equating the coefficients of like powers of ε to zero; one obtains the boundary value problem for $n = 0$ and $n = 1$ as,

$$\frac{d^4 u_0}{dy^4} = \left(a^2 \mp \phi\right)\frac{d^4\ u_0}{dy^4} \pm a^2\phi\frac{d^2\ u_0}{dy^2} \pm 48\ \ \phi a^2 \tag{38}$$

$$\frac{d^4 u_1}{dy^4} = \left(a^2 \mp \phi\right)\frac{d^4 u_1}{dy^4} \pm a^2\phi\frac{d^2 u_1}{dy^2} - a^2\left(\frac{du_0}{dy}\right)^2 \tag{39}$$

for the case of heat generation and absorption respectively. The boundary conditions of u_0 and u_1 by using the equation (37) simplify to

$$u_0 = \frac{d^2 u_0}{dy^2} = 0 \text{ at } y = \pm\frac{1}{4}$$

$$\frac{d^4 u_0}{dy^4} = 48\, a^2 - \frac{R_T \lambda a^2}{2} \text{ at } y = -\frac{1}{4}$$

$$\frac{d^4 u_0}{dy^4} = 48\, a^2 + \frac{R_T \lambda a^2}{2} \text{ at } y = \frac{1}{4} \tag{40}$$

$$u_1 = \frac{d^2 u_1}{dy^2} = \frac{d^4 u_1}{dy^4} = 0 \text{ at } y = \pm\frac{1}{4} \tag{41}$$

Equation (38) is ordinary linear differential equation and hence the exact solution can be solved easily. The solution of equation (38) obviously coincides with the solution of equation (14) in the case of $B = 0$. Evaluation of the exact solution for $n = 2$ becomes tedious and hence neglecting the terms for $n = 2$, zeroth and first order solutions are

$$u_0 = C_1 + C_2 y - 24\, y^2 + C_3 Coshay + C_4 Sinhay + C_5 Cos\sqrt{\phi}y + C_6 Sin\sqrt{\phi}y \tag{42}$$

for the case of heat generation and

$$u_0 = C_1 + C_2 y - 24\, y^2 + C_3 Coshay + C_4 Sinhay + C_5 Cosh\sqrt{\phi}y + C_6 Sinh\sqrt{\phi}y \tag{43}$$

for the case of heat absorption. The solution of equation (39) by using the equation (41) for $n = 1$ is

$$\begin{aligned} u_1 = {} & C_7 + C_8 y + C_9 Coshay + C_{10} Sinhay + C_{11} Cos\sqrt{\phi}y + C_{12} Sin\sqrt{\phi}y + \\ & l_1 Cosh2ay + l_2 Cos2\sqrt{\phi}y + l_3 Sinh2ay + l_4 Sin2\sqrt{\phi}y + l_5 Coshay \\ & Cos\sqrt{\phi}y + l_6 SinhaySin\sqrt{\phi}y + l_7 CoshaySin\sqrt{\phi}y + l_8 Sinhay \\ & Cos\sqrt{\phi}y + l_9 yCoshay + l_{10} ySinhay + l_{11} y^2 Coshay + l_{12} y^2 Sinhay \\ & + l_{13} y^2 Cos\sqrt{\phi}y + l_{14} y^2 Sin\sqrt{\phi}y + l_{15} yCos\sqrt{\phi}y + l_{16} ySin\sqrt{\phi}y \\ & + l_{17} y^4 + l_{18} y^3 + l_{19} y^2 \end{aligned} \tag{44}$$

