FULL-LENGTH ARTICLE

Application of Piecewise Mathematical Modeling to Analyse the Effect of HIV/AIDS on Working Class: The Case of Ethiopia

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ABSTRACT

In this study, we employed a novel combination of piecewise models to investigate the impact of IV/AIDS on the working class in Ethiopia. The piecewise model was changed from deterministic to stochastic, from fractal-fractional to a random, and from deterministic to fractal-fractional, then to stochastic. This innovative piecewise model takes into account data fitting for working-class HIV/AIDS cases for the period 2000–2022. According to numerical simulation, fractional model-follow stochastic models are more reliable than deterministic model-follow stochastic models. Numerical simulations show that decreasing the contact rate decreases the number of individuals infected with HIV/AIDS infection, decreasing the contact rate to eradicate HIV/AIDS infection before the estimated time, and increasing the contact rate prolongs the estimated time to end HIV/AIDS infection. According to the numerical simulation result, non-productivity increases the chance of being infected by HIV/AIDS. As the rate at which non-productive classes become productive increases, the productivity force of the population increases, and the non-productive class of the population decreases. This condition is used to quickly eradicate HIV/AIDS infection by decreasing the inequality due to the economic crisis.

Key words: ABC Operator; working class population; Fractional order; Fractal-Fractional; stochastic model

INTRODUCTION

HIV/AIDS is a severe condition that affects the immune system, leading to life-threatening infections and cancers (Page et al., 2020). It is transmitted through sexual contact, sharing needles, exposure to infected blood, and mother-to-child (Gupta & Saxena, 2021). The spread of HIV depends on several factors such as prevalence of the disease, human behaviours, access to treatment services, and social and economic conditions of the society. Treatments include antiretroviral therapy (ART), prophylaxis, management of opportunistic infections and cancers, and palliative care. Symptoms vary depending on the stage of infection and the individual's immune system (Dereje et al., 2019).

Mathematical modelling is used to describe the dynamic system using different mathematical models: deterministic, stochastic, fractional, fractal-fractional, piecewise, and many other types of mathematical models. We can easily apply this method to describe the dynamics of the epidemiology of infectious and non-infectious diseases. A deterministic model was studied by many researchers (Rana et al., 2024; Sado, 2019; Sado & Kotola, 2024). The fractional order derivative model and fractal-fractional order derivatives have

also been used to study HIV/AIDS by many scholars (Ahmad et al., 2023; Jamil et al., 2023; Mangal et al., 2023; Salah et al., 2024; Wu et al., 2024). Frequently, we require new definitions to better describe phenomena beyond the traditional concepts. In this study, we focus on a piecewise mathematical model: a mathematical model that contains different functions or equations for different intervals or segments of the input variable (Cao et al., 2021; Rezapour et al., 2022). A piecewise model can be useful for modeling HIV/AIDS because it can capture the non-linear and complex dynamics of the infection, such as changes in the transmission rate, progression of the disease stages, effects of interventions, and heterogeneity of the population (Li et al., 2021). A piecewise fractional differential equation model that uses the Atangana-Baleanu derivative to describe HIV/AIDS infection dynamics in a homogeneous population was studied by Zhao and colleagues (Zhao et al., 2022). This model fits the cases of HIV/AIDS in Indonesia better than the previous Caputo model.

The piecewise deterministic Markov process model incorporates the stochasticity and uncertainty of HIV/AIDS transmission and progression. This model was used to estimate the prevalence and incidence of HIV/AIDS in South Africa and to evaluate the impact of different prevention strategies. Here, we can observe the different procedures used to construct piecewise differential equations for different periods (Li et al., 2022). To increase the reliability of our model, we can divide our time into different intervals according to the nature of our real problem. Some real problems occur in the form of random processes rather than deterministic processes (Elangovan, 2023).

First, we formulated the HIV/AIDS model of working-class population as deterministic model and then extended to piecewise mathematical model by including random process (stochastic) and Mittag-Leffler function law for the generalization of the exponential function. Extending the model to piecewise form increases the reliability of the model in describing the model of HIV/AIDS of working-class populations. The numerical simulation was performed using MATLAB 2020a by using Runge Kutta for deterministic model and the numerical methods developed by Abdon Atangana and Sania Qureshi to solve fractal-fractional order. The Euler-Maruyama method was used to solve the stochastic parts of our model. After that we combined those numerical methods on piecewise form for mathematical modeling of effects of HIV/AIDS on working class population in case of Ethiopia.

The fundamental concepts of piecewise differential operators

The following list includes some basic definitions related to the idea of piecewise operators.

Definition: (Almalahi et al., 2024), Let H(t) be a continuous function on $[a, b]$ that is not necessarily differentiable. Next, the piecewise Atangana-Baleanu derivative is described as follows:**Error! Reference source not found.**

$$ {}^{PAB}_{\cdot}G(t) = \begin{cases} G'(t), & if \quad 0 \le t \le a, \\ {}^{AB}_{\cdot}D^{k}G(t) & if \ , a \le t \le b \end{cases} \tag{1} $$

Similarly, the definition of a normal derivative for functions is identical to the operator ${}^{PAB}_{\cdot}D^{\kappa}$ on the interval $[0, a]$, while on $[a, b]$, the definition is reduced to the operator ${}^{AB}_{\cdot}D^{\kappa}$, which is defined as

$$ {}^{AB}_{\cdot}D^{k}G(t) = \frac{AB(k)}{1-k}\int_0^t G'(\delta)E_{\kappa}\left[-\frac{\kappa}{1-\kappa}(t-\delta)^{\kappa}\right]d\delta \qquad (2) $$

$$ 0 \le \kappa \le 1, \text{and } AB(\kappa) = 1-\kappa+\frac{\kappa}{\Gamma(\kappa} $$

Composition of the model with piecewise operators

The piecewise form of model is formulated using one of the following scenarios: the model moves from a deterministic to a random process, from the Mittag-Leffler rule to a random process, and from a deterministic to a fractional order and then to a random process in which the underlying assumptions of piecewise differential and integral operators are taken into account.

Scenario I

Deterministic models are models that are based on precise inputs and produce the same output for a given set of inputs. They assume that the future can be predicted with certainty based on the current state. Stochastic models are models that incorporate randomness and uncertainty into the modeling process (Ndii & Supriatna, 2017). They consider the probability of different outcomes and provide various possible results. One advantage of stochastic models over deterministic models is that they can account for the uncertainty and variability inherent in many real-world situations, such as weather, stock prices, and epidemics. In this case, the model loses the property of a deterministic rate and gains the random rate of transmission and spread of the disease. This situation increases the complexity of the model but increases the reality and capacity to represent the real situations of the model. They can also offer a range of possible outcomes, enabling decision-makers to evaluate the likelihood of various scenarios and make informed choices. On the other hand, deterministic models may not capture the complexity and unpredictability of reality and may lead to potential inaccuracies or biases in predictions.

$$ \begin{cases} G_i'(t) = H(t, G_i), if\ 0 \le t \le a, \quad G_i(0) = G_i, 0, \ for\ 1 \le i \le n \\ dG_i(t) = H(t, G_i)dt + r_i G_i dB_i(t) \quad if\ b \le t \le c, \quad G_i(b) = G_{i,2}, for\ 1 \le i \le n \end{cases} \qquad (3) $$

Scenario II

Fractal-fractional order models can describe the nonlocal and memory effects that are observed in many complex and multiscale systems (Rezapour et al., 2022; Suzuki et al., 2023). They can also provide more flexibility and accuracy in fitting the experimental data and capturing the system dynamics (Rogosin, 2015). Stochastic models can capture the randomness and uncertainty inherent in many natural and social processes, such as epidemics, population dynamics, and stock markets. These models can also account for the effects of noise and external disturbances on a system's behavior. However, stochastic models may require more data and computational resources to estimate the parameters and simulate the outcomes. They may also be less interpretable and more difficult to analyze mathematically than deterministic models (Ndii & Supriatna, 2017).

