FULL-LENGTH ARTICLE

Zero-free Regions for Fractional Hypergeometric Zeta Functions on the Left half of the Complex Plane

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Abstract

Some zero-free regions were known on the right half of the complex plane in the form of vertical strips for fractional hypergeometric zeta functions. In this paper, we describe and demonstrate zero-free regions on the left half of the complex plane for fractional hypergeometric zeta functions. The fractional hypergeometric zeta function of order "a" has no zeros to the left half of the complex plane except the trivial zeros on the real axis.

Key Words: Analytic continuation; fractional hypergeometric zeta functions; Riemann zeta function; zero free regions.

INTRODUCTION

The study of the location of zeros and zero-free regions of the families of zeta functions has been studied by so many scholars, (Albeverio & Cebulla, 2007; Fekih-Ahmed, 2011; Garunkstis & Steuding, 2007; Apostol, 1976; Hassen & Nguyen, 2011). The families of hypergeometric zeta functions and fractional hypergeometric zeta functions are known only in their integral representations as generalizations of the classical Riemann zeta function via integral representation (Geleta & Hassen, 2016; Hassen & Nguyen, 2010). It was also discovered that both the hypergeometric zeta functions $\zeta_N(s)$ of order "N", and the fractional hypergeometric zeta functions $\zeta_a(s)$ of order "a", can be continued meromorphically to the whole complex plane (Geleta & Hassen, 2016; Hassen & Nguyen, 2010). Following these authors, it was described and demonstrated that zero free region on the right half plane $\mathcal{H} = \{s = \sigma + it \in \mathbb{C}: \sigma > 1\}$ in the form of vertical strips $V_a = \{1 \leq \sigma < 2 - a\}$ for fractional hypergeometric zeta functions $\zeta_a(s)$ and $V_N = \{1 \leq \sigma < 2\}$ for hypergeometric zeta functions $\zeta_N(s)$, where a is a positive real number between 0 and 1 and "N" is a natural number (Birmechu & Gelete, 2022). It was shown that the fractional hypergeometric zeta functions $\zeta_a(s)$ are zero-free for infinitely many positive real numbers "a" in the delta neighborhood of 1. Specifically, there is a positive real number δ so that $\zeta_a(s)$ is zero-free where the classical Riemann zeta function $\zeta(s)$ is zero-free for $a \in (1-\delta, 1+\delta)$ (Geleta, 2022).

The fractional hypergeometric zeta function $\zeta_a(s)$ has an infinite number of simple poles at $1, 0, -1, -2, \ldots$, and an infinite number of zeros at $1-a, -a, -(1+a), -(2+a), \ldots$ (Geleta, 2014; Geleta & Hassen, 2016). These zeros are called trivial zeros for the fractional hypergeometric zeta function. Observe that if $a > 1$ is fixed, then all the trivial zeros for the fractional hypergeometric zeta function are on the negative real axis, and if $0 < a < 1$, then "$1 - a$" is the only positive trivial zero for the fractional hypergeometric zeta function. The question of whether or not these zeros are the only zeros of fractional hypergeometric zeta function $\zeta_a(s)$ is unsettled. It has been suggested that it is possible to extend zero-free regions for these families of zeta functions to both the right and the left half of the complex plane with some evidence but no proof (Birmechu & Gelete, 2022). Motivated by this suggestion, in this paper, we show that the fractional hypergeometric zeta functions $\zeta_a(s)$ have no zeros for $a > 1$ on $\{\sigma + it \in \mathbb{C}: \sigma < 1\}$ and for $0 < a < 1$ on $\{\sigma + it \in \mathbb{C}: \sigma < 0\}$ except the trivial zeros aforementioned. For $0 < a < 1$ at present time we are not sure whether or not "$1 - a$" is the only zero on the critical strip of the Riemann zeta function $\{\sigma + it \in \mathbb{C}: 0 < \sigma < 1\}$ for the fractional hypergeometric zeta functions. To prove our results we use the method of analytic continuation for fractional hypergeometric zeta functions $\zeta_a(s)$ strip-by-strip (Geleta & Hassen, 2016) to the left half of the complex plane; positivity results on oscillatory integrals and monotonicity of real-valued functions (Albeverio & Cebulla, 2007).

