REVIEW ARTICLE

Interpolation of generalized Biaxisymmetric potentials

D. Kumar* G.L. `Reddy**

ABSTRACT

In this paper we study the chebyshev and interpolation error for a real valued generalized biaxisymmetric Potential (GBASP) which is regular in the open hyper sphere about the origin. The lower (p, q) order and lower generalized (p, q) – type have been characterized in terms of these approximation errors.

***Key words**: Transfinite diameter, proximate order, Lagrange interpolation polynomials and extremal polynomials.*

---

*2001 Mathematics Subject classification: Primary 30 E10, Secondary 41A10
on EOL from University of Hyderabad, India –e-mail: gglreddy@yahoo.com
**Department of Mathematics
Addis Ababa University, Addis Ababa, Ethiopia

INTRODUCTION

Let $F^{\alpha,\beta}$ be a real-valued regular solution to the generalized biaxially symmetric potential equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{(2\alpha+1)}{x}\frac{\partial}{\partial x}+\frac{(2\beta+1)}{y}\frac{\partial}{\partial y}\right]F^{\alpha,\beta} = 0, \qquad \alpha > \beta > -\frac{1}{2},$$

subject to the Cauchy data $F_x^{\alpha,\beta}(0,y) = F_y^{\alpha,\beta}(x,y) = 0$ which is satisfied along the singular lines in the open hyper sphere $\sum_r^{\alpha,\beta}$; $x^2 + y^2 < r^2$. Such functions with even harmonic extensions are referred to as generalized biaxisymmetric potentials (GBASP) having local expansions of the form

$$F^{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty} a_n\ R_n^{\alpha,\beta}(x,y)$$

interms of the complete set

$$R_n^{\alpha,\beta}\ (x,y) = (x^2+y^2)^n\ P_n^{\alpha,\beta}\left[(x^2-y^2)/(x^2+y^2)\right]/\,P_n^{\alpha,\beta}(1),\ \ n = 0, 1, 2, 3, \ldots,$$

of biaxisymmetric harmonic potentials, where $P_n^{\alpha,\beta}$ are Jacobi Polynomials ([1], [15]).

Let the operator $K_{\alpha,\beta}$ uniquely associated even analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n},\ z = x + iy \in \mathbb{C},\ \text{onto GBASP}\ F^{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty} a_n R_n^{\alpha,\beta}(x,y).$$

Following McCoy [11], for Karoonwinder's integral for Jacobi polynomials,

$$F^{\alpha,\beta}(x,y) = K_{\alpha,\beta}(f) = \int_0^1\int_0^\pi f(\varsigma)\,\mu_{\alpha,\beta}(t,s)\,ds\,dt$$

Where $\mu_{\alpha,\beta}(t, s) = \gamma^1_{\alpha,\beta}\,(1 - t^2)^{\alpha - \beta - 1} t^{2\beta+1} (\sin s)^{2\alpha}$

$$\zeta^2 = x^2 - y^2t^2 - i2xyt + \cos s$$

$$\gamma^1_{\alpha,\beta} = 2\Gamma(\alpha + 1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha - \beta)\Gamma\left(\beta + \frac{1}{2}\right).$$

The inverse operator $K^{-1}_{\alpha,\beta}$ applies orthogonality of Jacobi polynomials ([1], p.8) and Poisson Kernel ([1], p.11) to uniquely define the transform

$$f(z) = K^{-1}_{\alpha,\beta}(F^{\alpha,\beta}) = \int_{-1}^{+1} F^{\alpha,\beta}\left(r\xi, r(1-\xi^2)^{1/2}\right) \upsilon_{\alpha,\beta}\left(\left(z/r\right)^2, \xi\right) d\xi$$

where $\upsilon_{\alpha,\beta}(\Im, \xi) = S_{\alpha,\beta}(\Im,\xi)\,(1 - \xi)^{\alpha}\,(1 + \xi)^{\beta}$

$$S_{\alpha,\beta}(\Im, \xi) = \eta_{\alpha,\beta}\frac{1-\Im}{(1+\Im)^{\alpha+\beta+2}}\, F\left(\frac{\alpha+\beta+2}{2}; \frac{\alpha+\beta+3}{2}; \beta+1; \frac{2\Im(1+\xi)}{(1+\Im)^2}\right)$$

$$\eta_{\alpha,\beta} = \Gamma(\alpha + \beta + 2)\,/\,2^{\alpha+\beta+1}\,\Gamma(2\alpha+1)\Gamma(\beta + 1).$$

Here the normalization $K_{\alpha,\beta}(1) = K^{-1}_{\alpha,\beta}(1) = 1$ is taken place. The Kernel $S_{\alpha,\beta}(\Im, \xi)$ is analytic on $\|\Im\| < 1$ for $-1 \le \xi \le 1$.