for the case of heat generation and

$$
\begin{aligned}
u_1 = {} & C_7 + C_8 y + C_9 Coshay + C_{10}\ Sinhay + C_{11}\ Cosh\sqrt{\phi}y + C_{12}\ Sinh\sqrt{\phi}y + \\
& l_1 Cosh2ay \ + l_2 Cosh2\sqrt{\phi}y + l_3 Sinh2ay \ + l_4 Sinh2\sqrt{\phi}y + l_5 Coshay \\
& Cosh\sqrt{\phi}y + l_6 SinhaySinh\sqrt{\phi}y + l_7 CoshaySinh\sqrt{\phi}y + l_8 Sinhay \\
& Cosh\sqrt{\phi}y + l_9 yCoshay + l_{10}\ ySinhay + l_{11}\ y^2 Coshay + l_{12}\ y^2 Sinhay \\
& + l_{13}\ y^2 Cosh\sqrt{\phi}y + l_{14}\ y^2 Sinh\sqrt{\phi}y + l_{15}\ yCosh\sqrt{\phi}y + l_{16}\ ySinh\sqrt{\phi}y \\
& + l_{17}\ y^4 + l_{18}\ y^3 + l_{19}\ y^2
\end{aligned}
\tag{45}
$$

for the case of heat absorption. The dimensionless temperature field is obtained from equation (17) considering velocity fields defined as in equations (42) to (46) is

$$
\theta = \frac{1}{\lambda}\left(
\begin{aligned}
& P_5\phi(C_5 Cos\sqrt{\phi}y + C_6 Sin\sqrt{\phi}y) + \\
\varepsilon & \left[
\begin{aligned}
& P_5\phi(C_{11}\ Cos\sqrt{\phi}y + C_{12}\ Sin\sqrt{\phi}y) + 12\ a^2(l_1 Cosh2ay \ + l_3 Sinh2ay\ ) \\
& + P_6(l_4 Sin2\sqrt{\phi}y + l_2 Cos2\sqrt{\phi}y) + P_7 CoshayCos\sqrt{\phi}y \\
& + P_8 SinhaySin\sqrt{\phi}y + P_9 CoshayCos\sqrt{\phi}y + P_{10}\ SinhayCos\sqrt{\phi}y \\
& + P_{11}\ yCoshay + P_{12}\ ySinhay + P_{13}\ Coshay + P_{14}\ Sinhay + P_{15}\ y^2 Cos\sqrt{\phi}y \\
& + P_{16}\ y^2 Sin\sqrt{\phi}y + P_{17}\ yCos\sqrt{\phi}y + P_{18}\ ySin\sqrt{\phi}y + P_{19}\ Cos\sqrt{\phi}y + P_{20} \\
& Sin\sqrt{\phi}y + 24\ \frac{l_{17}}{a^2} + 12\ l_{17}\ y^2 + 6 l_{18}\ y + 2 l_{19}
\end{aligned}
\right]
\end{aligned}
\right)
\tag{46}
$$

for the case of heat generation and

$$
\theta = \frac{1}{\lambda}\left(
\begin{aligned}
& P_5\phi(C_5 Cosh\sqrt{\phi}y) + C_6 Sinh\sqrt{\phi}y) + \\
\varepsilon & \left[
\begin{aligned}
& P_5\phi(C_{11}\ Cosh\sqrt{\phi}y + C_{12}\ Sinh\sqrt{\phi}y) + 12\ a^2(l_1 Cosh2ay \ + l_3 Sinh2ay\ ) \\
& + P_6(l_4 Sinh2\sqrt{\phi}y + l_2 Cosh2\sqrt{\phi}y) + P_7 CoshayCosh\sqrt{\phi}y \\
& + P_8 SinhaySinh\sqrt{\phi}y + P_9 CoshayCosh\sqrt{\phi}y + P_{10}\ SinhayCosh\sqrt{\phi}y \\
& + P_{11}\ yCoshay + P_{12}\ ySinhay + P_{13}\ Coshay + P_{14}\ Sinhay + P_{15}\ y^2 Cosh\sqrt{\phi}y \\
& + P_{16}\ y^2 Sinh\sqrt{\phi}y + P_{17}\ yCosh\sqrt{\phi}y + P_{18}\ ySinh\sqrt{\phi}y + P_{19}\ Cosh\sqrt{\phi}y + P_{20} \\
& Sinh\sqrt{\phi}y + 24\ \frac{l_{17}}{a^2} + 12\ l_{17}\ y^2 + 6 l_{18}\ y + 2 l_{19}
\end{aligned}
\right]
\end{aligned}
\right)
\tag{47}
$$

or the case of heat absorption.