From the fractal-fractional Mittag-Leffler law to the randomness point of view of the model, it is given as:

$$\begin{cases} {}^{ABC}_{0}D_{t}^{\omega,\xi} H(t,G_i), if\ \mathrm{a} \le\ t \le\ b, \quad G_i(\mathrm{a}) = G_i, 1,\ \ for\ \ 0 \le \kappa \le 1,\ \ \text{for } 1 \le i \le\ n \\ dG_i(t) = H(t,G_i)dt + r_i G_i dB_i(t) \quad if\ \ b \le t \le c, \quad G_i(b) = G_{i,2}, for\ \ 1 \le i \le n \end{cases} \tag{4}$$

Scenario III

This scenario is a mix of the first two scenarios in the case of transforming a deterministic model to a fractal-fractional order model and then to a stochastic model. It is given as:

$$\begin{cases} G_i'(t) = H(t,G_i), if\ 0 \le\ t \le\ a,\ \ G_i(0) = G_i, 0,\ \ for\ 1 \le i \le\ n \\ {}^{ABC}_{0}D_{t}^{\omega,\xi} H(t,G_i), if\ \mathrm{a} \le\ t \le\ b,\ \ G_i(\mathrm{a}) = G_i, 1,\ \ for\ \ 0 \le \kappa \le 1,\ \ \text{for } 1 \le i \le\ n \\ dG_i(t) = H(t,G_i)dt + r_i G_i dB_i(t) \quad if\ \ b \le t \le c,\ \ G_i(b) = G_{i,2}, for\ \ 1 \le i \le n \end{cases} \tag{5}$$

Model formulation

In this section, we formulated the model of interest applicable to the problem under investigation. We took into consideration a uniformly mixed working-class population with a size of N, split into eight compartments that are mutually exclusive. There are four basic classes in the population: those who are susceptible, those who are infectious, those in the treatment class, and those with full-blown AIDS and sufficiently altered their sexual behaviours to be, in a sense, immune to HIV infection through their life. We then segmented each of the infectious and susceptible groups into productive and non-productive subclasses (with subscripts p and n denoting productive and non-productive, respectively) to examine the dynamics of productivity changes in the workforce. The entire population $N(t)$ is split into eight compartments: $S_p(t)$ denotes the number of susceptible productive individuals; $S_n(t)$ denotes the number of susceptible non-productive individuals; $I_p(t)$ denotes the number of HIV-positive but productive individuals in the stage of HIV infection; $I_n(t)$ denotes the number of HIV-positive but nonproductive individuals in the stage of HIV infection; $T(t)$ denotes the number of individuals undergoing treatment; $R(t)$ denotes the number of people who have sufficiently altered their sexual behaviors to be, in a sense, immune to HIV infection through safe sexual intercourse; and $A(t)$ reflects the number of people with full-blown AIDS. All other parameters in the model were given in

Table *1*. Susceptible infected individuals were in contact to infected individuals. AIDS treatment is crucial even if there is no known cure for it. Therefore, we studied HIV treatment for the working-class population in the sense of productive and non-productive classes in this study. Following therapy, there is a decreased likelihood that someone will contract AIDS, and it also lowers the chance that future generations will contract it. It should be noted that the members of the $R(t)$ class are those who established and maintained lifelong safe sexual behaviors. The total population $N(t)$ is given by $N = S_p(t) + S_n(t) + I_p(t) + I_n(t) + T(t) + A(t) + R(t)$

Table 1: Parameter descriptions

Parameter	Description
$\mathbf{1-Q}$	Fraction of new populations recruits in the S_p class
$\boldsymbol{Q}$	Fraction of new populations recruits in the S_n class
$\boldsymbol{\rho}$	Rate of recruitment
$\boldsymbol{a_1}$	Rate at which non-productive susceptible become productive
$\boldsymbol{a_{10}}$	Rate at which non-productive recovered produces
$\boldsymbol{a_4}$	Rate at which non-productive Infectives go to Productive class
$\boldsymbol{\beta}$	Contact rate between Susceptible and Infectives
$\boldsymbol{a_2}$	Rate at which productive Susceptible become Productive recovered
$\boldsymbol{a_3}$	Rate at which non-Productive susceptible become non-productive recovered
$\boldsymbol{a_6}$	Rate of progression of the I_p class into Treatments
$\boldsymbol{a_5}$	Rate of progression of the I_n class into Treatments
$\boldsymbol{\mu}$	Natural death rate
$\boldsymbol{d}$	AIDS-induced death rate
$\boldsymbol{a_7}$	Rate of progression of the T class into A
$\boldsymbol{a_8}$	Rate of progression of the $I_p$class into A
$\boldsymbol{a_9}$	Rate of progression of the $I_n$class into A

The corresponding dynamical system of the model or the flow diagram (Fig. 1) is given as:

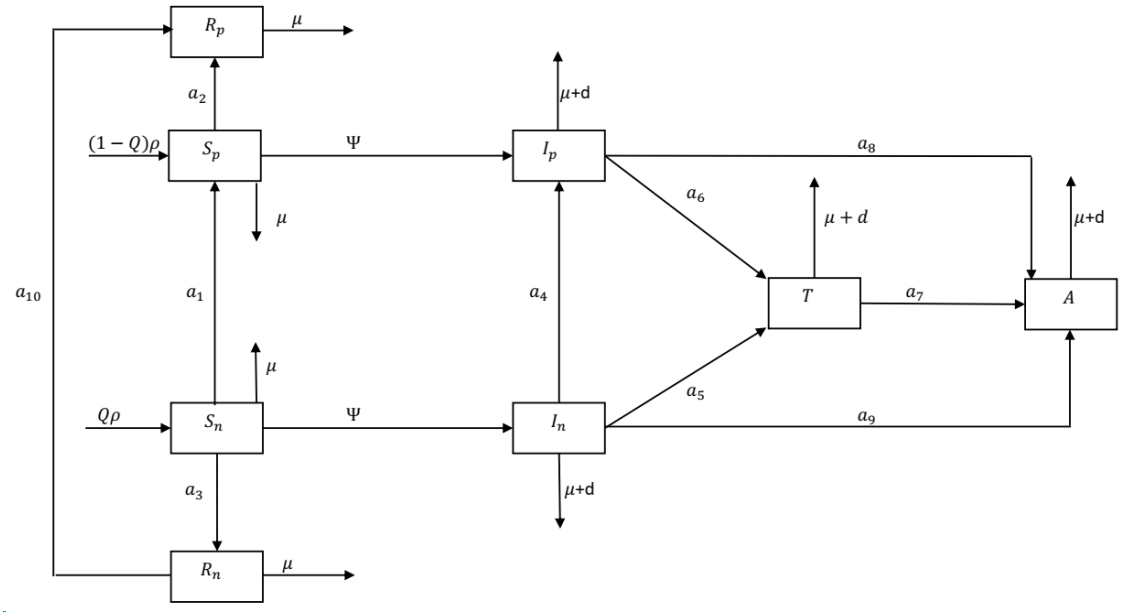

Figure 1: Flow diagram of the model

$$\begin{aligned}
\frac{dS_p}{dt} &= (1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu)S_p \\
\frac{dS_n}{dt} &= Q\rho - (\Psi + a_1 + a_3 + \mu)S_n \\
\frac{dI_p}{dt} &= \Psi S_p + a_4 I_n - (a_6 + a_8 + \mu + d)I_p \\
\frac{dI_n}{dt} &= \Psi S_n - (a_4 + a_5 + a_9 + \mu + d)I_n \\
\frac{dT}{dt} &= a_6 I_p + a_5 I_n - (a_7 + \mu + d)T \\
\frac{dA}{dt} &= a_7 T + a_8 I_p + a_9 I_n - (\mu + d)A \\
\frac{dR_p}{dt} &= a_2 S_p + a_{10} R_n - \mu R_p \\
\frac{dR_n}{dt} &= a_3 S_n - (a_{10} + \mu)R_n
\end{aligned} \tag{6}$$

Force of Infection is given by $\Psi = \beta \frac{I_p + I_n + A}{N}$. All initial values are non-negative, $S_p(0) > 0, S_n(0) > 0, I_p(0) > 0, I_n(0) > 0, A(0) > 0, T(0) > 0, R_p > 0, R_0 > 0$ and $t_0 > 0$ for all closed intervals $[0, t_0]$. Model analysis, well possessness, positivity, the existence of and uniqueness of solutions, the existence of equilibrium points and all necessary analyses of deterministic models are presented in the next section. Before extending the model to a piecewise model, the real-ability of the model can be improved, and numerical simulations can be performed. For simplification, we can write the system as $m_1 = a_2 + \mu, m_2 = a_1 + a_3 + \mu, m_3 = a_6 + a_8 + \mu + d, m_4 = a_4 + a_5 + a_9 + \mu + d, m_5 = a_7 + \mu + d, m_6 = \mu + d, m_7 = a_{10} + \mu$

$$\begin{aligned}
\frac{dS_p}{dt} &= (1-Q)\rho + a_1 S_n - (\Psi + \mathrm{m}_1)S_p \\
\frac{dS_n}{dt} &= Q\rho - (\Psi + \mathrm{m}_2)S_n \\
\frac{dI_p}{dt} &= \Psi S_p + a_4 I_n - \mathrm{m}_3 I_p \\
\frac{dI_n}{dt} &= \Psi S_n - \mathrm{m}_4 I_n \\
\frac{dT}{dt} &= a_6 I_p + a_5 I_n - \mathrm{m}_5 T \\
\frac{dA}{dt} &= a_7 T + a_8 I_p + a_9 I_n - \mathrm{m}_6 A \\
\frac{dR_p}{dt} &= a_2 S_p + a_{10} R_n - \mu R_p \\
\frac{dR_n}{dt} &= a_3 S_n - \mathrm{m}_7 R_n
\end{aligned} \qquad (7)$$

Mathematical model analysis of HIV/AIDS in the working-class population

Model properties

Theorem 1. (Kubra & Ali, 2023), The model(7) with the given initial conditions has non-negative solutions and the solution of the system will remain positive for all $t > 0$.