Let E be a compact set is the complex plane and $\xi^{(n)} = \{\xi_{n0}, \xi_{n1}, \xi_{nn}\}$ be a system of (n + 1) points of the set E such that

$$V(\xi^{(n)}) = \prod_{o\le j\le k\le n} |\xi_{nj} - \xi_{nk}| \text{ and } \Delta^{(j)}(\xi^{(n)}) = \prod_{\substack{k=0\\k\ne j}} |\xi_{nj} - \xi_{nk}|,\ j = 0, 1, \ldots n.$$

Again, let $\eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \eta_{n2}, \ldots \eta_{nn}\}$ be the system of (n + 1) points in E such that

$$V_n \equiv V(\eta^{(n)}) = \sup_{\xi^{(n)}\subset E} V(\xi^{(n)}) \text{ and } \Delta^{(0)}(\eta^{(n)}) \le \Delta^{(j)}(\eta^{(n)}) \text{ for } j = 1, 2, \ldots, n.$$

Such a system always exists and is called the n^{th} extremal system of E. The polynomials

$$L^j(z, \eta^{(n)}) = \prod_{\substack{k=0\\k\ne j}} \left(\frac{z-\eta_{nk}}{\eta_{nj}-\eta_{nk}}\right),\ j = 0, 1, 2, \ldots, n$$

are called Lagrange extremal polynomials and the limit d ≡ d(E) = $\lim_{n\to\infty} V_n^{2/n(n+1)}$ is called the transfinite diameter of E.

Let C(E) (f∈C(E) is holomorphic in the interior of E and continuous on E) denote the algebra of analytic functions on the set E. Let the Chebyshev norm be defined for f∈C(E) and $F^{\alpha,\beta} \in C\left(\sum_r^{\alpha,\beta}\right)$ as follows:

$$e_n(f: E) \equiv e_n(f) = \inf_{g\in h_n} \|f - g\|,$$

where $\|f - g\| = \sup_{x\in E} |f(x) - g(x)|$

and $E_n\left(F^{\alpha,\beta}; \sum_r^{\alpha,\beta}\right) \equiv E_n(F^{\alpha,\beta}) = \inf\{\|F^{\alpha,\beta} - G^{\alpha,\beta}\|, G^{\alpha,\beta} \in H_n^{\alpha,\beta}\}$ n = 0, 1, 2, …

$$\|F^{\alpha,\beta} - G^{\alpha,\beta}\| = \sup_{x^2+y^2=r^2} |F^{\alpha,\beta}(x, y) - G^{\alpha,\beta}(x, y)|.$$

The set h_n contains all real polynomials of degree at most 2n and the set $H_n^{\alpha,\beta}$ contains all real biaxisymmetric harmonic polynomials of degree at most 2n. The operators $K_{\alpha,\beta}$ and $K_{\alpha,\beta}^{-1}$ establish one-one equivalence of sets h_n and $H_n^{\alpha,\beta}$.

McCoy [11] connected classical order and type of real-valued entire functions GBASP $F^{\alpha,\beta}$ and the associate f respectively with even polynomial approximation error defined in [-1, 1]. He obtained the results for GBASP of Sato index k [13] using the results obtained by Reddy [12]. It has been noticed that these results fail to compare the growth of those entire GBASPs which have same positive finite order but their types are infinity. For the view point of including this class of entire GBASPS we shall utilize the concept of proximate order. Recently, Kumar and Kasana [9] studied the (p, q)-order and generalized (p, q)-type of GBASP for even polynomial approximation error defined on E, to include the real valued entire GBASPs of slow growth and fast growth. These results obviously leave a big class of real-valued entire GBASP, such as study of lower (p, q)-order and generalized lower (p, q) type which have not been considered earlier. The aim of this paper is to extend the results of [9] to lower (p, q)-order and generalized lower (p, q)-type.