Isoflux-isothermal $(q_1 - T_2)$ walls

In this case the left wall is maintained with a constant heat flux and the right wall is at a uniform temperature. The thermal boundary condition for the channel walls can be written in the dimensional form as

$$q_1 = -K\frac{dT}{dY} \text{ at } Y = -\frac{L}{2}$$

$$T = T_2 \text{ at } Y = \frac{L}{2} \tag{48}$$

It is convenient to non-dimensional the thermal boundary conditions by employing the equation (10) with $\Delta T = q_1 D / K$ to give

$$\frac{d\theta}{dy} = -1 \qquad \text{at } y = -\frac{1}{4}$$

$$\theta = R_q \qquad \text{at } y = \frac{1}{4} \tag{49}$$

where $R_q = (T_2 - T_0)/\Delta T$ is the thermal ratio parameter. Differentiating the equation (2) with respect to Y, we get the boundary condition of velocity field in dimensional form as,

$$\frac{d^3U}{dY^3} - \frac{\eta}{\mu}\frac{d^5U}{dY^5} + \frac{\beta g}{\nu}\frac{dT}{dY} = 0 \tag{50}$$

Equation (50) is non-dimensional zed by applying the equation. (10) to give

$$\frac{d^3u}{dy^3} - \frac{1}{a^2}\frac{d^5u}{dy^5} + \lambda\frac{d\theta}{dy} = 0 \tag{51}$$

Evaluating the equation (51) at the left wall $(y = -1/4)$ yields

$$\frac{d^3u}{dy^3} - \frac{1}{a^2}\frac{d^5u}{dy^5} = \lambda \quad \text{at} \quad y = -\frac{1}{4}. \tag{52}$$

The other boundary condition at the right wall can be shown to be the same as that given for the isothermal-isothermal case with R_T replaced by R_q such that

$$\frac{d^4u}{dy^4} = 8a^2 + \frac{R_T \lambda a^2}{2} \text{ at } y = \frac{1}{4} \tag{53}$$

The integrating constants in equations (42) to (47)are obtained using boundary conditions (52), (53) along with (40) and (41).

Isothermal-isoflux $(T_1 - q_2)$ walls

Here, the left wall is kept at a uniform temperature while the right wall is maintained at a uniform heat flux. The thermal boundary condition for the channel walls can be written in the dimensional form as,

$$q_2 = -K\frac{dT}{dY} \text{ at } Y = \frac{Y}{2}$$

$$T = T_1 \text{ at } Y = -\frac{Y}{2} \qquad (54)$$

The dimensionless form of the equation (54) can be obtained by using the equation (10) with $\Delta T = q_2 D / K$ to give

$$\frac{d\theta}{dy} = -1 \text{ at } y = \frac{1}{4}$$

$$\theta = R_q \qquad \text{at } y = -\frac{1}{4} \qquad (55)$$

where $R_q = (T_1 - T_0)/\Delta T$ is the thermal ratio parameter for the isothermal-isoflux case. Similar to the procedure done in the previous case on isoflux-isothermal walls, the dimensionless form of the boundary conditions obtained from equation (2) and applying equation (55) can be written as

$$\frac{d^3u}{dy^3} - \frac{1}{a^2}\frac{d^5u}{dy^5} = \lambda \quad \text{at} \quad Y = \frac{1}{4} \qquad (56)$$

The other boundary condition at the right wall can be shown to be the same as that given for the isothermal-isothermal case with R_T replaced by R_q such that

$$\frac{d^4u}{dy^4} = 8\, a^2 + \frac{R_T \lambda a^2}{2} \quad \text{at} \quad y = \frac{1}{4} \qquad (57)$$

Using these boundary conditions, the integrating constants are obtained from equations (43) to (46) and (52) up to $O\left(\varepsilon^1\right)$.

