FULL-LENGTH ARTICLE

Third Refinement Generalized Jacobi Iterative Method for Solving Linear System of Equations

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ABSTRACT
The Jacobi and Gauss-Seidel algorithms are among the stationary iterative methods for solving linear system of equations. Obtaining an approximation for the majority of sparse linear systems found in engineering and applied sciences requires efficient iteration approaches. Solving such linear systems using iterative techniques is possible, but the number of iterations is high. To acquire approximate solutions with rapid convergence, the need arises to redesign or make changes to the current approaches. In this study, a modified approach, termed the "third refinement Generalized" of the Jacobi algorithm, for solving linear systems is proposed. The primary objective of this research is to optimize for convergence speed by reducing the number of iterations and the spectral radius. Decomposing the coefficient matrix using a standard splitting strategy and performing an interpolation operation on the resulting simpler matrices led to the development of the proposed method. The study points out that, using the third refinement generalized of Jacobi method, we obtain a solution of a problem with a minimum number of iteration and obtain a greater rate of convergence than other previous methods like Jacobi, refinement Jacobi, refinement generalized Jacobi and second refinement generalized Jacobi.

Keywords and phrases: Generalized of Jacobi (GJ); Refinement of generalized Jacobi (RGJ) method; Second-refinement of generalized Jacobi (SRGJ) iterative method; Third-refinement of generalized Jacobi (3^{rd}RGJ) iterative method.

INTRODUCTION

We consider third-refinement of generalized Jacobi iterative method (3^{rd}RGJ). It is a refinement of second-refinement of generalized Jacobi iterative method (RSRGJ), hence here after we call third- refinement generalized Jacobi iterative method (3^{rd}RGJ). In many application one face with the problem of large and sparse linear systems of the form

$$Ax = b \qquad (1)$$

where $A = (a_{ij})$ is nonsingular real matrix of order n, b is a given n dimensional real vector and x is an n dimensional vector to be determined. Iterative methods, based on splitting A into $A = T_m - E_m - F_m$, where T_m is a banded matrix with band width 2m+1 , $a_{ii} \neq 0$ and E_m and F_m are strictly lower and upper triangular part of $T_m - A$ respectively, can compute successive approximations to obtain more accurate solutions to a linear system at each iteration step n. Third-refinement of generalized Jacobi (3^{rd}RGJ) iterative method is used to accelerate the convergence of basic Jacobi iterative method. It has been proved that, if A is strictly diagonally dominant (SDD) or irreducibly diagonally dominant (IDD), then the associated Jacobi iteration converges for any initial guess.

The Jacobi iteration (J) for first degree is

$$x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b \qquad (2)$$

PRELIMINARY

Let $A = (a_{ij})$ be an nxn matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth 2m + 1 defined as :

$$t_{ii} = \begin{cases} a_{ii}, |i-j| \leq m \\ 0, otherwise \end{cases}$$

We consider the decomposition $A = T_M - E_M - F_M \quad where \;\; -E_m \;\; and \;\; -F_m$ are the strict lower and upper part of the matrix A T_m, respectively.

$$T_m = \begin{pmatrix} a_{11} & . & . & . & a_{1,m+1} & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 0 \\ .a_{m+1,1} & . & . & . & . & . & . & . & a_{n-m,n} \\ 0 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & 0. & a_{n,n-m} & . & . & . & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ .-a_{m+2,1} & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ -a_{n,1} & . & . & . & -a_{n-m-1,n} & . & . & . & 0 \end{pmatrix},$$

$$F_m = \begin{pmatrix} 0 & . & . & . & -a_{1,m+2} & . & . & . & -a_{1,n} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ .. & . & . & . & . & . & . & . & a_{n-m-1,n} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \end{pmatrix}$$

Definition 1: (varga,2000). For n x n real matrices A, M, and N, A = M - N is a regular splitting of the matrix A if M is nonsingular with $M^{-1} \geq 0$ and $N^{-1} \geq 0$. Similarly, A = M - N is weak regular splitting of the matrix A if M is nonsingular with $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

The following definitions, lemmas and theorems are important for our study used (Young, 1971, Varga,2000, Datta,1995, Hackbusch,2016 and Saad,2003).

Definition 2: If a matrix A is strictly diagonally dominant or irreducibly diagonally dominant, then it is nonsingular.

Definition 3: A complex matrix $A \in C^{nxn}$ is reducible if and only if there exist a permutation matrix P (i.e., P is obtained from the identity I by a permutation of the rows of I) and an integer such

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

Where A_{11} is kxk and A_{22} is (n-k)x(n-k) . If A is not reducible, then A is said to be irreducible.
Definition 4: An nxn marix $A = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if

$$|a_{ii}| > \sum_{j=1, j\neq 1}^{n} |a_{ij}| \cdot$$

Definition 5: If an nxn marix $A = (a_{ij})$ is said to be diagonally dominant (DD) if

$$|a_{ii}| \geq \sum_{j=1, j\neq 1}^{n} |a_{ij}|$$

Definition 6: A is irreducibly diagonally dominant (IDD) if A is irreducible and diagonally dominant, with strict inequality holding in definition 2 for at least one i.
Definition 7: An nxn matrix A = $(a_{i\,j})$ is said to be symmetric positive definite (SPD) if A is symmetric, ($A = A^T$) and positive definite $x^T Ax > 0$ for all $x \neq 0$.