The maximum modulii of GBASP and associate are defined as in complex function theory

$$M(r, f) = \max_{|z|=r} |f(z)| \text{ and } M(r, F^{\alpha,\beta}) = \max_{x^2+y^2=r^2} |F^{\alpha,\beta}(x, y)|.$$

A real entire GBASP is said to be of (p, q)-order $\rho(p, q)$ and lower (p, q) – order $\lambda(p, q)$ if it is of index-pair (p, q) such that

$$\lim_{r\to\infty} \sup_{\inf} \frac{\log^{[p]} M(r, F^{\alpha,\beta})}{\log^{[q]} r} = \begin{cases} \rho(p,q) \\ \lambda(p,q) \end{cases},$$

and the function f(z) having (p, q)-order $\rho(p, q)$ ($b < \rho(p, q) < \infty$) is said to be of (p, q) – type T(p, q) and lower (p, q) – type t(p, q) if

$$\lim_{r\to\infty} \sup_{\inf} \frac{\log^{[p-1]} M(r, F^{\alpha,\beta})}{\left(\log^{[q-1]} r\right)^{\rho(p,q)}} = \begin{cases} T(p,q) \\ t(p,q) \end{cases}, \ 0 \le t(p, q) \le T(p, q) \le \infty.$$

where b = 1 if p = q, b = 0 if p > q.

A positive function $\rho_{p,q}(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]}1$, is said to be proximate order of an entire function GBASP with index-pair (p, q) if

(i) $\rho_{p,q}(r) \to \rho(p, q)$ as $r \to \infty$, $b < \rho < \infty$:

(ii) $\wedge_{[q]}(r)\, \rho'_{p,q}(r) \to 0$ as $r \to \infty$, where $\rho'_{p,q}(r)$ denotes the derivative of $\rho_{p,q}(r)$ and for convenience $\Delta_{[q]}(r) = \prod_{i=0}^{q} \log^{[i]} r$.

The existence of such comparison functions on (p, q)-scale has been established in [8]. We now define generalized (p, q)-type T*(p, q) and generalized lower (p, q)-type t*(p, q) of $F^{\alpha,\beta}$ with respect to a given proximate order $\rho_{p,q}(r)$ as

$$\lim_{r\to\infty} \sup_{\inf} \frac{\log^{[p-1]} M(r, F^{\alpha,\beta})}{\left(\log^{[q-1]} r\right)^{\rho_{p,a}(r)}} = \begin{cases} T^*(p,q) \\ t^*(p,q) \end{cases}, \ 0 \le t^*(p, q) \le T^*(p, q) \le \infty.$$

If the quantity t*(p, q) is different from zero and infinite then $\rho_{p,q}(r)$ is said to be the lower proximate order of a given entire function GBASP and t*(p, q) as its generalized lower (p, q)-type. Clearly, lower proximate order and corresponding generalized lower (p, q)-type of $F^{\alpha,\beta}$ are not uniquely determined. For example, if we add $c/\log^{[q]}r$, $0< c < \infty$, to the proximate order $\rho_{p,q}(r)$ then $\rho_{p,q}(r) + c/\log^{[q]}r$ is also a proximate order satisfying (i) and (ii) and consequently, the generalized lower (p, q)-type turns out to be $e^c t^*(p, q)$.

Since $\left(\log^{[q-1]} r\right)^{\rho(r)-A}$ is a monotonically increasing function [2] of r for $r > r_0$, we can define $\phi(x)$ to be the unique solution of the equation.

$$x = \left(\log^{[q-1]} r\right)^{\rho(r)-A} \text{ if and only if } \phi(x) = \log^{[q-1]}r,$$

where A = 1, if (p, q) = (2, 2) and A = 0, otherwise.

Consequently, it can be shown that [2]

$$\lim_{x\to\infty} \frac{\phi(\eta x)}{\phi(x)} = \eta^{1/(\rho_{p,q}-A)} \text{ uniformly for every } \eta,\ 0 < \eta < \infty.$$

Let E_r be the largest equipotential curve of E defined by $E_r = \{z \in \mathbb{C} / |\varphi(z)|d = r\}$ (if r = d then $E_r = E$) where $w = \varphi(z)$ is holomorphic and maps the unbounded component of the complement of E on $|w| > 1$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. Also, we set

$$\overline{M}(r, F^{\alpha,\beta}) = \sup_{z\in Er} |F^{\alpha,\beta}(z, 0)| \text{ for } r > 1, \text{ and } \overline{M}(r, f)\ \sup_{z\in Er} |f(z)|.$$

2. Some Basic Results.

Lemma 1. Let $F^{\alpha,\beta}$ be real valued entire functions GBASP with map $K_{\alpha,\beta}$ associate f. Then the (p, q) orders and lower (p, q)-orders of $F^{\alpha,\beta}$ and f are identical. Further the respective generalized (p, q)-type and generalized lower (p, q)-type of $F^{\alpha,\beta}$ and f are also equal.