RESULTS AND DISCUSSION

The theory of couple stress fluid due to Stokes is used to formulate a set of boundary layer equations for a flow of incompressible, couple stress fluid in a vertical channel for mixed convection. Analytical solutions are obtained using perturbation technique valid for small value of ε. Figures 1 and 2 show the effect of ε for $\lambda = \pm 500$. When the flow is upward, ε and λ are positive and on the other hand, the flow is downward when the ε and λ are negative. It is very interesting to note that there is a flow reversal at both the boundaries for λ positive, which is different from the result for viscous fluid where there is a flow reversal for positive λ only at cool wall. The phenomena of flow reversal is typical in buoyancy driven flows We observe that for purely viscous fluid for $\lambda = -500$ the flow reversal was at the hot wall whereas for couple stress fluid there is a flow reversal both at left and right walls. The effect of ε on the flow for couple stress fluid is dominating compare to viscous fluid both on velocity and temperature. The profile of temperature are significant for couple stress fluid for different ε where as the profiles were not sensible for different ε.

Figures 3 and 4 show the effect of heat generation coefficient ϕ on velocity and temperature. For both positive and negative values of λ, the flow suppresses as ϕ increases, which is opposite, result for viscous fluid. Figures 5 and 6 show the effect of couple stress parameter on the flow. For positive λ as $'a'$ increases velocity increases and it is notified that the maximum velocity occurs at both left and right walls for small $'a'$ and the maximum velocity is in the middle of the channel for large $'a'$ and also flow reversal occurs as $'a'$ increases. For negative λ velocity decreases as $'a'$ increases and here also the maximum velocity is at both the walls for small $'a'$ and moves to the middle of channels as $'a'$ increases. The temperature decreases as $'a'$ for both positive and negative λ. It is also noticed that for small values of ε the effect of $'a'$ is insignificant for upward and downward flows.

Figures 7 illustrate the influence of couple stress parameter $'a'$ with isoflux-isothermal and isothermal-isoflux wall conditions for $\lambda = \pm 500, \varepsilon = \pm 0.1$ and $R_q = R_q = 1$. It is seen that as $'a'$ increases the flow is assisted for positive λ and suppresses for negative λ at the reversal side. Also, as $'a'$ increases temperature decreases for both $\pm\varepsilon$. The effect of $'a'$ on velocity and temperature for isoflux-isothermal case is similar to that for isothermal-isoflux wall conditions as seen in Figure 7.

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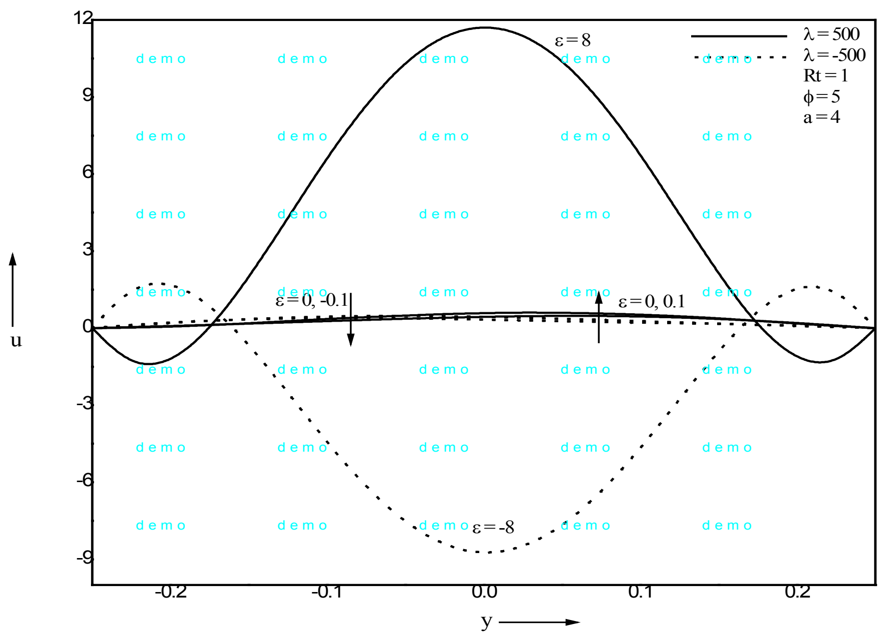