Throughout this paper, we use the following notations for vertical strips between the poles of the fractional hypergeometric zeta functions. For each natural number n we define the vertical strip V_n as follows:

$$V_n = \{s \in \mathbb{C}: s = \sigma + it, t \in \mathbb{R} \text{ and } 1 - n < \sigma < 2 - n\}$$

$$V_n^+ = \{s \in \mathbb{C}: s = \sigma + it, t > 0 \text{ and } 1 - n < \sigma < 2 - n\}$$

$$V_n^- = \{s \in \mathbb{C}: s = \sigma + it, t < 0 \text{ and } 1 - n < \sigma < 2 - n\}$$

$$V_n = V_n^+ \cup V_n^- \cup \mathbb{R}.$$

In this paper, for each natural number n we first show that,

$$F_{n,a}(s) = \sum_{k=1}^{n} L_k(a)\Gamma(s+k-2) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \sum_{k=1}^{n} L_k(a)\, x^{-a+k-1}\right) x^{s+a-2} e^{-x}\, dx$$

is the analytic continuation of

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \int_0^\infty \frac{x^{s+a-2}}{a\gamma(a,x)e^x}\, dx$$

on V_n. So that $\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = F_{n,a}(s)$ on each V_n. Then to prove that $\zeta_a(s)$ has no zeros on V_n, except for the trivial zeros mentioned, it is enough to show $\Im\left(F_{n,a}(s)\right)$ has no zeros on V_n^+ and V_n^-. This can be accomplished by showing that $\Im\left(\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)\right) = \Im\left(F_{n,a}(s)\right)$, where $\Im(z)$ represents an imaginary part of z.

Observe that the left-hand side $\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)$ is defined only for $\sigma > 1$, but the right-hand side $F_{n,a}(s)$ is defined for $\sigma \in \mathbb{R}\backslash\{1, 0, -1, -2, -3, \dots, 2-n\}$.

The main result of this paper is the following:

Theorem 3.1 Let a be a fixed positive real number. Then $\zeta_a(s)$ has no zeros on V_n except for infinitely many trivial zeros on the left side of $\sigma = 0$, one in each of the intervals $I_n = [-n, 1-n]$, for $a \in (0,1)$ and one in each of the intervals $I_n = [1-n, 2-n]$ for $a > 1$, where $n \in \mathbb{N}$.

In this theorem, the trivial zero "$1-a$" is not included, as we are not sure whether or not it is the only positive root in the critical strip $\{s \in \mathbb{C}: 0 < \sigma = \Re(s) < 1\}$ of the classical Riemann zeta function. We present this issue in the conclusion part as a conjecture based on some evidence.

The structure of the present work is as follows. In section 2 we review some of the main results obtained so far regarding fractional hypergeometric zeta functions $\zeta_a(s)$ which we think are important for the coming sections. In section 3 we reveal and prove our main result and demonstrate that $\zeta_a(s)$ are zero-free on the left half of the complex plane, except for the aforementioned trivial zeros. In section 4 we give some concluding remarks.

Preliminaries

In this section, we review basic terms and results which we will use to prove the main result of this paper. As our work is the continuation of (Birmechu & Gelete, 2022; Geleta, 2022), we review some results concerning fractional hypergeometric zeta functions $\zeta_a(s)$ and its analytic continuation, to state and prove our results in perspective.

Definition 2.1 Let $s = \sigma + it$ be a complex variable. Then the Riemann zeta function $\zeta(s)$ for $\sigma > 1$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

Respectively by an Euler product over prime numbers P

$$\zeta(s) = \prod_{p\in \mathrm{P}} (1 - p^{-s})^{-1}.$$

For $\sigma > 1$, the classical zeta function $\zeta(s)$ is also defined by the integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx,$$

where $\Gamma(s)$ is the Gamma function given by

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx\ .$$

Since the convergent of the infinite product of non-zero factors is not zero, the zeta function does not vanish on the right half of the complex plane for $\sigma > 1$. Therefore, this product formula is one of the important tools to show that the Riemann zeta function is zero-free on the right half of the complex plane (actually for $\sigma > 1$).

The functional equation is also another important representation of the zeta function to locate the zeros of the zeta function. It is given by

$$\zeta(s) = 2(2\pi)^{s-1}\zeta(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$

and it is the celebrated functional equation for the Riemann zeta function. The reflection principle

$$\zeta(\bar{s}) = \overline{\zeta(s)} \text{ for } s \in \mathbb{C}$$

provides a further functional equation for the Riemann zeta function. The functional equation, together with the reflection principle, evokes a strong symmetry of the Riemann zeta function with respect to the so-called critical line $\sigma = \frac{1}{2}$.