Proof: From the first equation of the model system (7)

$$\frac{dS_p}{dt} = (1-Q)\rho + a_1 S_n - (\Psi + \mathrm{m}_1)S_p \geq (\Psi + \mathrm{m}_1)S_p$$

Thus, we have

$$S_p(t) \geq S_P(0) \exp\left[-\int_0^t (\Psi + m_1)dt\right] > 0$$

From the second equation of the model system [7] we obtain

$$\frac{dS_n}{dt} = Q\rho - (\Psi + \mathrm{m}_2)S_n \geq (\Psi + \mathrm{m}_2)S_n$$

Hence,

$$S_n(t) \geq S_n(0) \exp\left[-\int_0^t (\Psi + m_2)\, dt\right] > 0$$

Similarly, all $I_p(t) > 0, I_n(t) > 0, A(t) > 0, T(t) > 0, R_p(t) > 0$, and $R_n(t) > 0$ for all $t > 0$.

Theorem 2. The model 6 solutions are uniformly bounded in the set

$$\mho = \{(S_p(t) + S_n(t) + I_p(t) + I_n(t) + A(t) + T(t) + R_p(t) + R_n(t)) \in R_+^8 | 0 \leq N \leq \frac{\rho}{\mu}\}$$

Proof: All parameters and initial conditions in the system 6 are assumed to be positive. The sum of equations system 6 gives

$$\frac{dN}{dt} = \frac{dS_p(t)}{dt} + \frac{dS_n(t)}{dt} + \frac{dI_p(t)}{dt} + \frac{dI_n(t)}{dt} + \frac{dT(t)}{dt} + \frac{dA(t)}{dt} + \frac{R(t)}{dt}$$

$$\frac{dN}{dt} = \rho - \mu N - d(I_p + I_n + T + A)$$

$$\frac{dN}{dt} < \rho - \mu N$$

The solutions of the differential equations are given by, $N(t) \leq \frac{\rho}{\mu} + \left(N_0 - \frac{\rho}{\mu}\right) e^{-\mu t}$

If $N_0 > \frac{\rho}{\mu}$, the solutions of N decrease to $\frac{\rho}{\mu}$, and if $N_0 < \frac{\rho}{\mu}$, then N approaches $\frac{\rho}{\mu}$ as t approaches infinity. The region $\mho$ is thus positively invariant, and the solutions are bounded. This means that every solution of (6) with initial conditions in $\mho$ remains in $\mho$ for all $t \geq 0$. The model is thus epidemiologically and mathematically well-posed in the region.

$\mho = \{(S_p(t) + S_n(t) + I_p(t) + I_n(t) + A(t) + T(t) + R_p(t) + R_n(t)) \in R_+^8 | 0 \leq N \leq \frac{\rho}{\mu}$

Equilibrium points

To determine the HIV/AIDS disease free equilibrium point, we solve the system $\frac{dS_p(t)}{dt} = 0, \frac{dS_n(t)}{dt} = 0, \frac{dI_p(t)}{dt} = 0, \frac{dI_n(t)}{dt} = 0, \frac{dT(t)}{dt} = 0, \frac{dA(t)}{dt} = 0, \frac{R_p(t)}{dt} = 0, \frac{R_n(t)}{dt} = 0$ and the infected class

$I_p = 0, I_n = 0, A = 0, T = 0$. Then the disease-free equilibrium point is equal to

$$DFE = \left(\frac{(1-Q)\rho + a_1 S_n^*}{a_2 + \mu}, \frac{Q\rho}{a_1 + a_3 + \mu}, 0,0,0,0, \frac{a_2 S_p^* + a_{10} R_n^*}{\mu}, \frac{a_3 S_n^*}{a_{10} + \mu}\right)$$

By using the next generation matrix, we obtain the reproduction number R_0,

$$R_0 = \beta \frac{S_n^0(a_7\, a_6 a_4 + a_5 a_7 m_3 + a_4 m_5(a_8 + m_6) + m_3 m_5(a_9 + m_6)) + S_p^0(a_6 a_7 m_4 + m_4 m_5(a_8 + m_6))}{N m_3 m_4 m_5 m_6} \quad (8)$$

Endemic equilibrium points of the model

$$S_n^* = \frac{Q\rho}{\Psi^* + m_2}, S_p^* = \frac{\rho(1-Q)(\Psi^* + m_2) + a_1 Q\rho}{(\Psi^* + m_1)(\Psi^* + m_2)}, I_p^* = \frac{\Psi^*}{m_3}\left(\frac{\rho(1-Q)(\Psi^* + m_2) + a_1 Q\rho}{(\Psi^* + m_1)(\Psi^* + m_2)}\right)$$

$I_n^* = \frac{\Psi^* S_n^* + a_4 I_p^*}{m_4}$, $T^* = \frac{a_6 I_p^* + a_5 I_n^*}{m_5}, A^* = \frac{a_7 T^* + a_8 I_P^* + a_9 I_n^*}{m_6}, R_n^* = a_3 \frac{S_n^*}{m_7}, R_p^* = \frac{a_2\ S_p^*\ + a_{10} R_n^*}{\mu}$

interms of force of infection $\Psi^* = \frac{I_p^* + I_n^* + A^*}{N}$

Stability of the disease-free equilibrium point by using a Jacobian matrix

The Jacobian matrix can be computed by taking partial derivatives of the right-hand side of the differential equations with respect to each compartment (Almeida et al., 2021). The Jacobian matrix at the DFE point is evaluated,

$$J(DFE)=\begin{bmatrix} -m_1 & a_1 & -\frac{S_p^0\beta}{N} & -\frac{S_p^0\beta}{N} & \frac{S_p^0\beta}{N} & 0 & 0 & 0 \\ 0 & -m_2 & -\frac{S_p^0\beta}{N} & -\frac{S_p^0\beta}{N} & -\frac{S_p^0\beta}{N} & 0 & 0 & 0 \\ 0 & 0 & -m_3+\frac{S_p^0\beta}{N} & a_4+\frac{S_p^0\beta}{N} & \frac{S_p^0\beta}{N} & 0 & 0 & 0 \\ 0 & 0 & \frac{S_p^0\beta}{N} & -m_4+\frac{S_p^0\beta}{N} & \frac{S_P^0\beta}{N} & 0 & 0 & 0 \\ 0 & 0 & a_8 & a_9 & -m_6 & a_7 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & 0 & -m_5 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & -\mu & a_{10} \\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 & -m_7 \end{bmatrix}$$

From this, we have the eigenvalues$\lambda_8=-m_7\,,\lambda_7-\mu,\lambda_1=-m_1,\lambda_2=-m_2$. Hence the Jacobian matrix is reduced to $J^*=\begin{bmatrix} -m_3+\frac{S_p^0\beta}{N} & a_4+\frac{S_p^0\beta}{N} & \frac{S_p^0\beta}{N} & 0 \\ \frac{S_p^0\beta}{N} & -m_4+\frac{S_p^0\beta}{N} & \frac{S_p^0\beta}{N} & 0 \\ a_8 & a_9 & -m_6 & a_7 \\ a_6 & a_5 & 0 & -m_5 \end{bmatrix}$

The determinant of the characteristic equations is given as

$$Det(J)=1-\beta\left(\frac{S_n^0\left(a_7\,a_6a_4+a_5a_7m_3+a_4m_5(a_8+m_6)+m_3m_{5(a_9+m_6)}\right)}{Nm_3m_4m_5m_6}\right.$$
$$\left.-\beta\frac{S_p^0\left(a_6a_7m_4+m_4m_5(a_8+m_6)\right)}{Nm_3m_4m_5m_6}=1-R_0\right. \qquad (9)$$