Definition 8: A matrix is said to be an M-matrix if it satisfies the following four properties:

(1) $a_{ii} > 0 \;\; for\; i = 1..., n$
(2) $a_{ii} \leq 0 \; for \; i \neq j \;, i, j = 1,..., n$
(3) $A\, is\, non\sin gular$
(4) $A^{-1} \geq 0$

Alternatively, A matrix $A \in \Re^{nxn}$, n is said to be an M-matrix if A can be written as A = sI - B, where $B \geq 0$ and $s \geq \sigma(B)$.
Definition 9: The spectral radius matrix A is the largest absolute value of the eigenvalues of A:
$\sigma(A) = \max\{|\lambda : \lambda \in \sigma(A)|\}$

Lemma 1: The spectral radius satisfies the following rules

- $\rho(kA) = |k|\rho(A)$ for all $k \in C$ and $A \in C^{nxn}$.
- $\rho(A^k) = (\rho(A))^k$ for all $k \in N$ and $A \in C^{nxn}$.
- $\rho(A) = \rho(A^H) = \rho(A^T)$ for all $A \in C^{nxn}$.

Theorem 1: A linear iteration $\Phi(x,b) = Mx + Nb$ with the iteration matrix

M = M[A] is convergent if and only if $\rho(M) < 1$.
Theorem 2: Let A = M - N be a regular splitting of the matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and $A^{-1} \geq O$.
Theorem 3: Let $A = (a_{i\,j})$, $B = (b_{i\,j})$ be two matrices such that $A \leq B$ and $b_{i\,j} \leq O$ for all $i \neq j$. Then if A is an M-matrix, so is the matrix B.

GENERALIZED JACOBI (GJ) ITERATIVE METHOD

The system of linear equation (1) is solved by different iterative methods. One of them is GJ iterative method. This method was first proposed by D.K. Salkuyeh, 2007.
If equation (1) can be written as $(T_m - E_m - F_m)x = b$

$$\Rightarrow x^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b$$

$$x^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b \qquad (3)$$

This scheme is called Generalized Jacobi (GJ) iterative method for m = 0, 1, 2,

If m = 0, then GJ = J.

Refinement Generalized Jacobi (RGJ) Method

Generalized Jacobi (GJ) iterative method is a few modification of Jacobi iterative method and refinement of generalized Jacobi (RGJ) iterative method is similarly a few modification of generalized Jacobi iterative method. It is a method with a few computations. This method was first introduced by V. B. Kumar Vatti and G. G. Gonfa ,2011. Equation (1) with $A = T_m — E_m — F_m$ can be written as:

$$\Rightarrow (T_m - E_m - F_m)x = b$$
$$\Rightarrow T_m x = (E_m + F_m)x + b$$
$$\Rightarrow T_m x = (T_m - A)x + b \text{ , } where\ E_m + F_m = T_m - A$$
$$\Rightarrow T_m x = T_m x b - Ax$$
$$\Rightarrow x = x + T_m^{-1}(b - Ax)$$
$$\Rightarrow x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(b - A\tilde{x}^{(n+1)}) \ where\ \tilde{x}^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b$$
$$\Rightarrow x^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b + T_m^{-1}(b - AT_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b)$$
$$collect\ like\ terms\ and\ after\ simplification\ , we\ get$$

$$x^{(n+1)} = \left[x^{(n+1)} = T_m^{-1}(E_m + F_m)\right]^2 x^{(n)} + (I + T_m^{-1}(E_m + F_m)T_m^{-1}b)$$

The above equation is called refinement of generalized Jacobi (RGJ) iterative method for m = 0, 1, 2,… If m = 0, then RGJ = RJ. One can get by similar method (step) the second refinement generalized Jacobi (SRGJ) iterative method

$$x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^3 x^{(n)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2]T_m^{-1}b \qquad (4)$$

Third-Refinement Generalized Jacobi (3rdRGJ) Method

In this paper we need to introduce third-refinement of generalized Jacobi (3rd RGJ) iterative method. By taking equation $x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(b - A\tilde{x}^{(n+1)})$

Substitute equation (4) on $\tilde{x}^{(n+1)}$. We get

$$\Rightarrow x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b +$$
$$T_m^{-1}(b - A(\left[T_m^{-1}(E_m + F_m)\right]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b)$$
$$\Rightarrow x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b +$$
$$T_m^{-1}(b - (T_m - E_m - F_m)(\left[T_m^{-1}(E_m + F_m)\right]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b),$$
$$where\ \ A = T_m - E_m - F_m \text{ , } After\ some\ like\ term\ collection\ and\ simplification\ ,$$
$$we\ get\ the\ following\ formula$$
$$\Rightarrow x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^4 x^{(n)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3]T_m^{-1}b$$

Let $G = T_m^{-1}(E_m + F_m)$, then one can put the above equation as follow

$$x^{(n+1)} = G^4 x^{(n)} + [I + G + G^2 + G^3]T_m^{-1} b \qquad (5)$$

The above equation (5) is called third refinement generalized Jacobi (3^{rd}RGJ) method for m = 0,1,2,… If m = 0, then 3^{rd} RGJ = 3^{rd}RJ.