Fig. 1 Plots of u versus y in the case of asymmetric heating for different values of λ and ε

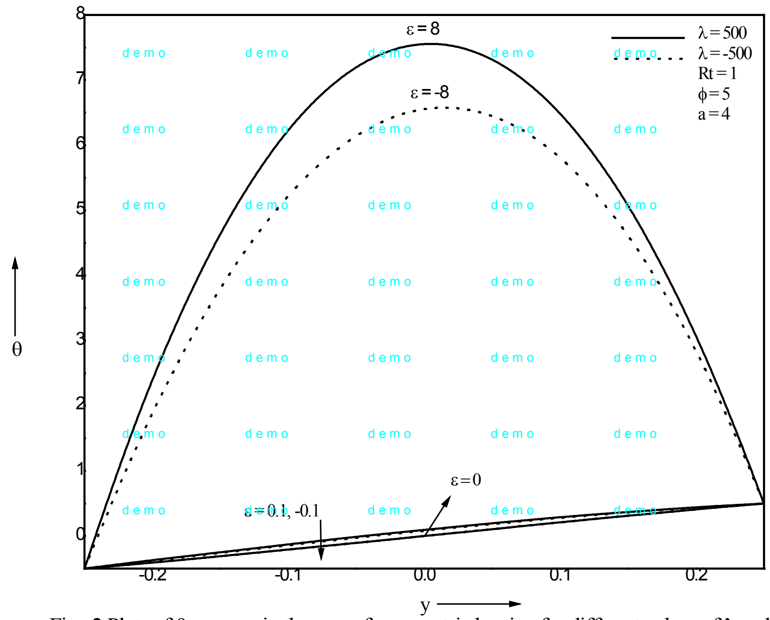

Fig.. 2 Plots of θ versus y in the case of asymmetric heating for different values of λ and ε

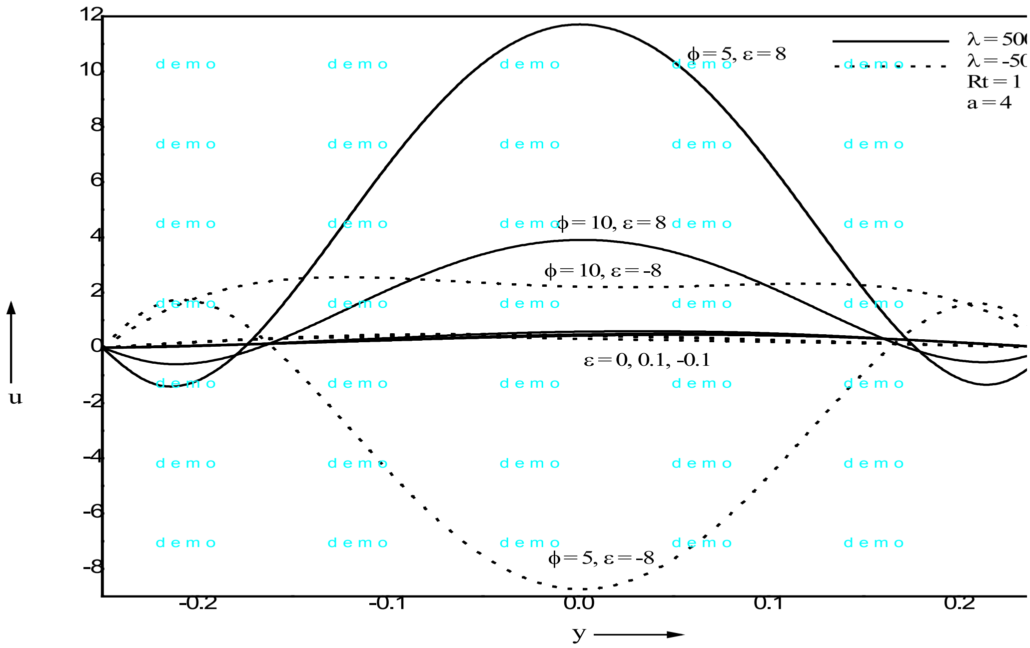

Fig. 3 Plots of u versus y in the case of asymmetric heating for different values of heat generation coefficient ϕ and ε