The trace of the Jacobian matrix is $Trace(J)=-m_3-m_4-m_5-m_6+\frac{S_n\beta}{N}+\frac{Sp\beta}{N}$ and the characteristic polynomial of the sub-matrix is given as

$$\lambda^4 + (m_3 + m_4 + m_5 + m_6 - \beta(S_n + S_p)/N\,\lambda^3 + (m_3 m_4 + m_3 m_5 + m_3 m_6 + m_4 m_5 + m_4 m_6 + m_5 m_6 - S_n \beta(a_4 + a_9 + m_3 + m_5 + m_6) - Sp(\beta(a_8 + m_4 + m_5 + m_6)\lambda^2)/N + (m_3 m_4 m_5 + m_3 m_4 m_6 + m_3 m_5 m_6 + m_4 m_5 m_6 - Sn\beta(a_4 a_8 + a_4 m_5 + a_4 m_6 + a_5 a_7 + a_9 m_3 + a_9 m_5 + m_3 m_5 + m_3 m_6 + m_5 m_6) - Sp\beta(a_6 a_7 + a_8 m_4 + a_8 m_5 + m_4 m_5 + m_4 m_6 + m_5 m_6)\lambda/N + (N m_3 m_4 m_5 m_6 - Sn\beta(a_4 a_6 a_7 + a_4 a_8 m_5 + a_4 m_5 m_6 + a_5 a_7 m_3 + a_9 m_3 m_5 + m_3 m_5 m_6) - Sp\beta(a_6 a_7 m_4 + a_8 m_4 m_5 + m_4 m_5 m_6)/N$$

Then, the characteristic polynomial equations are given as

$$P(\lambda) = b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0$$

were,

$$b_4 = 1$$

$$b_3 = m_3 + m_4 + m_5 + m_6 - \frac{\beta(S_n+S_p)}{N}$$

$$b_2 = (m_3m_4 + m_3 m_5 + m_3 m_6 + m_4 m_5 + m_4 m_6 + m_5 m_6 - S_n \beta(a_4 + a_9 + m_3 + m_5 + m_6) - Sp\frac{\beta(a_8+ m_4+ m_5 + m_6)}{N}$$

$$b_1 = (m_3 m_4 m_5 + m_3 m_4 m_6 + m_3 m_5 m_6 + m_4 m_5 m_6 - Sn\beta(a_4a_8 + a_4m_5 + a_4m_6 + a_5a_7 + a_9m_3 + a_9m_5 + m_3 m_5 + m_3 m_6 + m_5 m_6) - Sp\beta\frac{a_6a_7+a_8m_4+ a_8m_5+m_4 m_5 + m_4 m_6 + m_5 m_6}{N}$$

$$b_0 = (N m_3 m_4 m_5 m_6 - S_n\beta(a_4a_6a_7 + a_4a_8m_5 + a_4m_5m_6 + a_5a_7m_3 + a_9 m_3 m_5 + m_3m_5m_6) - S_p\beta\frac{a_6a_7m_4+ a_8m_4 m_5+ m_4 m_5 m_6}{N} = 1 - R_0$$

The condition $R_0 < 1$ should be equivalent to the condition $b_0 > 0$. Hence, we define R_0 as follows:

$$R_0 = \beta\frac{S_n^0(a_7 a_6a_4+a_5a_7m_3+a_4m_5(a_8+m_6)+m_3m_5(a_9+m_6))+S_p^0(a_6a_7m_4+m_4m_5(a_8+m_6))}{Nm_3m_4m_5m_6}$$

(10)

The conditions $R_0 < 1$ and $b_0 > 0$ are equivalent. In addition, $b\_0 < 0$ if $R_0 > 1$. The equation $p(\lambda) = 0$ has a real positive solution because $\lim_{\lambda\to\infty} p(\lambda) = \infty$, and as a result, the disease-free equilibrium is unstable. It is still to be shown if the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$.

Sensitivity analysis of parameter parameters to the reproduction number

The extent to which the value of the reproduction number (R_0) fluctuates in reaction to changes in particular parameters or variables is determined by the sensitivity index of R_0. It is useful to know which variables most significantly affect how a disease spread. The explicit expression of R_0 is given by equation 10, and we derive an analytical expression for its sensitivity to each parameter in R_0.The normalized forward sensitivity index of R_0, which depends on a parameter p, is defined as:

$$SI_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}$$

These sensitivity indices allow us to determine the relative importance of different parameters in HIV/AIDS transmission and prevalence. We use the values in Table 3; then, we obtain the results in the sensitivity indices values.

Table 2 and Figure 2 according to the sensitivity indices values.

Table 2: Sensitivity indices of parameters

Parameters	Sensitivity index	Parameters	Sensitivity index
$\boldsymbol{\beta}$	1	a_6	0.0307
$\boldsymbol{a_1}$	-0.4638	a_7	-0.0183
$\boldsymbol{a_2}$	-0.2031	a_8	-0.0398
$\boldsymbol{a_3}$	0.0401	a_9	0.2039
$\boldsymbol{a_4}$	-0.2775	d	-0.4168
$\boldsymbol{a_5}$	0.1227		

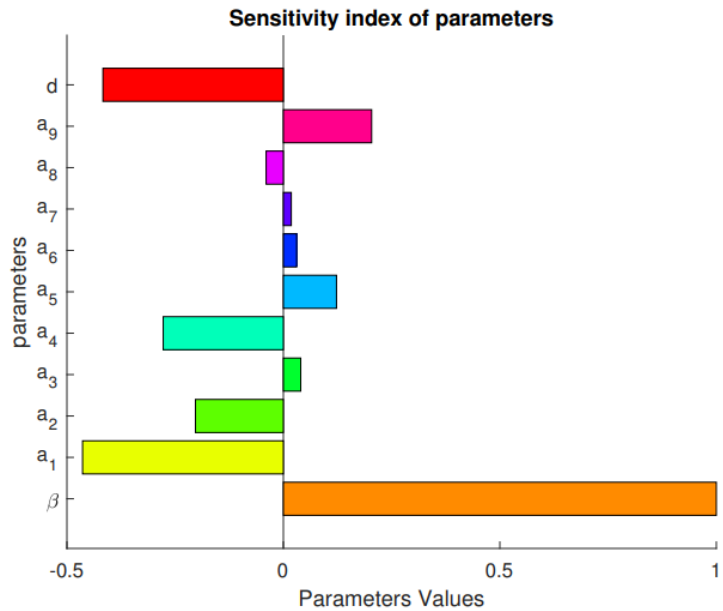

Figure 2: Sensitivity indices of parameter

Fractal-Fractional version of the model

$$
\begin{aligned}
{}^{ABC}_{0}D_t^{\omega,\xi} S_p &= (1-\mathrm{Q})\rho\ + \mathrm{a_1 S_n} - (\Psi + \mathrm{a_2} + \mu)\mathrm{S_p} \\
{}^{ABC}_{0}D_t^{\omega,\xi} S_n &= \ Q\rho - (\Psi\ + a_1 + a_3 + \mu)S_n \\
{}^{ABC}_{0}D_t^{\omega,\xi} I_p &= \Psi\, S_p\ + a_4 I_n - (a_6 + a_8 + \mu)I_p \\
{}^{ABC}_{0}D_t^{\omega,\xi} I_n &= \Psi\, S_n - (a_4 +\ a_5 + a_9 + \mu)I_n \\
{}^{ABC}_{0}D_t^{\omega,\xi} T &= a_6\, I_p + a_5\, I_n - (a_7 + \mu + d)T
\end{aligned}
$$

$$
\begin{aligned}
{}^{ABC}_{0}D_t^{\omega,\xi} A &= a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d)A \\
{}^{ABC}_{0}D_t^{\omega,\xi} R_p &= a_2\, S_p + a_{10} R_n - \mu\, R_p \\
{}^{ABC}_{0}D_t^{\omega,\xi} R_n &= a_3 S_n - (a_{10} + \mu) R_n
\end{aligned}
$$

All initial values are positive, $S_p(0) \geq 0, S_n(0) \geq 0, I_p(0) \geq 0, I_n(0) \geq 0, A(0) \geq 0, T(0) \geq 0, R_p \geq 0, R_0 \geq 0$ and $t_0 > 0$ for all closed intervals $[0, t_0]$.