Proof: Let us consider the relation $N_{\alpha,\beta}(\Upsilon) = \max\left\{\eta_{\alpha,\beta}^{-1}\left|s_{\alpha,\beta}(\gamma,\beta)\right| / -1 \le \xi \le 1\right\}$. For any $\varepsilon > 0$, we have [11, p.158]

(2.1) $M(r, F^{\alpha,\beta}) \le M(r, f) \le M(\varepsilon^{-1}r, F^{\alpha,\beta})\, N_{\alpha,\beta}(\varepsilon^2)$.

Using (2.1) and definition of (p, q)-order, lower (p, q)-order, generalized (p, q)-type and generalized lower (p, q)-type of $F^{\alpha,\beta}$ and f, we conclude the result.

Lemma 2: Let $F^{\alpha,\beta}$ be a real valued entire function GBASP of (p, q)-order ρ(p, q) and lower (p, q)-order λ(p, q). Then

$$\lim_{r\to\infty} \sup_{\inf} \frac{\log^{[p-1]} \overline{M}(r, F^{\alpha,\beta})}{\log^{[q]} r} = \begin{cases} \rho(p,q) \\ \lambda(p,q) \end{cases},$$

and for ρ(p, q) (b < ρ(p, q) < ∞), T*(p, q) and t*Ip, q) are given by

$$\lim_{r\to\infty} \sup_{\inf} \frac{\log^{[p-1]} \overline{M}(r, F^{\alpha,\beta})}{\left(\log^{[q-1]} r\right)^{\rho_{p,q}(r)}} = \begin{cases} T^*(p,q) \\ t^*(p,q) \end{cases} 0 \le t^*(p, q) \le T^*(p, q) \le \infty.$$

Proof: Taking the definition of (p, q)-order, lower (p, q)-order, generalized (p, q)-type and generalized lower (p, q) –type of entire GBASP into account the proof follows on the lines of Lemma 1 in [6].

Theorem A. If f ∈ C(E) can be extended to an entire function with index-pair (p, q), lower (p, q)-order λ(p, q) (b < λ(p, q) < ∞) and generalized lower (p, q)-type t*(p, q), then for $e_n(f)$, there exists an entire function g(z) = $\sum_{n=0}^{\infty} e_n(f) z^{n+1}$ such that

λ(p, q, f) = λ(p, q, g) and t* (p, q, f) = β*t*(p, q, f),

where β* = $d^{-\rho(p, q)}$ for q = 1, β* = 1 for q > 1.

Proof: It has been shown in Lemma 3 and 4 of [6] that the function g(z) = $\sum_{n=0}^{\infty} e_n(f) z^{n+1}$ is an entire function. Winiarski [16, p. 266] has proved that for any ε > 0

(2.2) $e_n(f) \le K \overline{M}(r, f) \left(\dfrac{de^{\varepsilon}}{r}\right)^n$,

where K is a constant and d > 0 is the transfinite diameter of E.

Using (2.2) in the power series expansion of g(z), it is inferred that

$$g\left(\frac{r}{de^{2\varepsilon}}\right) = \sum_{n=0}^{\infty} e_n(f) \left(\frac{r}{de^{2\varepsilon}}\right)^{n+1} \le \frac{Kr\overline{M}(r, f)}{de^{2\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{e^{n\varepsilon}} \le \frac{Kr\overline{M}(r, f)}{de^{2\varepsilon}\left(e^{\varepsilon}-1\right)},$$

or $\quad \log g\left(\frac{r}{de^{z\epsilon}}\right) \leq 0(1) + \log \overline{M}\,(r, f) + \log r.$

Thus, in view of above inequality and Lemma 1, for $p \geq 2$ and $q = 1$,

$$\lambda(p, 1, g) \leq \lambda(p, 1, f) \text{ and } t^*(p, 1, g) \leq e^{2\varepsilon\rho(p,1)}\, d^{\rho(p,1)} t^*(p, 1, f),$$

and for $p \geq 2$ and $q > 1$, $\lambda(p, q, g) \leq \lambda(p, q, f)$ and $t^*(p, q, g) \leq t^*(p, q, f)$.