Definition 2.2 The points $s = -2, -4, -6, \ldots$ are called the "trivial" zeros of the zeta function $\zeta(s)$, and the vertical strip $V = \{s \in \mathbb{C}: 0 \leq \sigma \leq 1\}$ is called the critical strip.

Regarding the zeros inside the critical strip, it is conjectured that these nontrivial zeros all lay on the critical line at $\sigma = \frac{1}{2}$. This conjecture is known as Riemann's Hypothesis.

As a generalization of the Riemann zeta function $\zeta(s)$ via integral representation we have the following definition:

Definition 2.3 The fractional hypergeometric zeta function $\zeta_a(s)$ is defined for all positive real numbers "a" and $\sigma > 1$ as

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2} e^{-x}}{a\gamma(a,x)} dx,$$

where $\Gamma(s)$ is the Gamma function defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$$

and $\gamma(a,x)$ is the lower incomplete gamma function given by $\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt$, and they have the following relation,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt = \int_0^x t^{a-1} e^{-t}\, dt + \int_x^\infty t^{a-1} e^{-t}\, dt = \gamma(a,x) + \Gamma(a,x).$$

In the above expression $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\, dt$ is called the upper incomplete gamma function.

Observe that when $a = N$, the natural number we get the hypergeometric zeta functions $\zeta_N(s)$, and if $a = 1$, we get a classical Riemann zeta function $\zeta(s)$. The fractional hypergeometric zeta functions $\zeta_a(s)$ have poles at $s = 1, 0, -1, -2, \ldots$ and zeros at $s = 1 - a, -a, -(1+a), -(2+a), \ldots$ (Geleta & Hassen, 2016). These zeros are called the trivial zeros of $\zeta_a(s)$.

Concerning zero free regions for fractional hypergeometric zeta function $\zeta_a(s)$, the following results were known.

Theorem 2.1. Let $0 < a < 1$ be fixed. Then $\zeta_a(s) \neq 0$ in the vertical strip

$V_a = \{s = \sigma + it \in \mathbb{C}: 1 \leq \sigma < 2 - a\}$ (Birmechu & Gelete, 2022).

Theorem 2.2 There is a positive number δ such that the fractional hypergeometric zeta functions of order "a", $\zeta_a(s)$ is zero free for $a \in (1-\delta, 1+\delta)$ where the zeta function is zero-free (Geleta, 2022).

Positivity Properties of Integrals

Proposition 2.1 For some nonnegative integer k and positive real number t, let the function $h(r) \geq 0$ on $(0,\infty)$, $h \in L^1_{loc}(0,\infty), h$ is decreasing and strictly decreasing on some open sub-intervals of the type $\left(\frac{k\pi}{t}, \frac{(k+1)\pi}{t}\right)$. Then

$$\int_0^\infty h(r)\sin(tr)\,dr > 0.$$

Moreover,

$$\int_0^T h(r)\sin(tr)\,dr > 0$$

for any $T > 0$ provided, h satisfies the above assumptions where $(0,\infty)$ is replaced with $(0,T)$ (Albeverio & Cebulla, 2007).

Corollary 2.1 Let $t > 0$ and $\tilde{x}_{t,k} \geq 1$ be such that $t\ln(\tilde{x}_{t,k}) = 2\pi k$, for some positive integer k and let $g \geq 0$ on $[\tilde{x}_{t,k}, \infty), g \in L^1_{loc}(\tilde{x}_{t,k},\infty)$ such that $x \longmapsto xg(x)$ is decreasing on $[\tilde{x}_{t,k}, \infty)$, strictly decreasing on $\left(\ln\frac{j\pi}{t}, \ln\frac{(j+1)\pi}{t}\right)$ for some positive integer j. Then

$$\int_{\tilde{x}_{t,k}}^\infty g(x)\sin(t\ln x)\,dx > 0.$$

Moreover,

$$\int_{\tilde{x}_{t,k}}^T g(x)\sin(t\ln x)\,dx > 0$$

For any $T > \tilde{x}_{t,k}$, whenever $xg(x)$ is decreasing in $[\tilde{x}_{t,k}, T]$, strictly decreasing on $\left(\ln\frac{j\pi}{t}, \ln\frac{(j+1)\pi}{t}\right)$ for some positive integer j such that $\ln\frac{j\pi}{t} \geq \tilde{x}_{t,k,}$ and $\ln\frac{(j+1)\pi}{t} \leq T$ (Albeverio & Cebulla, 2007).