Stochastic version of the Model

$$
\begin{aligned}
dS_p &= ((1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu) S_p)dt + v_1(S_p)dB_1(t) \\
dS_n &= (Q\rho - (\Psi + a_1 + a_3 + \mu) S_n)dt + v_2(S_n)dB_2(t) \\
dI_p &= (\Psi\, S_p + a_4 I_n - (a_6 + a_8 + \mu) I_p)dt + v_3(I_p)dB_3(t) \\
dI_n &= (\Psi\, S_n - (a_4 + a_5 + a_9 + \mu) I_n)dt + v_4(I_n)dB_4(t) \\
dT &= (a_6\, I_p + a_5\, I_n - (a_7 + \mu + d)T)dt + v_5(A)dB_5(t) \\
dA &= (a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d)A)dt + v_6(A)dB_6(t) \\
dR_p &= (a_2\, S_p + a_{10} R_n - \mu\, R_p)dt + v_7(A_{pp})dB_7(t) \\
dR_n &= (a_3 S_n - (a_{10} + \mu) R_n)dt + v_8(A_{pn})dB_8(t)
\end{aligned}
$$

All initial values are positive, $S_p(0) > 0, S_n(0) > 0, I_p(0) > 0, I_n(0) > 0, A(0) > 0, T(0) > 0, R_p > 0, R_0 > 0$ and $t_0 > 0$ for all closed intervals $[0, t_0]$. Stochastic models are used to maintain the randomness of disease.

Piecewise mathematical modelling formulation
From deterministic to random process

For the following time interval and initial conditions, our model is deterministic.

For $0 \leq t \leq t_1$, $S_p(0) = S_{p11}, S_n(0) = S_{n12}, I_p(0) = I_{p13}, I_n(0) = I_{n14}, R_p = R_{p15}, R_n(0) = R_{n16}, A(0) = A_{17}$

$$
\begin{aligned}
\frac{dS_p}{dt} &= (1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu) S_p \\
\frac{dS_n}{dt} &= Q\rho - (\Psi + a_1 + a_3 + \mu) S_n \\
\frac{dI_p}{dt} &= \Psi\, S_p + a_4 I_n - (a_6 + a_8 + \mu + d) I_p \qquad (13) \\
\frac{dI_n}{dt} &= \Psi\, S_n - (a_4 + a_5 + a_9 + \mu + d) I_n \\
\frac{dT}{dt} &= a_6\, I_p + a_5\, I_n - (a_7 + \mu + d) T \\
\frac{dA}{dt} &= a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d) A \\
\frac{dR_p}{dt} &= a_2\, S_p + a_{10} R_n - \mu\, R_p \\
\frac{dR_n}{dt} &= a_3 S_n - (a_{10} + \mu) R_n
\end{aligned}
$$

For $t_1 \leq t < T$, $S_p(t_1) = S_{p21}, S_n(t_1) = S_{n22}, I_p(t_1) = I_{p23}, I_n(t_1) = I_{n24}, R_p(t_1) = R_{p25}, R_n(t_1) = R_{n26}, A(t_1) = A_{27}$

$$\begin{aligned}
dS_p &= ((1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu)S_p)dt + v_1(S_p)dB_1(t)\\
dS_n &= (Q\rho - (\Psi + a_1 + a_3 + \mu)S_n)dt + v_2(S_n)dB_2(t)\\
dI_p &= (\Psi S_p + a_4 I_n - (a_6 + a_8 + \mu)I_p)dt + v_3(I_p)dB_3(t)\\
dI_n &= (\Psi S_n - (a_4 + a_5 + a_9 + \mu)I_n)dt + v_4(I_n)dB_4(t)\\
dT &= (a_6 I_p + a_5 I_n - (a_7 + \mu + d)T +)dt + v_5(A)dB_5(t)\\
dA &= (a_7 T + a_8 I_p + a_9 I_n - (\mu + d)A)dt + v_6(A)dB_6(t)\\
dR_p &= (a_2 S_p + a_{10} R_n - \mu R_p)dt + v_7(A_{pp})dB_7(t)\\
dR_n &= (a_3 S_n - (a_{10} + \mu)R_n)dt + v_8(A_{pn})dB_8(t)
\end{aligned} \tag{14}$$

The mathematical model 13 has a deterministic character and is extended to stochastic models described in 14 by adding white noise type perturbations to the system. The parameters $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 are positive constants and the intensities of the random disturbance. $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t), B_5(t), B_6(t), B_7(t), B_8(t))$ is the white noise process.

From fractal-fractional to random process

For $0 \leq t \leq t_1$, $S_p(0) = S_{p11}, S_n(0) = S_{n12}, I_p(0) = I_{p13}, I_n(0) = I_{n14}, T = T_{15}, A(0) = A_{16}, R_p(0) = R_{p17}, R_n(0) = R_{n18}$

$$\begin{aligned}
{}^{ABC}_{0}D_t^{\omega,\xi} S_p &= (1-\mathrm{Q})\rho + \mathrm{a_1 S_n} - (\Psi + \mathrm{a_2} + \mu)\mathrm{S_p}\\
{}^{ABC}_{0}D_t^{\omega,\xi} S_n &= Q\rho - (\Psi + a_1 + a_3 + \mu)S_n\\
{}^{ABC}_{0}D_t^{\omega,\xi} I_p &= \Psi S_p + a_4 I_n - (a_6 + a_8 + \mu)I_p\\
{}^{ABC}_{0}D_t^{\omega,\xi} I_n &= \Psi S_n - (a_4 + a_5 + a_9 + \mu)I_n\\
{}^{ABC}_{0}D_t^{\omega,\xi} T &= a_6 I_p + a_5 I_n - (a_7 + \mu + d)T\\
{}^{ABC}_{0}D_t^{\omega,\xi} A &= a_7 T + a_8 I_p + a_9 I_n - (\mu + d)A\\
{}^{ABC}_{0}D_t^{\omega,\xi} R_p &= a_2 S_p + a_{10} R_n - \mu R_p\\
{}^{ABC}_{0}D_t^{\omega,\xi} R_n &= a_3 S_n - (a_{10} + \mu)R_n
\end{aligned} \tag{15}$$

We assume that stochastic perturbations of the variables around their interior equilibrium points are of the white noise type, which is proportional to the distances of $S_p, S_n, I_p, I_n, T, A, R_p, R_n$ from values $S_p^*, S_n^*, I_p^*, I_n^*, T^*, A^*, R_p^*, R_n^*$. Thus, system of model in 15 results in the following stochastic model.

For $t_1 \leq t < t_2$, $S_p(t_1) = S_{p21}, S_n(t_1) = S_{n22}, I_p(t_1) = I_{p23}, I_n(t_1) = I_{n24}, T(t_1) = T_{25},, A(t_1) = A_{26}, R_p(t_1) = R_{p27}, R_n(t_1) = R_{n28}$

$$\begin{aligned}
dS_p &= ((1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu)S_p)dt + v_1(S_p)dB_1(t)\\
dS_n &= (Q\rho - (\Psi + a_1 + a_3 + \mu)S_n)dt + v_2(S_n)dB_2(t)\\
dI_p &= (\Psi S_p + a_4 I_n - (a_6 + a_8 + \mu)I_p)dt + v_3(I_p)dB_3(t)\\
dI_n &= (\Psi S_n - (a_4 + a_5 + a_9 + \mu)I_n)dt + v_4(I_n)dB_4(t)\\
dT &= (a_6 I_p + a_5 I_n - (a_7 + \mu + d)T +)dt + v_5(A)dB_5(t)\\
dA &= (a_7 T + a_8 I_p + a_9 I_n - (\mu + d)A)dt + v_6(A)dB_6(t)
\end{aligned}$$

$$dR_p = \left(a_2\, S_p + a_{10} R_n - \mu\, R_p\right)dt + v_7\left(A_{pp}\right)dB_7(t)$$
$$dR_n = (a_3 S_n - (a_{10} + \mu)R_n)dt + v_8\left(A_{pn}\right)dB_8(t)$$

Deterministic fractal-fractional-stochastic model

For the following time interval and initial conditions, our model is deterministic:

For $0 \leq t \leq t_1$, $S_p(0) = S_{p11}, S_n(0) = S_{n12}, I_p(0) = I_{p13}, I_n(0) = I_{n14}, R_p = R_{p15}, R_n(0) = R_{n16}, A(0) = A_{17}$

$$\frac{dS_p}{dt} = (1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu)S_p$$
$$\frac{dS_n}{dt} = Q\rho - (\Psi + a_1 + a_3 + \mu)S_n$$
$$\frac{dI_p}{dt} = \Psi\, S_p + a_4 I_n - (a_6 + a_8 + \mu + d)I_p$$
$$\frac{dI_n}{dt} = \Psi\, S_n - (a_4 + a_5 + a_9 + \mu + d)I_n$$
$$\frac{dT}{dt} = a_6\, I_p + a_5\, I_n - (a_7 + \mu + d)T$$
$$\frac{dA}{dt} = a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d)A$$
$$\frac{dR_p}{dt} = a_2\, S_p + a_{10} R_n - \mu\, R_p$$
$$\frac{dR_n}{dt} = a_3 S_n - (a_{10} + \mu)R_n$$