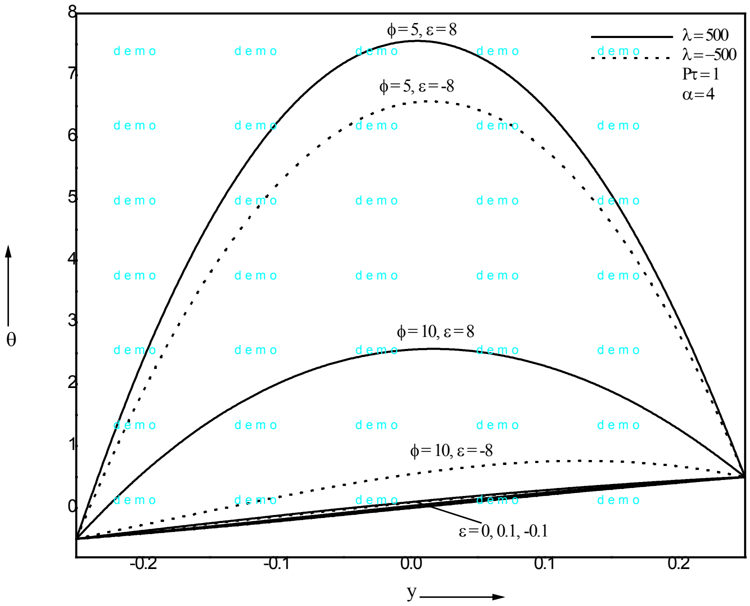

Fig. 4 Plots of θ versus y in the case of asymmetric heating for different values of heat generation coefficient φ and ε

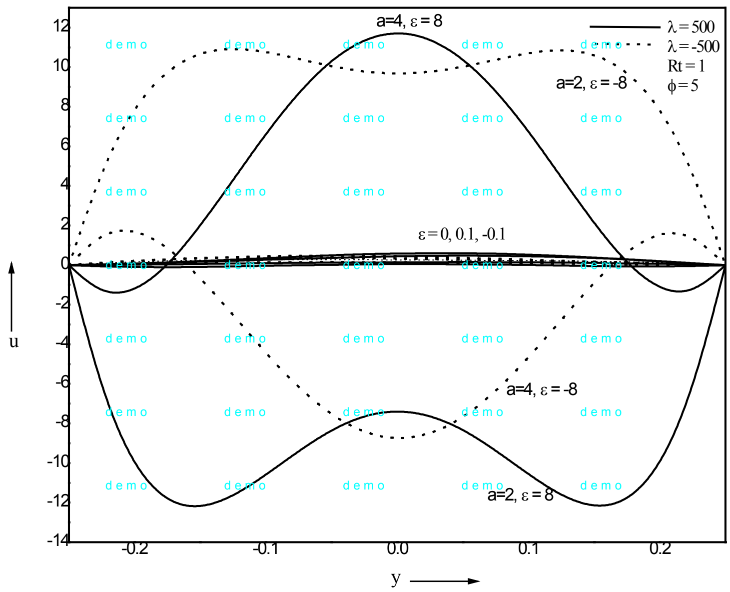

Fig. 5 Plots of u versus y in the case of asymmetric heating for different values of couple stress parameter a