Analytic Continuation

Next, we review the analytic continuation. The analytic continuation of an analytic function is a process of extending the domain of the function to a larger domain.

Definition 2.4 Let f and g both be analytic in domains D_1 and D_2 respectively. If $D_1 \cap D_2 \neq \emptyset$ and $f(s) = g(s)$ for all s in $D_1 \cap D_2$, then we call g a direct analytic continuation of f to D_2.

The analytic continuation of the fractional hypergeometric zeta functions $\zeta_a(s)$ has been shown strip-by-strip in stages (Geleta & Hassen, 2016). Concerning the analytic continuation of $\zeta_a(s)$, the following results were also known.

Theorem 2.3 For $0 < \Re(s) = \sigma < 1$,

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)}\left[\Gamma(s-1) + \int_0^\infty \left(\frac{1}{a\,\gamma(a,\ x)} - \frac{1}{x^a}\right) x^{s+a-2} e^{-x}\, dx\right]$$

(Geleta & Hassen, 2016).

In general this analytic continuation is based on the series representation of $\frac{1}{a\gamma(a,x)}$, where $\gamma(a,x)$ is the lower incomplete gamma function given by $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}\,dt$ and, $\Gamma(a,x)$ is the upper incomplete gamma function given by $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt$ and they have the following relation, $\Gamma(s) = \gamma(s,x) + \Gamma(s,x)$.

$$\frac{1}{a\gamma(a,x)} = x^{-a}\left[1 + \frac{a}{1+a}x + \left(\frac{a^2}{(1+a)^2} - \frac{a}{2!\,(2+a)}\right)x^2 + \left(\frac{a^3}{(1+a)^3} - \frac{a^2}{(1+a)(2+a)} + \frac{a}{3!\,(3+a)}\right)x^3 + \cdots\right].$$

To simplify things put $L_1(a) = 1, L_2(a) = \frac{a}{1+a}, L_3(a) = \frac{a^2}{(1+a)^2} - \frac{a}{2!(2+a)}$,

$L_4(a) = \frac{a^3}{(1+a)^3} - \frac{a^2}{(1+a)(2+a)} + \frac{a}{3!(3+a)}$, ... the coefficients of x^{k-1} for each $k = 1, 2, 3, \ldots$.

Using the above notation observe that,

$$\frac{1}{a\gamma(a,x)} = \frac{1}{a\gamma(a,x)} - \sum_{k=1}^{n} L_k(a)x^{-a+k-1} + \sum_{k=1}^{n} L_k(a)x^{-a+k-1}.$$

From this multiplying both sides by $x^{s+a-2}e^{-x}$ we obtain,

$$\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} = \left(\frac{1}{a\gamma(a,x)} - \sum_{k=1}^{n} L_k(a)x^{-a+k-1}\right)x^{s+a-2}e^{-x} + x^{s+a-2}e^{-x}\sum_{k=1}^{n} L_k(a)x^{-a+k-1}, \text{for } x \in (0,\infty).$$

Extending zero free Region to the left half of the Complex plane for $\zeta_a(s)$

In this section, we prove our main result and demonstrate that the fractional hypergeometric zeta function $\zeta_a(s)$ has no zeros on the left half of the complex plane except the trivial zeros.

Theorem 3.1 Let a be a fixed positive real number. Then $\zeta_a(s)$ has no zeros on V_n except for infinitely many trivial zeros on the left side of $\sigma = 0$, one in each of the intervals $I_n = [-n, 1-n]$, for $a \in (0,1)$ and one in each of the intervals $I_n = [1-n, 2-n]$ for $a > 1$, where $n \in \mathbb{N}$.

Proof: From the analytic continuation of $\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)$ on each $V_{n,}$ we have,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)=\int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,\ x)}dx=F_{n,a}(s),\text{where}$$

$$F_{n,a}(s)=\sum_{k=1}^{n}L_k(a)\Gamma(s+k-2)+\int_0^\infty\left(\frac{1}{a\gamma(a,x)}-\sum_{k=1}^{n}L_k(a)\,x^{-a+k-1}\right)x^{s+a-2}e^{-x}\,dx.$$

$F_{n,a}(s)$ is obtained as an analytic continuation of $\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)$ as follows:

The analytic continuation of $\zeta_a(s)$ is based on the series representation of $\frac{1}{a\gamma(a,x)}$, where $\gamma(a,x)$ is the lower incomplete gamma function given by $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}\,dt$ and, $\Gamma(a,x)$ is the upper incomplete gamma function given by $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}\,dt$.