Convergence of Third-Refinement Generalized Jacobi (3^{rd}RGJ) Method

Theorem 4: If A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix, then the associated Jacobi iterations converge for any $x^{(0)}$.

See the proof in R. S. Varga,2000.

Theorem 5: If A and 2D - A are symmetric and positive definite matrices, then the Jacobi method is convergent for any initial guess.

Given:- A and 2D - A are symmetric and positive definite matrices.

Required:- the Jacobi method is convergent for any initial guess.

Proof: et A and 2D - A be SPD. We know $x^* Ax > 0$ and $x^*(2D - A)x > 0$,

where $A = D - L - L^t$.

$$\Rightarrow D^{-1}(L + L^t)x = \lambda x$$

$$\Rightarrow (L + L^t)x = \lambda Dx$$

$$\Rightarrow x^*(L + L^t)x = \lambda x^* Dx$$

$$\Rightarrow x^* Dx - x^* Ax = \lambda x^* Dx$$

$$\Rightarrow x^* Ax = (1 - \lambda)x^* Dx$$

$$\Rightarrow 1 - \lambda > 0$$

$$\Rightarrow \lambda < 1$$

$\lambda < 1$ .... *

And we consider

$$x^*(2D - A)x > 0$$

$$\Rightarrow 2x^* Dx - x^* Ax > 0$$

$$\Rightarrow x^* Ax < 2x^* Dx$$

$$\Rightarrow (1 - \lambda)x^* Dx < 2x^* Dx \Rightarrow 1 - \lambda < 2$$

$$\Rightarrow \lambda > -1$$

$$\therefore \lambda > -1 \quad \ldots \quad **$$

From * and **, we get $-1 < \lambda < 1$.Where λ is the eigenvalues of $D^{-1}(L + L^t)$.

Hence, $\sigma(D^{-1}(L + L^t)) < 1$.

Theorem 6: (Salkuyeh,2007): If A is an M-matrix, then the Jacobi iterative method is convergent for any initial guess x^0.

Given:- A is an M-matrix.

Required:- The Jacobi iterative method is convergent for any initial guess x^0.

Proof:- Given A is M-matrix. Let A = M - N. $A = D - L - U \Rightarrow M = D$ and $N = L + U \Rightarrow A \le M$ so by theorem 8 M is M-matrix.

$\Rightarrow M^{-1} > 0$. On the other hand $N^{-1} \ge 0$.

$\therefore A = M - N$ is a regular splitting of the matrix A. Having in mind that $A^{-1} \ge 0$ and by theorem 7, we deduce that $\sigma(G_J) < 1$.

Theorem 7: Let A be an SDD matrix. Then for any natural number $m < n$ the generalized Jacobi (GJ) iterative method is convergent for any initial guess $x^{(0)}$.
Given:- A be an SDD matrix.
Required:- For any natural number $m < n$ the generalized Jacobi (GJ) iterative method is convergent for any initial guess $x^{(0)}$.
Proof:- See the proof in D. K. Salkuyeh,2007.

Theorem 8: If A and $2T_m - A$ are symmetric and positive definite matrices, then the Generalized Jacobi (GJ) iterative method converges for any initial guess $x^{(0)}$.
Given:- A and $2T_m - A$ are symmetric and positive definite matrices.
Required:- The Generalized Jacobi (GJ) iterative method converges for any initial guess $x^{(0)}$.
Proof: Let A and $2T_m - A$ be SPD.
We know that $x^* Ax > 0$ *and* $x^*(2T_m - A)x > 0$, where $A = T_m - E_m - E_m^{\ t}$

$$\Rightarrow T_m^{-1}(E_m + E_m^{\ t})x = \lambda x$$
$$\Rightarrow (E_m + E_m^{\ t})x = \lambda T_m x$$
$$\Rightarrow x^*(E_m + E_m^{\ t})x = \lambda x^* T_m x$$
$$\Rightarrow x^* T_m x - x^* Ax = \lambda x^* T_m x$$
$$\Rightarrow x^* Ax = (1-\lambda) x^* T_m x$$
$$\Rightarrow 1-\lambda > 0$$
$$\Rightarrow \lambda < 1$$
$$\therefore \lambda < 1 \qquad \ldots \quad ***$$

And we consider

$$x^*(2T_m - A)x > 0$$
$$\Rightarrow 2x^* T_m x - x^* Ax > 0$$
$$\Rightarrow x^* Ax < 2x^* T_m x$$
$$\Rightarrow (1-\lambda) x^* T_m x < 2x^* T_m x$$
$$\Rightarrow 1-\lambda < 2$$
$$\Rightarrow \lambda > -1$$
$$\therefore \lambda > -1 \qquad \ldots \quad ****$$

From *** and ****, we get $-1 < \lambda < 1$. where λ is the eigenvalues of $T_m^{-1}(E_m + E_m^{\ t})$
Hence, $\sigma(T_m^{-1}(E_m + E_m^{\ t})) < 1$

Theorem 9: Let A be an M-matrix. Then for a given natural number $m < n$, the GJ method is convergent for any initial guess $x^{(0)}$.
Given:- A be an M-matrix.
Required:- For a given natural number $m < n$, the GJ method is convergent for any initial guess $x^{(0)}$.
Proof: (Salkuyeh, 2007). Let $M_m = T_m$ and $N_m = E_m + F_m$ in the GJ method.