For $t_1 \leq t < t_2$, $S_p(t_1) = S_{p21}, S_n(t_1) = S_{n22}, I_p(t_1) = I_{p23}, I_n(t_1) = I_{n24}, T(t_1) = T_{25},, A(t_1) = A_{26}, R_p(t_1) = R_{p27}, R_n(t_1) = R_{n28}$, our model is fractal fractional order model with ABC operator.

$$\begin{aligned}
{}^{ABC}_{0}D_t^{\omega,\xi} S_p &= (1-\mathrm{Q})\rho + \mathrm{a_1 S_n} - (\Psi + \mathrm{a_2} + \mu)\mathrm{S_p} \\
{}^{ABC}_{0}D_t^{\omega,\xi} S_n &= Q\rho - (\Psi + a_1 + a_3 + \mu)S_n \\
{}^{ABC}_{0}D_t^{\omega,\xi} I_p &= \Psi\, S_p + a_4 I_n - (a_6 + a_8 + \mu)I_p \\
{}^{ABC}_{0}D_t^{\omega,\xi} I_n &= \Psi\, S_n - (a_4 + a_5 + a_9 + \mu)I_n \\
{}^{ABC}_{0}D_t^{\omega,\xi} T &= a_6\, I_p + a_5\, I_n - (a_7 + \mu + d)T \\
{}^{ABC}_{0}D_t^{\omega,\xi} A &= a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d)A \\
{}^{ABC}_{0}D_t^{\omega,\xi} R_p &= a_2\, S_p + a_{10} R_n - \mu\, R_p \\
{}^{ABC}_{0}D_t^{\omega,\xi} R_n &= a_3 S_n - (a_{10} + \mu)R_n
\end{aligned} \tag{15}$$

For $t_2 \leq t < T$, $S_p(t_2) = S_{p31}, S_n(t_2) = S_{n32}, I_p(t_2) = I_{p33}, I_n(t_2) = I_{n34}, T(t_2) = T_{35},, A(t_2) = A_{36}, R_p(t_2) = R_{p37}, R_n(t_2) = R_{n38}$ the model transformed to stochastic model.

$$dS_p = ((1-Q)\rho + a_1 S_n - (\Psi + a_2 + \mu)S_p)dt + v_1\left(S_p\right)dB_1(t)$$
$$dS_n = (Q\rho - (\Psi + a_1 + a_3 + \mu)S_n)dt + v_2(S_n)dB_2(t)$$
$$dI_p = \left(\Psi\, S_p + a_4 I_n - (a_6 + a_8 + \mu)I_p\right)dt + v_3\left(I_p\right)dB_3(t)$$
$$dI_n = (\Psi\, S_n - (a_4 + a_5 + a_9 + \mu)I_n)dt + v_4(I_n)dB_4(t)$$
$$dT = \left(a_6 I_p + a_5\, I_n - (a_7 + \mu + d)T + \right)dt + v_5(A)dB_5(t)$$

$$dA = \left(a_7\, T + a_8\, I_p + a_9\, I_n - (\mu + d)A\right)dt + v_6(A)dB_6(t)$$
$$dR_p = \left(a_2\, S_p + a_{10} R_n - \mu\, R_p\right)dt + v_7\left(A_{pp}\right)dB_7(t)$$
$$dR_n = (a_3 S_n - (a_{10} + \mu)R_n)dt + v_8\left(A_{pn}\right)dB_8(t)$$

Numerical Simulation

The values of the parameters used in the numerical simulations are estimated from the data for Ethiopia estimated by UNAIDS for twenty-three years. MATLAB R2020a was used to fit the data via least squares methods (Martcheva, 2015). We fit the data to the model with a minimum error of $9.5964e-04$, and the estimated parameter values are given in Table 3.

Table 3: Parameter values and descriptions

Parameter	values	sources	Parameter	values	Sources
$1-Q$	0.500	Estimated	a_3	0.0734	Fitted
Q	0.500	Calculated	a_6	0.0368	Fitted
ρ	0.0758	Fitted	a_5	0.0368	Fitted
a_1	0.140	Fitted	μ	0.0148	Calculated
a_{10}	0.020	Fitted	d	0.100	Estimated
a_4	0.1400	Fitted	a_7	0.007	Fitted
β	0.1017	Fitted	a_8	0.0012	Fitted
a_2	0.0734	Fitted	a_9	0.0012	Fitted

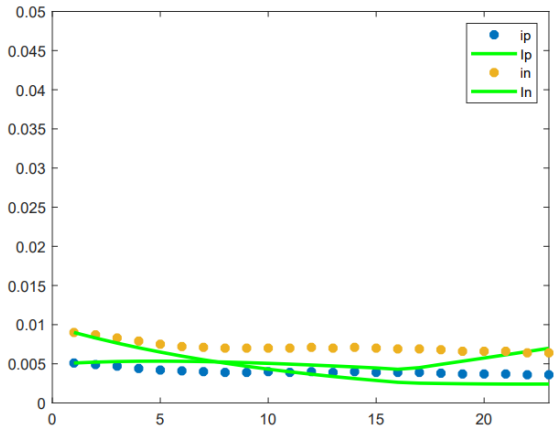

Figure 3: Actual data fitted to the model to estimate the parameters in Table 3

Numerical methods for solving the deterministic model

The deterministic model version was solved using the embedded Runge Kutta order four and five in MATLAB ode45 with the parameters in Table 3. Figure 4 shows that the recovered

productive population increases above the susceptible population after fifteen years, which is unrealistic.

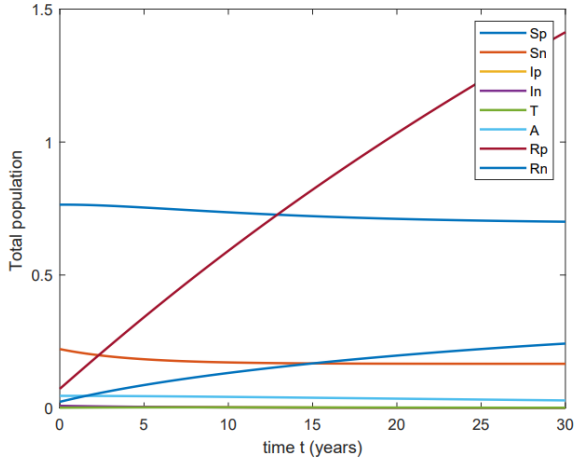

Figure 4: Solution of the deterministic HIV/AIDS model

Numerical methods for solving fractal-fractional order models

The fractal-fractional model of the HIV/AIDS model based on the working-class population in Ethiopia was solved by using the method of (Atangana & Qureshi, 2019). We plot *Figure 5* by the parameter in Table3, with the fractal dimension ($\tau = 0.8$) and fractional order ($\eta = 0.8$). Unlike the deterministic model, the recovered productive class of the population is less than the susceptible productive class of the population, even after thirty years. We understand that the fractal-fractional model is more realistic than the deterministic model.

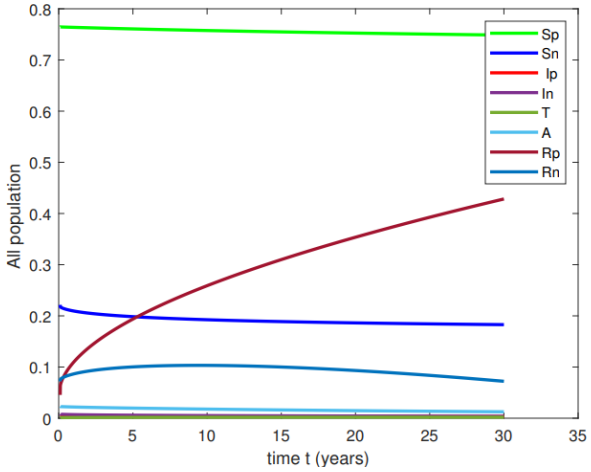

Figure 5: Fractal-fractional HIV/AIDS model solution

Numerical scheme for solving the stochastic model

The Euler-Maruyama method (Huang et al., 2023) is a simple and widely used numerical scheme for solving stochastic differential equations (SDEs) of the form:

$$dX(t) = f(X(t), t)\, dt + g(X(t), t)\, dW(t)$$

where $X(t)$ is the unknown solution, f and g are given functions, and $W(t)$ is a Wiener process. The idea of the method is to discretize the time interval $[0, T]$ into N equal subintervals of length $\Delta t = \frac{T}{N}$ and approximate the solution $X(t)$ at the points $t_n = n\,\Delta t, n = 0, 1, \ldots, N$. The method is based on the following approximation:

$$X(t_{n+1}) \approx X(t_n) + f(X(t_n), t_n)\,\Delta t + g(X(t_n), t_n)\,\Delta W_n$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ is the increment of the Wiener process, which can be sampled from a normal distribution with mean zero and variance Δt. The Euler-Maruyama method then defines the numerical solution X_n as:

$$X_{n+1} = X_n + f(X_n, t_n)\,\Delta t + g(X_n, t_n)\,\Delta W_n$$

with a given initial condition $X_0 = X(0)$. This recursive formula can be implemented in a loop to generate the numerical solution. The Euler-Maruyama method is easy to implement and has a strong convergence order of 0.5, which means that the error between the true solution and the numerical solution is proportional to $\sqrt{\Delta t}$ as Δt approaches zero (Higham, 2001; Wu, 2023)

By using the selected methods, we find the solution of the stochastic model, as shown in Figure 6. In the stochastic version of the model, the susceptible productive and recovered productive classes of the population cross each other for approximately ten years, but the productive recovered class of the population looks like a sigmoid graph. The fully blown AIDS class population approaches zero very quickly when we compare it with the above two versions of the models.