The series representation of $\frac{1}{a\gamma(a,x)}$ is given by

$$\frac{1}{a\gamma(a,x)}=x^{-a}\left[1+\frac{a}{1+a}x+\left(\frac{a^2}{(1+a)^2}-\frac{a}{2!\,(2+a)}\right)x^2+\left(\frac{a^3}{(1+a)^3}-\frac{a^2}{(1+a)(2+a)}+\frac{a}{3!\,(3+a)}\right)x^3+\cdots\right].$$

To simplify things put $L_1(a)=1, L_2(a)=\frac{a}{1+a}, L_3(a)=\frac{a^2}{(1+a)^2}-\frac{a}{2!(2+a)}, L_4(a)=\frac{a^3}{(1+a)^3}-\frac{a^2}{(1+a)(2+a)}+\frac{a}{3!(3+a)}$, ... the coefficients of x^{k-1} for each $k=1,2,3,\ldots$.

Using the above notation observe that,

$$\frac{1}{a\gamma(a,x)}=\frac{1}{a\gamma(a,x)}-\sum_{k=1}^{n}L_k(a)x^{-a+k-1}+\sum_{k=1}^{n}L_k(a)x^{-a+k-1}.$$

From this multiplying both sides by $x^{s+a-2}e^{-x}$ we obtain,

$$\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}=\left(\frac{1}{a\gamma(a,x)}-\sum_{k=1}^{n}L_k(a)x^{-a+k-1}\right)x^{s+a-2}e^{-x}+x^{s+a-2}e^{-x}\sum_{k=1}^{n}L_k(a)x^{-a+k-1},\text{for } x\in(0,\infty).$$

But then

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)=\int_0^\infty\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx=\int_0^1\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx+\int_1^\infty\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx$$

$$= \int_0^1 \left(\frac{1}{a\gamma(a,x)} - \sum_{k=1}^{n} L_k(a) x^{-a+k-1} \right) x^{s+a-2} e^{-x}\, dx$$
$$+ \int_0^1 x^{s+a-2} e^{-x} \sum_{k=1}^{n} L_k(a) x^{-a+k-1}\, dx + \int_1^\infty \frac{x^{s+a-2} e^{-x}}{a\gamma(a,x)} dx,$$

where

$$\int_0^1 x^{s+a-2} e^{-x} \sum_{k=1}^{n} L_k(a) x^{-a+k-1}\, dx = \int_0^1 \sum_{k=1}^{n} L_k(a) x^{s+k-3} e^{-x}\, dx$$
$$= \sum_{k=1}^{n} L_k(a) \int_0^1 x^{s+k-3} e^{-x}\, dx$$
$$= \sum_{k=1}^{n} L_k(a)\, \gamma(s+k-2, 1)$$
$$= \sum_{k=1}^{n} L_k(a)\, [\Gamma(s+k-2) - \Gamma(s+k-2, 1)], (\text{since } \Gamma(s) = \gamma(s,x) + \Gamma(s,x))$$
$$= \sum_{k=1}^{n} L_k(a)\, \Gamma(s+k-2) - \sum_{k=1}^{n} L_k(a) \Gamma(s+k-2, 1)$$
$$= \sum_{k=1}^{n} L_k(a)\, \Gamma(s+k-2) - \sum_{k=1}^{n} L_k(a) \int_1^\infty x^{s+k-3}\, e^{-x} dx$$
$$= \sum_{k=1}^{n} L_k(a)\, \Gamma(s+k-2) - \int_1^\infty \sum_{k=1}^{n} L_k(a)\, x^{-a+k-1} x^{s+a-2}\, e^{x} dx.$$

Therefore,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)} \zeta_a(s) = \int_0^\infty \frac{x^{s+a-2} e^{-x}}{a\gamma(a,x)} dx$$
$$= \sum_{k=1}^{n} L_k(a) \Gamma(s+k-2)$$
$$+ \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \sum_{k=1}^{n} L_k(a)\, x^{-a+k-1} \right) x^{s+a-2} e^{-x} dx = F_{n,a}(s).$$

The right-hand side converges for $\sigma \in \bigcup_{k=1}^{n} (-k, -(k-1))$ for $a > 1$ and

$\sigma \in \bigcup_{k=1}^{n} (1-k, 2-k)$ for $0 < a < 1$.