Obviously, in this case we have $A \le M_m$. Hence by Theorem 3, we conclude that the matrix M_m is an M-matrix. On the other hand we have $N_m \ge O$. Therefore,
$A = M_m - N_m$ is a regular splitting of the matrix A. Having in mind that $A^{-1} \ge 0$ and by theorem 2, we deduce that $\sigma(B_{GJ}{}^{m}) < 1.$

Theorem 10: If A is strictly diagonally dominant matrix, then the refinement generalized Jacobi method converges for any choice of the initial approximation $x^{(0)}$.
Given:- A is strictly diagonally dominant matrix.
Required:- The refinement generalized Jacobi method converges for any choice of the initial approximation $x^{(0)}$.
Proof: (Vatti and et al,2011). Assuming $\tilde{x}$ is the real solution of (1), as A is SDD by Theorem 7, generalized Jacobi method is convergent.
Let $x^{(n+1)} \to \tilde{x}$ (exact solution). Then we have
$$\left\|x^{(n+1)} - \tilde{x}\right\|_\infty \le \left\|x^{(n+1)} - \tilde{x}\right\|_\infty + \left\|T_m{}^{-1}\right\|_\infty \left\|(b - Ax^{(n+1)}\right\|_\infty .$$
From the fact $\left\|x^{(n+1)} - \tilde{x}\right\|_\infty \to 0$, we have $\left\|b - Ax^{(n+1)}\right\|_\infty \to 0$. Therefore, $\left\|x^{(n+1)} - \tilde{x}\right\|_\infty \to 0$. Hence refinement of generalized Jacobi method is convergent.

Theorem 11: If A and $2T_m - A$ are SPD matrix, then the refinement generalized Jacobi iterative method is convergent for any initial guess $x(0)$.
Given:- A and $2T_m - A$ are SPD matrix.
Required:- The refinement generalized Jacobi iterative method is convergent for any initial guess $x(0)$.
Proof: Using equation (3) and by theorem 8, we have $\sigma(T_m{}^{-1}(E_m + E_m{}^{t})) < 1.$

Theorem 12: Let $A = (a_{i\,j})$ be an M-matrix. Then for a given natural number $m < n$, the refinement of generalized Jacobi method converges for any choice of initial approximation $x^{(0)}$.
Given:- $A = (a_{i\,j})$ be an M-matrix.
Required:- For a given natural number $m < n$, the refinement of generalized Jacobi method converges for any choice of initial approximation $x^{(0)}$.
Proof: It follows from theorem 9. See Vatti and et al,2011.
Theorem 13: If A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix, then the second-refinement generalized Jacobi iterations converge for any $x^{(0)}$.
Given:- A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix.
Required:- The second-refinement of generalized Jacobi iterations converge for any $x^{(0)}$.
Proof: Let X be the real solution of (1). Given that A is SDD, using theorem 4, 7, and 10, the J,GJ and RGJ methods are convergent and hence $x^{(n+1)}$ X (exact Solution). As we mentioned above by theorem 10 one can prove the second-refinement of generalized Jacobi iterations method (SRGJ) converge for any $x^{(0)}$.
Theorem 14: If A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix, then the third-refinement of generalized Jacobi iterations (3rdRGJ) converge for any $x^{(0)}$.
Given:- A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix.
Required:- The third-refinement of generalized Jacobi iterations converge for any $x^{(0)}$.
Proof: Let X be the real solution of (1). Given that A is SDD, using theorem 4, 7, 10 and 13, the J,GJ , RGJ and SRGJ methods are convergent and hence $x^{(n+1)} \to x$ (exact Solution). As we mentioned above by theorem 13,
$$x^{(n+1)} = \left[T_m{}^{-1}(E_m + F_m)\right]^{4n+4} x^{(0)} + [I + T_m{}^{-1}(E_m + F_m) + (T_m{}^{-1}(E_m + F_m))^2 + (T_m{}^{-1}(E_m + F_m))^3 + \ldots + (T_m{}^{-1}(E_m + F_m))^{4n+3}]T_m{}^{-1}b.$$

$$x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(Eb - A\tilde{x}^{(n+1)}) \ \ or \ \ x^{(n+1)} - x = \tilde{x}^{(n+1)} - x + T_m^{-1}(b - A\tilde{x}^{(n+1)}).$$

Then ,

$$\left\|x^{(n+1)} - x\right\|. = \left\|\tilde{x}^{(n+1)} - x + T_m^{-1}(b - A\tilde{x}^{(n+1)})\right\| \le \left\|\tilde{x}^{(n+1)} - x\right\| + \left\|T_m^{-1}(b - A\tilde{x}^{(n+1)})\right\|$$

$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le \left\|\tilde{x}^{(n+1)} - x\right\| + \left\|T_m^{-1}(b - A\tilde{x}^{(n+1)})\right\| \le \left\|\tilde{x}^{(n+1)} - x\right\| + \left\|T_m^{-1}\right\| + \left\|(b - A\tilde{x}^{(n+1)})\right\|$$