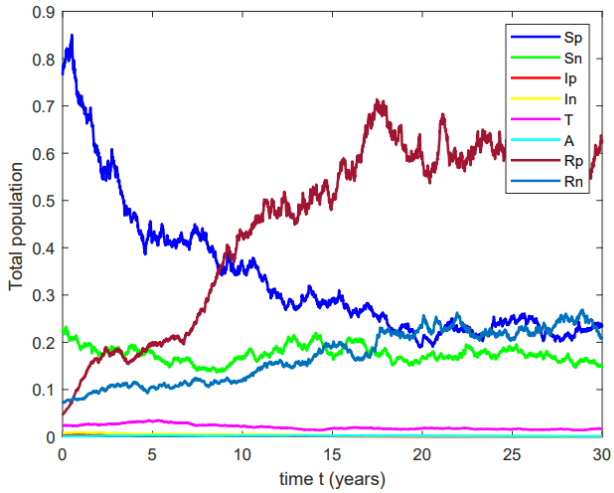

Figure 6: Solution of the stochastic HIV/AIDS model

Simulations of deterministic model to stochastic model

In this simulation, the deterministic characteristics and assumptions of the model were applied for fifteen years, followed by the stochastic characteristics and assumptions for the other fifteen years.

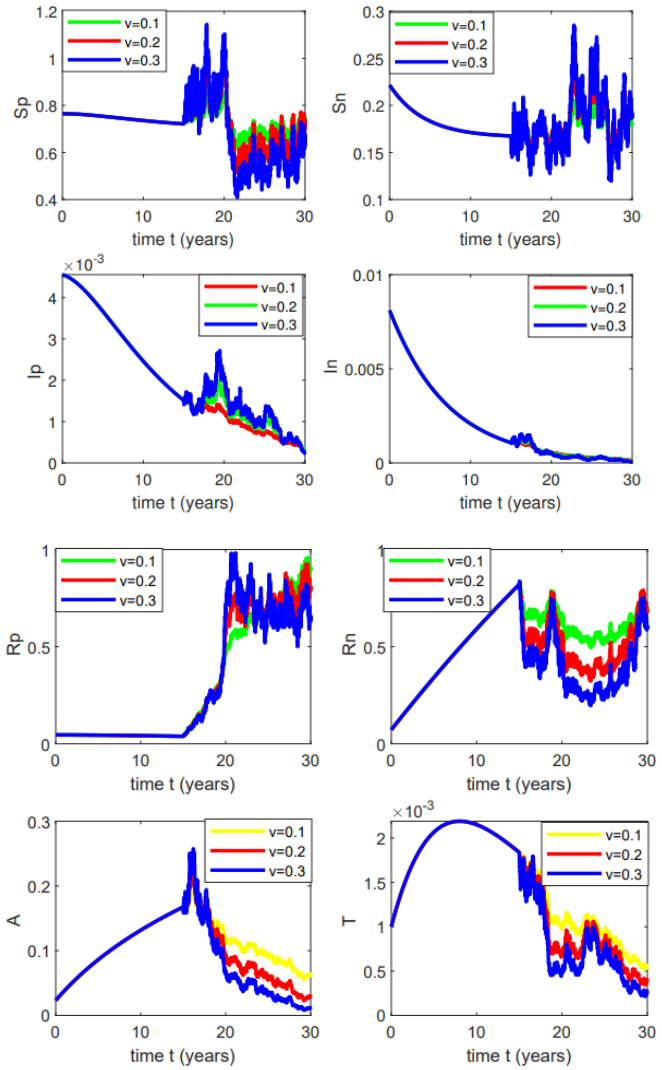

Figure 7: Deterministic to stochastic distributions with different values of the intensities of the random disturbance

Here, our model changes from a deterministic model to a stochastic model. As shown in Figure 7, the number of susceptible individuals decreased and then increased. The number of Recovered populations of both productive and non-productive population increases and reduces the number of susceptible populations. The AIDS class decreases and approaches zero.

Simulations of fractal-fractional order ABC model to stochastic model

In this case, the model changed from the fractal-fractional order model to stochastic by losing memory effects and holding the random process of epidemic nature.

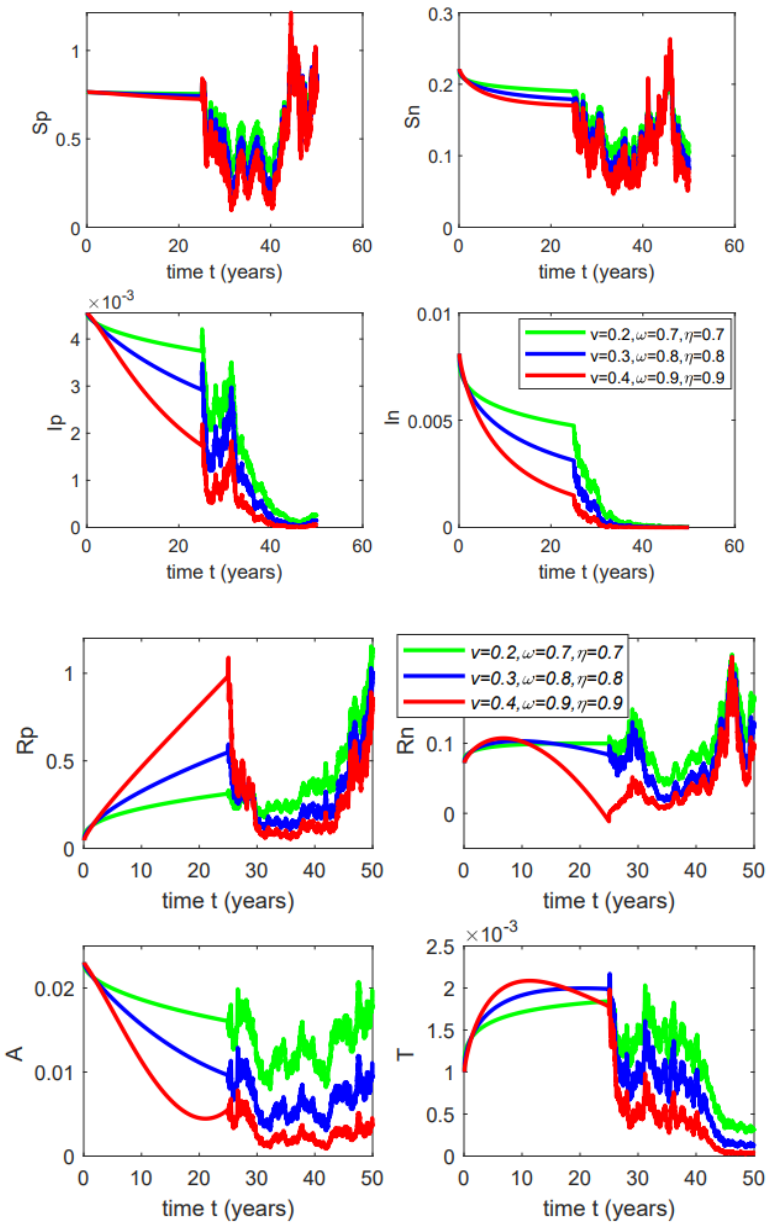

Figure 8: From fractal fractional to stochastic, with different values of fractional order and fractal dimension, including different intensities of random disturbance

The piecewise model explains the problem and shows the chance of disease extinction. In Figure 8, the numbers of susceptible productive, infected productive, and pre-AIDS productive populations increased. However, the number of susceptible non-productive, infected non-productive, and pre-AIDS non-productive decreased in a stochastic model where the full-blown AIDS class approaches zero.