Since ε is arbitrary both inequalities imply that for all (p, q),

(2.3) $\lambda(p, q, g) \leq \lambda(p, q, f)$ and $\beta^* t^*(p, q, g) \leq t^*(p, q, f)$.

Further, using the inequality $\overline{M}\,(r, f) \leq a_0\; 2g(r/d)$, we observe that for $q = 1$, $\lambda(p, 1, f) \leq \lambda(p, 1, g)$ and $t^*(p, 1, f) \leq d^{-\rho(p,1)}\, t^*(p, 1, g)$, and for $q > 1$, $\lambda(p, q, f) \leq \lambda(p, q, g)$ and $t^*(p, 1, f) \leq t^*(p, q, g)$. Hence, for all index-pairs (p, q),

(2.4) $\lambda(p, q, f) \leq \lambda(p, q, g)$ and $t^*(p, q, f) \leq \beta^* t^*(p, q, g)$.

Combining (2.3) and (2.4), we have

$$\lambda(p, q, f) = \lambda(p, q, g) \text{ and } t^*(p, q, f) = \beta^* t^*(p, q, g).$$

Theorem B: Let $f(z) \in C(E)$. Then $f(z)$ can be extended to an entire function of lower (p, q)-order $\lambda(p, q)$ $(b < \lambda(p, q) < \infty)$, if and only if, for $(p, q) \neq (2, 2)$,

(2.5) $\lambda(p, q) = \max\limits_{\{n_k\}} [P_\chi(\ell(p, q))]$ and

(2.6) $\lambda(p, q) = \max\limits_{\{n_k\}} [P_\chi(\ell^*(p, q))]$,

where $\ell(p, q) = \lim\limits_{k\to\infty} \dfrac{\log^{[p-1]} n_{k-1}}{\log^{[q-1]} e_{n_k}^{-1/n_k}(f)}$

and $\ell^*(p, q) = \lim\limits_{k\to\infty} \dfrac{\log^{[p-1]} n_{k-1}}{\log^{[q-1]}\left(\dfrac{1}{n_k - n_{k-1}} \log \dfrac{e_{n_{k-1}}(f)}{e_{n_k}(f)}\right)}$.

such that $P_\chi(L(p, q)) = \begin{cases} L(p,q) & if\ \ q < p < \infty \\ \chi + L(p,q) & if\ \ p = q = 2 \\ \max(1, L(p,q)) & if\ 3 \le p = q \\ \infty & if\ \ p = q = \infty \end{cases}$

and $\chi \equiv \chi\{n_k\} = \lim\limits_{k\to\infty} \dfrac{\log n_{k-1}}{\log n_k}$.

Further (2.5) and (2.6) hold for (p, q) = (2, 2) also provided $\{n_k\}$ be the sequence of principal indices such that $\log n_{k-1} \cong \log n_k$ as $k \to \infty$.

Proof: By Lemma 3 & 4 of [6] we conclude that $f \in C(E)$ can be extended to an entire function if and only if g(z) is an entire function. Moreover, by Theorem A, f(z) and g(z) have the same lower (p, q)-order. Applying Theorem 2 by Juneja et al. [4, p. 62] to the function $g(z) = \sum_{n=0}^{\infty} e_n(f) z^{n+1}$ the result follows.

Remarks.

(a) For E = [-1, 1], and (p, q) = (2, 1) the result (2.5) includes a Theorem by Singh [14] and a result (1.2) by Massa [10] and in addition for (p, q) = (2,2), (2.6) gives Theorem 5 by Reddy [12].

(b) Also, for E = [-1, 1] the results (2.5) and (2.6) give Theorem 1 and 2 by Juneja [3] for entire functions of Sato growth [13] i.e. (p, q) = (p, 1).