In particular, for $n = 1, 2,$ and 3 we have the following expressions respectively,

$$F_{1,a}(s) = \Gamma(s-1) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right) x^{s+a-2} e^{-x}\, dx,$$

$$F_{2,a}(s) = \frac{a}{a+1}\Gamma(s) + \Gamma(s-1) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \frac{1}{x^a} - \frac{a}{(a+1)x^{a-1}}\right) x^{s+a-2} e^{-x}\, dx,$$

$$F_{3,a}(s) = \sum_{k=1}^{3} L_k(a)\Gamma(s+k-2) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \sum_{k=1}^{3} L_k(a)\, x^{-a+k-1}\right) x^{s+a-2} e^{-x}\, dx.$$

Now we have shown that

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = F_{n,a}(s).$$

Observe that $F_{n,a}(s)$ is obtained by subtracting the terms

$$\int_0^1 x^{s+a-2} e^{-x} \sum_{k=1}^{n} L_k(a) x^{-a+k-1}\, dx$$

and adding the same terms

$$\int_0^1 x^{s+a-2} e^{-x} \sum_{k=1}^{n} L_k(a) x^{-a+k-1}\, dx$$

to the term $\int_0^1 \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx$ in the following:

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx = \int_0^1 \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx + \int_1^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx.$$

Therefore, the sign of the imaginary part of $\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)$ is the same as the sign of the imaginary part of $F_{n,a}(s)$. Their difference is the integral on the left-hand side is convergent only for $\sigma > 1$, but the integral on the right-hand side converges for

$\sigma \in \mathbb{R}\backslash\{1, 0, -1, -2, -3, \dots, 2-n\}$.

To apply positivity properties of the oscillatory integral, for $x \in (0,\ \infty)$, put

$$h(x) = \frac{x^{s+a-2}}{a\gamma(a,\ x)e^x} > 0.$$

Then since for $1 < \sigma + a - 2 < 0$, the function $h(x)$ is nonnegative, $h(x) \in L^1_{loc}(0,\infty)$ and strictly decreases on any sub-interval $(0,\infty)$ hence by Müntz formula we have

$$\int_0^\infty \frac{x^{\sigma+a-2}e^{-x}}{a\gamma(a,\ x)} \sin(t \ln x)\, dx > 0.$$

This implies that

$$\Im\left(\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)\right) > 0.$$

But as explained above,

$$\Im\left(\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)\right) = \Im\big(F_{n,a}(s)\big) > 0.$$

Thus, $\zeta_a(s) \neq 0$ on V_n^+. But, then by reflection principle $\zeta_a(s) = \overline{\zeta_a(\bar{s})}$, so that $\zeta_a(s) \neq 0$, on V_n^- as well. Therefore, $\zeta_a(s) \neq 0$ on the left half of the complex plane except the aforementioned trivial zeros on the real axis.

CONCLUSION

In this paper, we described zero free regions on the left half of the complex plane. Moreover, for $0 < a < 1$, we showed that $\zeta_a(s)$ has no zeros on V_n except for infinitely many trivial zeros on the left side of $\sigma = 0$, one in each of the intervals $I_n = [-n, 1-n]$, for $a \in (0,1)$ and one in each of the intervals $I_n = [1-n, 2-n]$ for $a > 1$, where $n \in \mathbb{N}$. Generally, if $n < a < (n+1)$, then on each interval $[-n,\ 1-n]$ we have exactly one trivial zero for fractional hypergeometric zeta function $\zeta_a(s)$ of order "a". But for $0 < a < 1$, we are not sure whether or not "$1-a$" is the only zero of $\zeta_a(s)$ on $I_1 = (0,\ 1)$. However, by analyzing the paper (Geleta, 2022) we conjecture that there are more zeros of $\zeta_a(s)$ on I_1 besides the trivial zero "$1-a$". And we call these zeros the nontrivial zeros of the fractional hypergeometric zeta functions. We expect and hope that a proof of such conjecture may shed light either to prove or disprove the Riemann Hypothesis.

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