$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le \left\|\tilde{x}^{(n+1)} - x\right\| + \left\|T_m^{-1}\right\| + \left\|(b - A\tilde{x}^{(n+1)})\right\| = 0 + \left\|T_m^{-1}\right\|.0 \ , \sin ce \ A\tilde{x}^{(n+1)} \to 0$$

$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le 0 + \left\|T_m^{-1}\right\|.0 = 0$$

$$\Rightarrow x^{(n+1)} - x = 0$$

Hence $x^{(n+1)} \to x$

$x^{(n+1)} = [T_m^{-1}(E_m + F_m)]^3 x^{(n)} + [I + T_m^{-1}(E_m + F_m) + \left(T_m^{-1}(E_m + F_m)^2\right)]T_m^{-1}b$ is convergent, so $\tilde{x}^{(n+1)} \to x$.

$\sigma\left[(T_m^{-1}(E_m + F_m))^4\right] = (\sigma(T_m^{-1}(E_m + F_m)))^4 < 1$

Therefore, the SRGJ iterative method is convergent.

Theorem 15: If A and $2T_m - A$ are SPD matrices, then the third refinement generalized Jacobi (3rdRGJ) iterative method is convergent for any initial guess $x^{(0)}$.

Given:- A and $2T_m - A$ are SPD matrices.

Required:- The third refinement generalized Jacobi (3rdRGJ) iterative method is convergent for any initial guess $x^{(0)}$.

Proof:- Using equation (3) and by theorem 8, we have $\sigma(T_m^{-1}(E_m + E_m^{\ t}) < 1$.

Let x be the exact solution of (1). Then the generalized Jacobi iterative method can be written as

$x = (I - T_m^{-1}(E_m + F_m))^{-1}T_m^{-1}b \ if \ \ x^{(n+1)} \to x$. Using equation (5):

$$x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^4 x^{(n)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3]T_m^{-1}b.$$

Now using equation (4) and the exact solution x, we have:

$$\Rightarrow x = (I - (T_m^{-1}(E_m + F_m))^4)^{-1}[I + (T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 +$$
$$(T_m^{-1}(E_m + F_m))^3]T_m^{-1}b.$$

$$\Rightarrow x = [(I + (T_m^{-1}(E_m + F_m))^4 + (T_m^{-1}(E_m + F_m))^8 + ...][I + (T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 +$$
$$(T_m^{-1}(E_m + F_m))^3]T_m^{-1}b. \ \sin ce \ (1 - x^3)^{-1} = 1 + x^3 + x^6 + ...$$

$$\Rightarrow x = [I + (T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3 + (T_m^{-1}(E_m + F_m))^3 + ...]T_m^{-1}b.$$

$\Rightarrow x = [I - (T_m^{-1}(E_m + F_m)]T_m^{-1}b.$ is consistent to (1) and generalized Jacobi method. On the other hand,

$$x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^4 x^{(n)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3]T_m^{-1}b.$$

.We are given that A is SPD then $\sigma(T_m^{-1}(E_m + E_m^{\ t}) < 1.$

$$x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^8 x^{(n-1)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3 + ... +$$
$$(T_m^{-1}(E_m + F_m))^7]T_m^{-1}b.$$

$$x^{(n+1)} = \left[T_m^{-1}(E_m + F_m)\right]^{16} x^{(n-1)} + [I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2 + (T_m^{-1}(E_m + F_m))^3 + ... +$$
$$(T_m^{-1}(E_m + F_m))^{15}]T_m^{-1}b.$$

Thus

$$\lim_{x\to\infty}[T_m^{-1}(E_m+F_m)]^{4n+4}=0$$

$$\lim_{x\to\infty}x^{(n+1)}=\lim_{x\to\infty}\left[T_m^{-1}(E_m+F_m)\right]^{4n+4}x^{(0)}+[I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2+(T_m^{-1}(E_m+F_m))^3+\dots+(T_m^{-1}(E_m+F_m))^{4n+3}]T_m^{-1}b.$$

$$\lim_{x\to\infty}x=0+(I-T_m^{-1}(E_m+F_m))^{-1}T_m^{-1}b$$

$$\lim_{x\to\infty}x=(I-T_m^{-1}(E_m+F_m))^{-1}T_m^{-1}b$$

$$(I-T_m^{-1}(E_m+F_m))^{-1}T_m^{-1}b\to x$$

$$\sigma[(T_m^{-1}(E_m+F_m))^4]=[\sigma(T_m^{-1}(E_m+F_m))]^4<1$$

Therefore, the third-refinement of generalized Jacobi (3^{rd}RGJ) iterative method is convergent.