Theorem C. Let $f(z) \in C(E)$. Then f can be extended to an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ and generalized lower (p, q)-type $t^*(p, q)$ $(0 < t^*(p, q) < \infty)$ if and only if

(2.7) $$t^*(p, q) = \beta^* \max_{\{m_k\}} \left\{ \lim_{k\to\infty} \left(\frac{\phi\left(\log^{[p-2]} m_{k-1}\right)}{\log^{[q-1]}\left(\log^{[q-1]} e_{m_k}^{-1/m_k}(f)\right)} \right)^{\rho(p,q)} \right\}, \ p \ge 2$$

and further, if the sequence of principal indices $\{m_k\}$ satisfies

$m_{k-1} \cong m_k$ as $k \to \infty$, then, for p = 2,

$$\frac{t^*(2,q)}{\wp(2,q)} = \beta^* \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(m_{k-1})}{\log^{[q-1]}\left(\log^{[A]} e_{m_k}^{-1/m_k}(f)\right)} \right)^{\rho(2,q)-A} \right\},$$

where maximum is taken over all increasing sequence of positive integers and

$$\wp(p,q) = \begin{cases} (\rho(2,2)-1)^{\rho(2,2)-1} / (\rho(2,2))^{\rho(2,2)} & if \quad (p,q)=(2,2) \\ 1/e\rho(2,1) & if \quad (p,q)=(2,1) \\ 1 & otherwise. \end{cases}$$

Proof: Applying Theorem 2 of Kasana et al [7] to the function $g(z) = \sum_{n=0}^{\infty} e_n(f) z^{n+1}$

and the resulting characterization of $t^*(p, q, g)$ in terms of $e_n(f)$ and the relation $t^*(p,q,f) = \beta^* t^*(p, q, g)$ taking together prove the theorem.

Taking $\rho_{p,q}(r) = \rho(p, q)$ for all $r > r_0$ and $\phi(x) = x^{1/(\rho(p,q)-A)}$, we have the following corollary which gives a formula for lower (p, q)-type t(p, q) in terms of approximation errors of an entire function f(z).

Corollary 1. Let $f(z) \in C(E)$. Then f(z) is the restriction of an entire function having (p, q)-order $\rho(p, q)$ ($b < \rho(p, q) < \infty$) and lower (p, q)-type t(p, q) ($0 < t(p, q) < \infty$) if and only if

$$\frac{t(p,q)}{\wp(p,q)} \beta^* \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \frac{\log^{[p-2]} m_{k-1}}{\left(\log^{[q-1]} e_{m_k}^{-1/m_k}(f)\right)^{\rho(2,q)-A}} \right\}.$$

On the domain E = [-1, 1] and for approximation error $e_n(f)$, this corollary also includes some results of Reddy[12] respectively for (p, q) = (2, 1) and (p, q) = (2, 2).

3. Main Results:

Polynomial Approximation of GBASP

In this section we examine the global existence of GBASP $F^{\alpha,\beta}$ and growth of its c-norm $E_{2n}(F^{\alpha,\beta})$.

McCoy[11] proved the following theorem.

Theorem E. For each GBASP $F^{\alpha,\beta}$ regular in the hyper sphere $\sum_r^{\alpha,\beta}$ thee is a unique map $K_{\alpha,\beta}$ associated with an even function f analytic in the disc D_R and conversely.

Now we prove the following

Theorem 1. Let $F^{\alpha,\beta}$ be real valued GBASP regular in $\sum_r^{\alpha,\beta}$ and continuous on $\left(\overline{\sum_r^{\alpha,\beta}}\right)$.

Then $F^{\alpha,\beta}$ can be extended to an entire function G ASP of lower (p, q)-order $\lambda(p, q)$ $(b < \lambda(p, q) < \infty)$ if and only if, for $(p, q) \neq (2, 2)$.

3.1 $\lambda(p, q) = \max\limits_{\{n_k\}} [p_{\chi^*}(\ell'(p, q))]$ and

3.2 $\lambda(p, q) = \max\limits_{\{n_k\}} [p_{\chi^*}(\ell^{**}(p, q))]$,

where $$\ell'(p, q) = \liminf_{k\to\infty} \frac{\log^{[p-1]} 2n_{k-1}}{\log^{[q-1]} E_{2n_k}^{-1/2n_k}(F^{\alpha,\beta})}$$

and $$\ell^{**}(p, q) = \liminf_{k\to\infty} \frac{\log^{[p-1]} 2n_{k-1}}{\log^{[q-1]}\left(\frac{1}{n_k - n_{k-1}}\log\frac{E_{2n_{k-1}}\left(F^{\alpha,\beta}\right)}{E_{2n_k}\left(F^{\alpha,\beta}\right)}\right)}$$

such tht $$\chi^* \equiv \chi^*\{n_k\} = \liminf_{k\to\infty}\left(\frac{\log 2n_{k-1}}{\log 2n_k}\right).$$

Further (3.1) and (3.2) hold for (p, q) = (2, 2) also provide $\{n_k\}$ be the sequence of principal indices such that $\log 2n_{k-1} \cong \log 2n_k$ as $k \to \infty$.