Transformation of the model from deterministic-fractal-fractional ABC-Stochastic process

The trajectory of the dynamical systems changes from one form of model to another, which increases the relative accuracy of the representation of a real problem by using a mathematical model. Here, we solve the group of three interconnected models via piecewise mathematical modeling. For the first fifteen years, we used a deterministic model; for the next fifteen years, we used a fractal-fractional order model; and for the last thirty years, we used a stochastic model, including the randomness property of HIV/AIDS infection. Infected productive and non-productive class population decrease due to treatments as shown in Figure 9.

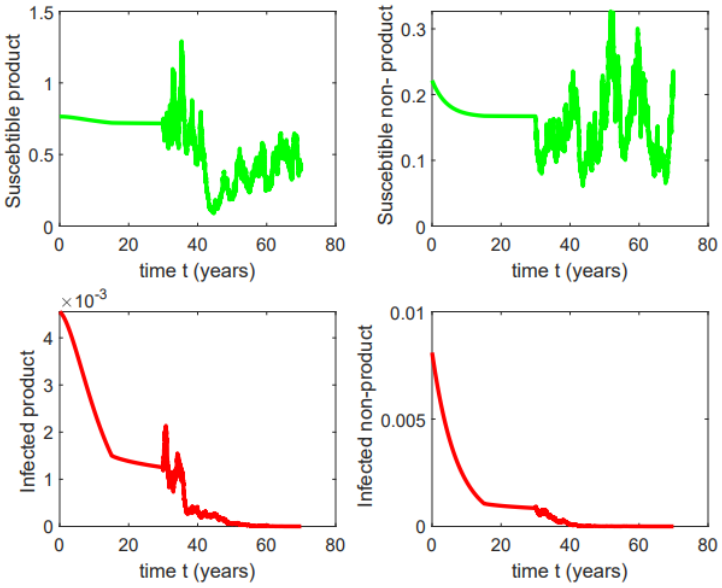

Figure 9: Migration of the system from deterministic to fractal-fractional order then to stochastic

As shown in Figure 10, the nature of HIV/AIDS in the population approaches zero after fifty years from the starting point of our data (2000). According to these numerical simulations and as per plan of World Health Organizations both infected and full-blown AIDS class population approaches to zero.

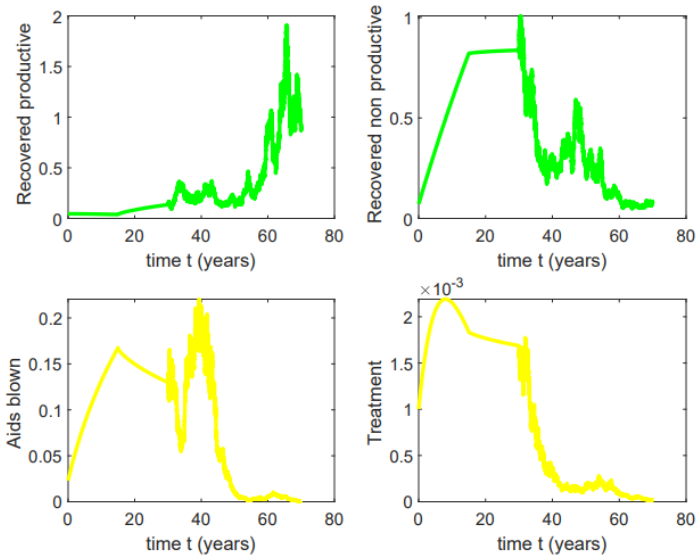

Figure 10: Migration of the system from deterministic to fractal-fractional order then to stochastic

The effect of contact rate

HIV/AIDS infection is transmitted from infected HIV/AIDS patients to susceptible individuals through contact via sexual intercourse made without any protection. Increasing the contact rate increases the number of individuals infected with HIV/AIDS, and decreasing the contact rate reduces the number of susceptible individuals. Accordingly, in Figure 11, decreasing the contact rate before the estimated time and increasing the contact rate prolonged the estimated time, as shown in Figure 10.

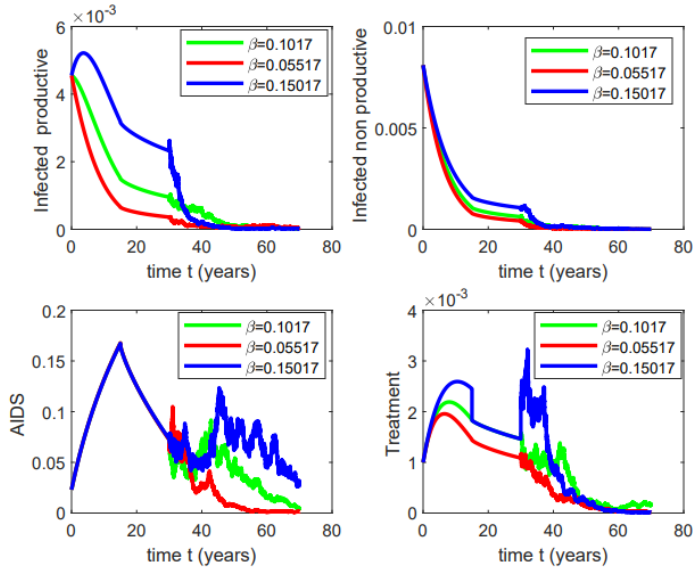

Figure 11: Effect of contact rate to end the HIV/AIDS epidemic

The effect of HIV/AIDS on working class

In this section, we show the effects of decreasing non-productivity and increasing productivity to control the spread and transmission of HIV/AIDS. As the rate at which non-productive classes become productive increases (Figure 12), the productivity force of the population increases, and the non-productive class of the population decreases and is inversely proportional to the AIDS class. Increasing productivity of working class reduce the number of populations infected by HIV/AIDS infections.

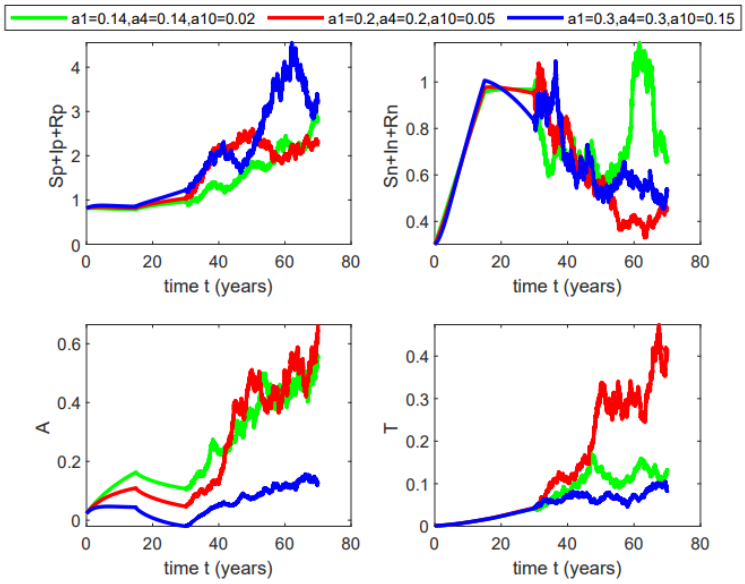

Figure 12: Effect of the rate at which the non-productive class becomes productive

CONCLUSION

In this paper, we examined HIV/AIDS models that are based on the Ethiopian working-class population and applied the concept of piecewise operators. Thus far, research has been conducted on the use of piecewise operators for stochastic models in epidemiology and deterministic, fractal-fractional Atangana-Baleanu-Caputo sense models. Additionally, several corresponding numerical simulations have been created for various parameter values. It is confirmed that the disease-free equilibrium point is regionally asymptotically stable. This novel piecewise model takes into account data fitting for working-class HIV/AIDS cases for the period 2000-2022 G.C. The numerical simulation results graphically show that increasing the contact rate increases the number of individuals infected with HIV/AIDS, and decreasing the contact rate reduces the number of infected individuals, decreasing the contact rate eradicate HIV/AIDS before the estimated time to end HIV/AIDS infection. And increasing the contact rate prolonged the HIV/AIDS more than time estimated by WHO. As the rate at which non-productive classes become productive increases, the productive class population increases, then non-productive class of the population decreases, similar to the AIDS class. Hence, the productivity of infected working-class population increases and reduce the effect of HIV/AIDS on working-class populations. The novel idea of piecewise model yields better results to support the

theoretical solutions. Working on working-class populations used to reduce the non-productivity of infected population and reduce inequality due to economic crisis. Reducing inequality reduce the number of populations infected by HIV/AIDS infection. Our result confirms the result reported by Sia and colleques (Sia et al., 2020), who suggested that reducing inequalities is a potential strategy to reduce HIV incidence in the sub-Saharan Africa region.

Declaration of competing interest

The authors declare that none of the work reported in this study have been influenced by any known competing financial interests or personal relationships.

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