Theorem 16:- If A is an M-matrix, then the third-refinement generalized Jacobi iterative method is convergent for any initial guess $x^{(0)}$.
Given:- A is an M-matrix.
Required:- The third refinement generalized Jacobi (3^{rd}RGJ) iterative method is convergent for any initial guess $x^{(0)}$.
Proof: We are given that A is an M-matrix. We want to show that 3^{rd}RGJ iterative method is convergent. From theorem 9 one can see that GJ iterative method is convergent.
$\sigma(T_m^{-1}(E_m+F_m)<1$. Using theorem 1 and 12,

$$\sigma(B_{RGJ})=\sigma([T_m^{-1}(E_m+F_m)]^2)=[\sigma(T_m^{-1}(E_m+F_m))]^2<1.$$

$$\sigma(B_{RGJ})<1.$$

$$\sigma(B_{3^{rd}\,RGJ})=\sigma([T_m^{-1}(E_m+F_m)]^4)=[\sigma(T_m^{-1}(E_m+F_m))]^4<1.$$

Using theorem 12,

$$\therefore \sigma(B_{3^{rd}\,RGJ})<1.$$

Therefore 3^{rd}RGJ iterative method is convergent if A is an M-matrix.
Theorem 17:-The third-refinement generalized Jacobi method converges faster than the generalized Jacobi, refinement of generalized Jacobi method and second refinement generalized Jacobi method when generalized Jacobi method is convergent.
Given:- when generalized Jacobi method is convergent.
Required:- The third-refinement generalized Jacobi method converges faster than the generalized Jacobi, refinement generalized Jacobi method and second refinement generalized Jacobi method.

Proof: We have by equation 3, $x^{(n+1)}=Gx^{(n)}+k$, by equation 4 , $x^{(n+1)}=G^2x^{(n)}+k_1$ and by equation 5, $x^{(n+1)}=G^4x^{(n)}+k_2$ where $G=T_m^{-1}(E_m+F_m)$, $k=T_m^{-1}b$, $k_1=(I+G)T_m^{-1}b$ *and* $k_2=(I+G+G^2+G^3)T_m^{-1}b.$ Given that $\|G\|<1$

Let x be the exact solution of (1), so we have

$$x=Gx^{(n)}+k\ ,x=G^2x^{(n)}+k_1\ \text{and}\ x=G^4x^{(n)}+k_2$$

Let us consider generalized Jacobi method:

Now let us consider refinement generalized Jacobi method:

$$i.e.\ x^{(n+1)} = Gx^{(n)} + k$$
$$\Rightarrow x^{(n+1)} - x = Gx^{(n)} - x + k$$
$$\Rightarrow x^{(n+1)} - x = G(x^{(n)} - x) + Gx + k - x$$
$$\Rightarrow x^{(n+1)} - x = G(x^{(n)} - x)$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| = \left\|G(x^{(n)} - x)\right\| \le \|G\|\left\|x^{(n)} - x\right\| \le \left\|G^2\right\|\left\|(x^{(n-1)} - x)\right\| \le \ldots \le \left\|G^n\right\|\left\|(x^{(1)} - x)\right\|$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le \left\|G^n\right\|\left\|(x^{(1)} - x)\right\| = \left\|G^n\right\|\left\|(x^{(1)} - x)\right\|$$

$$i.e.\ x^{(n+1)} = G^2x^{(n)} + k_1$$
$$\Rightarrow x^{(n+1)} - x = G^2x^{(n)} - x + k_1$$
$$\Rightarrow x^{(n+1)} - x = G^2(x^{(n)} - x) + G^2x + k_1 - x$$
$$\Rightarrow x^{(n+1)} - x = G^2(x^{(n)} - x)$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| = \left\|G^2(x^{(n)} - x)\right\| \le \left\|G^2\right\|\left\|x^{(n)} - x\right\| \le \left\|G^4\right\|\left\|(x^{(n-1)} - x)\right\| \le \ldots \le \left\|G^{2n}\right\|\left\|(x^{(1)} - x)\right\|$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le \left\|G^{2n}\right\|\left\|(x^{(1)} - x)\right\| = \left\|G^{2n}\right\|\left\|(x^{(1)} - x)\right\|$$

Again let us consider third-refinement generalized Jacobi method:

According to the coefficients of the above inequalities, we have
$\|G\|^{4n} < \|G\|^{2n} < \|G\|^{n}$ since $\|G\| < 1$.

$$i.e.\ x^{(n+1)} = G^4x^{(n)} + k_2$$
$$\Rightarrow x^{(n+1)} - x = G^4x^{(n)} - x + k_2$$
$$\Rightarrow x^{(n+1)} - x = G^4(x^{(n)} - x) + G^4x + k_2 - x$$
$$\Rightarrow x^{(n+1)} - x = G^4(x^{(n)} - x)$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| = \left\|G^4(x^{(n)} - x)\right\| \le \left\|G^4\right\|\left\|x^{(n)} - x\right\| \le \left\|G^8\right\|\left\|(x^{(n-1)} - x)\right\| \le \ldots \le \left\|G^{4n}\right\|\left\|(x^{(1)} - x)\right\|$$
$$\Rightarrow \left\|x^{(n+1)} - x\right\| \le \left\|G^{4n}\right\|\left\|(x^{(1)} - x)\right\| = \left\|G^{4n}\right\|\left\|(x^{(1)} - x)\right\|$$

Therefore, the third-refinement generalized Jacobi method converges faster than the generalized Jacobi method and refinement generalized Jacobi method.

Numerical Examples

Example 1: Consider the following system of linear equations whose coefficient matrix is both SDD and SPD with tolerance 0.0001.

$$\begin{aligned} 6x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 + 8x_2 + 2x_3 &= 6 \\ 2x_1 + 2x_2 + 10x_3 &= 7 \end{aligned}$$

Given:- The matrix SDD and SPD with tolerance 0.0001.