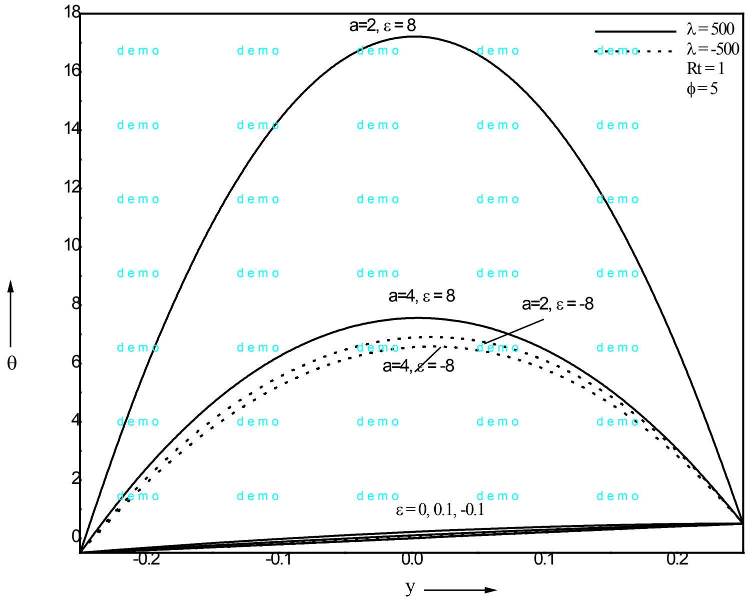

Fig. 6 Plots of θ versus y in the case of asymmetric heating for different values of couple stress parameter a

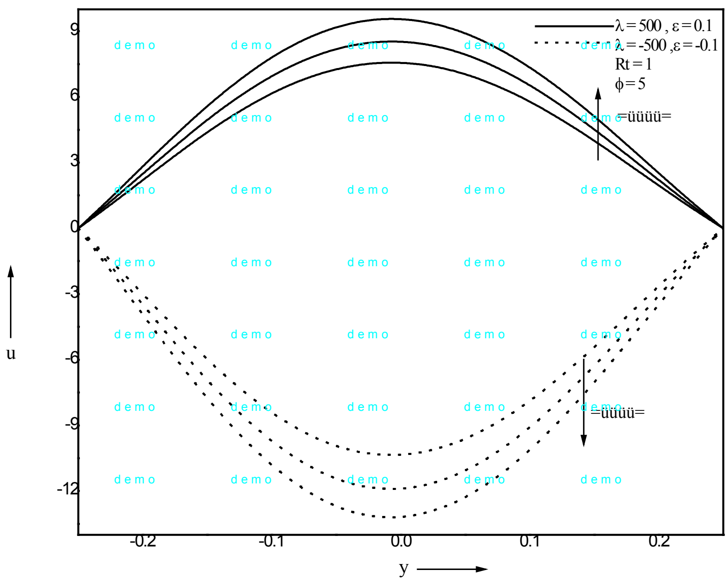

Fig. 7 plots of u versus y for different values of couple stress parameter a for isoflux-isothermal case

Nomenclature

A Constant defined in equation (4)
C_p specific heat at constant pressure
D = 2L, hydraulic parameter
g acceleration due to gravity
λ dimensionless parameter (Gr/Re) defined in equation (10)
G Grashof number defined in equation (10)
B_r Brinkman number defined in equation (10)
K thermal conductivity
k non-dimensional material parameter
a couple stress parameter as defined in equation (10)
L channel width
η cross Viscosity
p pressure
P $= p + \rho_o g X$, difference between the pressure and the hydrostatic pressure
R Reynolds number defined in equation (10)
R_T temperature difference ratio defined in equation (10)
T temperature
T_1, T_2 prescribed boundary temperatures
T_o reference temperature
u dimensionless velocity component in the X- direction
U velocity component in the X-direction
U_0 reference velocity
V velocity component in the Y-direction
X, Y space coordinates
y dimensionless transverse coordinate

Greek symbols

α $= \dfrac{k}{\rho_0 C_p}$, thermal diffusivity
β thermal expansion coefficient
ΔT reference temperature difference defined by equation (12)
θ dimensionless temperature defined in equation (10)
μ dynamic viscosity
ν $= \dfrac{\mu}{\rho_0}$, kinematic viscosity

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