Proof: By Theorem E, $F^{\alpha,\beta}$ is entire if and only if the associate f is entire. Moreover, lower (p, q)-order of the associate agrees by Theorem B. Using Corollary 1 ([4], p. 62) and Lemma 1, Theorem 1 follows.

Theorem 2. Let $F^{\alpha,\beta}$ be real valued GBASP regular in $\sum_r^{\alpha,\beta}$ and continuous on $\left(\overline{\sum_r^{\alpha,\beta}}\right)$.

Then $F^{\alpha,\beta}$ can be extended to an entire function GBASP of (p, q)-order $\rho(p, q)$ ($b < \rho(p, q) < \infty$) and generalized lower (p, q)-type $t^*(p, q)$ ($0 < t^*(p, q) < \infty$) if and only if

$$t^*(p,q) = \beta^* \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(\log^{[p-2]} 2m_{k-1})}{\log^{[q-1]} E_{2m_k}^{-1/2m_k}(F^{\alpha,\beta})} \right)^{\rho(2,q)} \right\}, p \geq 3,$$

and further, if the sequence of principal indices $\{m_k\}$ satisfies

$2m_{k-1} \cong 2m_k$ as $k \to \infty$, then, for p = 2

$$\frac{t(2,q)}{\wp(2,q)} = \beta^* \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(2m_{k-1})}{\log^{[A]} E_{2m_k}^{-1/2m_k}(F^{\alpha,\beta})} \right)^{\rho(2,q)-A} \right\}.$$

where maximum is taken over all increasing sequence of positive integers.

Proof: Inequalities (2.4) and (2.8) of [9] show that the associate f meets the same limiting requirements as GBASP. Using Theorem C for even f and Lemma 2, the proof of the theorem follows.

REFERENCES

1. Askey, R., Orthogonal Polynomials and Special Functions, Regional conference series in Applied Mathematics, SIAM, Philadelphia, 1975.

2. Einstein-Mathews, S.M. and Kasana, H.S., Proximate order and type of entire functions of several complex variables, Israel J. Math (Jerusalem) 92(1995), no. 1-3, 273 – 284.
3. Juneja, O.P., Approximation of an entire function, J. Approx. Theory No. 4, 11(1974), 343 – 349.
4. Juneja, O.P., Kapoor, G.P., and Bajapi, S.K., On the (p, q)-order and lower (p, q)-order of an entire function. J.Reine angew. Math, 282(1976), 53 – 57.
5. Kasana, H.S., The generalized type of entire functions with index-pair (p, q). Comment . Math. No2, 29(1990), 101 – 108.
6. Kasana, H.S. and Kumar, D., On approximation and interpolation of entire functions with index-pair (p, q), Publications Mathemtiques, 38(1994), 255 – 267.
7. Kasana, H.S., Kumar, D. and Srivastava, G.S., On the generalized lower type of entire gap power series with index-pair (p, q), U.U.D.M. Report (1990), 1 – 12.
8. Kasana, H.S. and Sahai, A. The proximate order of entire Dirichlet series, Complex Variables: Theory and Application (New York – USA), 9(1987), no. 1, 49 – 62.
9. Kumar D. and Kasana, H.S., Approximation and interpolation of generalized Biaxisymmetric Potentials, Panamer Mathematical J. (1999), no. 1, 55 – 62.
10. Massa, S., Remarks on the growth of an entire function and the degree of approxiamtion, Riv. Mat, Univ. Parma(4), 7(1981), 1- 6
11. McCoy, P.A., Polynomial approximation of generalized biaxisymmetric potentials. J. Approx. Theory 25(1979) 153 – 168.
12. Reddy, A.R., Approximation of an entire function, J. Approx. Theory 3(1970), 128 – 137.
13. Sato, D. On the rate of growth of entire functions of fast growth. Bull, Amer. Math Soe. 69(1963), 411 – 414.
14. Singh, J.P., Approxmaition of entire functions. Yokohama.Math.J. 19(1971), 105 – 108.
15. Szego, G. Orthogonal Polynomial, Colloquium Publications, 23, Amer.Math. Soc. Providence, R.I. 1967.
16. Winiarski, T. N., Approxiamtion and interpolation of entire functions. Ann. Polon. Math, 29(1970), 259 – 273.