Required:- Find the spectral radius and the iteration of the given matrix.

Solution:- Let us consider spectral radius and solution

Table 1. Spectral Radius

Method	J	GJ	RGJ	SRGJ	3rdRGJ
Spectral Radius	0.5146	0.2972	0.0884	0.0263	0.0078

Table 1 shows that the 3rdRGJ method has small spectral radius than J, GJ,RGJ and SRGJ whereas Table 2 shows that the third- refinement of generalized Jacobi (3rdRGJ) iterative method is much better than generalized Jacobi (GJ) method, refinement generalized Jacobi (RGJ) method and second refinement generalized Jacobi (SRGJ) method.

Example 2: Consider the following system of linear equations whose coefficient matrix is SDD but not SPD with tolerance 0.0001.

$$\begin{aligned} 6x_1 + 4x_2 - x_3 &= 9 \\ 3x_1 + 7x_2 + 2x_3 &= 12 \\ -4x_1 + 3x_2 + 8x_3 &= 7 \end{aligned}$$

Given:- The matrix SDD but not SPD with tolerance 0.0001.
Required:- Find the spectral radius and the iteration of the given matrix.
Solution:- Let us consider spectral radius and solution:

Table 2. Numerical results of example 7.1 and comparison between GJ, RGJ , SRGJ and 3^{rd}RGJ.

	GJ			RGJ			SRGJ			3^{rd}RGJ		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.6923	0.4231	0.6154	0.4541	0.5222	0.4571	0.5166	0.4932	0.5105	0.4958	0.5000	0.4963
2	0.4541	0.5222	0.4571	0.4958	0.5020	0.4963	0.4996	0.5002	0.4992	0.5000	0.5000	0.5000
3	0.5166	0.4932	0.5103	0.4996	0.5002	0.4992	0.5000	0.5000	0.5000			
4	0.4958	0.5020	0.4963	0.5000	0.5000	0.5000						
5	0.5014	0.4994	0.5010									
6	0.4996	0.5002	0.4997									
7	0.5001	0.4999	0.5001									
8	0.5000	0.5000	0.5000									

Table 3. Spectral radius

Method	J	GJ	RGJ	SRGJ	3^{rd}RGJ
Spectral Radius	0.7937	0.4844	0.2346	0.1136	0.050

Table 3 shows that the 3^{rd}RGJ has small spectral radius than J, GJ,RGJ and SRGJ whereas Table 4 shows that the third- refinement of generalized Jacobi (3^{rd}RGJ) iterative method is much better than Jacobi method, generalized Jacobi (GJ) method , refinement generalized Jacobi (RGJ) method and second refinement generalized Jacobi (SRGJ) method . We can also compare the iteration number, i.e, GJ at 14, RGJ at 7 , SRGJ at 5 and 3^{rd}RGJ at 4. So our new method is better than others.

Table 4. Numerical results of example 7.2 and comparison between GJ, RGJ , SRGJ and 3rdRGJ.

	GJ			RGJ			SRGJ			3rdRGJ		
n	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.5980	1.3529	0.3676	0.7820	1.1690	0.7357	0.9010	1.0824	0.8601	0.9500	1.0398	0.9356
2	0.7820	0.1690	0.7357	0.9502	1.0398	0.9356	0.9884	1.0093	0.9847	0.9973	1.0022	0.9964
3	0.9010	1.0824	0.8601	0.9884	1.0093	0.9847	0.9987	1.0011	0.9982	0.9999	1.0001	0.9998
4	0.9502	1.0398	0.9356	0.9973	1.0022	0.9964	0.9999	1.0001	0.9998	1.0000	1.0000	1.0000
5	0.9764	1.0193	0.9699	0.9994	1.0005	0.9992	1.0000	1.0000	1.0000			
6	0.9884	1.0093	0.9847	0.9999	1.0001	0.9998						
7	0.9944	1.0045	0.9925	1.0000	1.0000	1.0000						
8	0.9973	1.0022	0.9964									
9	0.9987	1.0011	0.9982									
10	0.9994	1.0005	0.9992									
11	0.9997	1.0002	0.9996									
12	0.9999	1.0001	0.9998									
13	0.9999	1.0001	0.9999									
14	1.0000	1.0000	1.0000									

Example 3: Consider the following system of linear equations whose coefficient matrix is SPD but not SDD with tolerance 0.0001.

$$6x_1 + 4x_2 + 3x_3 = 13$$
$$4x_1 + 5x_2 + 2x_3 = 11$$
$$3x_1 + 2x_2 + 2x_3 = 7$$

Given:- The matrix SPD but not SDD with tolerance 0.0001.
Required:- Find the spectral radius and the iteration of the given matrix.
Solution:- Let us consider spectral radius and solution:-

Table 5. Spectral radius

Method	J	GJ	RGJ	SRGJ	3^{rd}RGJ
Spectral Radius	1.4900	12.8739	165.7364	2133.6854	$2.74\text{x}10^4$

The iterative solution of the above equation diverges from the exact solution. The system has no solution when we apply Jacobi, Generalized Jacobi method, refinement of generalized Jacobi method, second refinement of generalized Jacobi method and third refinement of generalized Jacobi method. Since the eigenvalues of iteration matrix is greater than one. We know that the Jacobi method to be convergent the matrix should satisfy the following conditions:

(1) A must be SPD, and
(2) $2T_m — A$ must be SPD

Example 4: Consider the following system of linear equations whose coefficient matrix is SDD but not PD and SPD with tolerance 0.0001.

$$\begin{aligned} 5x_1 + 3x_2 + x_3 &= 9 \\ 4x_1 - 6x_2 + x_3 &= -1 \\ 2x_1 + x_2 + 4x_3 &= 7 \end{aligned}$$

Given:- The matrix SDD but not PD and SPD with tolerance 0.0001.
Required:- Find the spectral radius and the iteration of the given matrix.
Solution:- Let us consider spectral radius and solution

Table 6. Spectral radius

Method	J	GJ	RGJ	SRGJ	3^{rd}RGJ
Spectral Radius	0.6227	0.2939	0.0863	0.0254	0.0175

Table 6 shows spectral radius of the methods whereas Table 7 shows that the Third- Refinement of Generalized Jacobi (3^{rd}RGJ) iterative method is much better than Jacobi(J),Generalized Jacobi (GJ) method, Refinement of Generalized Jacobi (RGJ) method and Second Refinement of Generalized Jacobi (SRGJ) method. We can also conclude that 3^{rd}RGJ method minimizes iteration number to half as compared to GJ method.

Example 5: Consider the following system whose coefficient matrix is an M-matrix (or 2-cyclic matrix), which arises from the discretization

of the Poisson equation, on the unit $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f$ square as considered

by Vatti and Genanew,2011, Datta,1995 and Dafchahi,2008, with tolerance 0.00001. Now consider $Ax = b$ where $m = 1$, $X = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T$ and $b = (1\ 0\ 0\ 0\ 0\ 0)^T$ or

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Table 7. Numerical results of example 7.4 and comparison between GJ, RGJ ,SRGJ and 3rdRGJ.

	GJ			RGJ			SRGJ			3rdRGJ		
n	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$	$X_1^{(n)}$	$X_2^{(n)}$	$X_3^{(n)}$
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1.1098	1.1503	1.4624	0.9370	0.9509	0.9574	1.0040	1.0076	1.0296	0.9959	0.9970	0.9988
2	0.9370	0.9509	0.9574	0.9959	0.9970	0.9988	0.9997	0.9998	1.0000	1.0000	1.0000	1.0000
3	1.0040	1.0076	1.0296	0.9997	0.9998	1.0000	1.0000	1.0000	1.0000			
4	0.9959	0.9970	0.9988	1.0000	1.0000	1.0000						
5	1.0000	1.0004	1.0020									
6	0.9997	0.9998	1.0000									
7	1.0000	1.0000	1.0001									
8	1.0000	1.0000	1.0000									

Table 8. Spectral radius

Method	J	GJ	RGJ	SRGJ	3rdRGJ
Spectral Radius	0.6036	0.3867	0.1496	0.0578	0.0224

Table 9. (a) Numerical results of Example 5 and comparison between GJ, RGJ and SRGJ.

GJ, for m = 1						
	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$
	0	0	0	0	0	0
	0.2679	0.0714	0.0179	0	0	0
	0.2679	0.0714	0.0179	0.0772	0.0408	0.0147
	0.2917	0.0897	0.0261	0.0772	0.0408	0.0147
	0.2917	0.0897	0.0261	0.0850	0.0483	0.0186
	0.2944	0.0926	0.0278	0.0850	0.0483	0.0186
	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
	0.2948	0.0931	0.0281	0.0860	0.0495	0.0193
	0.2948	0.0931	0.0281	0.0861	0.0497	0.0194
	0.2948	0.0932	0.0281	0.0861	0.0497	0.0194
1	0.2948	0.0932	0.0281	0.0861	0.0497	0.0195
1	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

(b)

RGJ Iterative method, for m = 1						
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$
0	0	0	0	0	0	0
1	0.2679	0.0714	0.0179	0.0772	0.0408	0.0147
2	0.2917	0.0897	0.0261	0.0850	0.0483	0.0186
3	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
4	0.2948	0.0931	0.0281	0.0861	0.0497	0.0194
5	0.2948	0.0932	0.0281	0.0861	0.0497	0.0195
6	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

(c).

SRGJ Iterative method, for m = 1						
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$
0	0	0	0	0	0	0
1	0.2917	0.0897	0.0261	0.0772	0.0408	0.0147
2	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
3	0.2948	0.0932	0.0281	0.0861	0.0497	0.0194
4	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

(d)

3rdRGJ Iterative method, for m = 1						
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$
0	0	0	0	0	0	0
1	0.2917	0.0897	0.0261	0.0850	0.0481	0.0156
2	0.2948	0.0931	0.0271	0.0860	0.0497	0.0194
3	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

Table 8 shows spectral radius of the methods whereas Table 9(a)-(d) shows that the third- refinement of generalized Jacobi (3^{rd}RGJ) iterative method is much better than Jacobi method (J), generalized Jacobi (GJ) method, refinement of generalized Jacobi (RGJ) and second refinement of generalized Jacobi (SRGJ) method. So our new method is better than the others.

CONCLUSION

In this study, the third refinement generalized of Jacobi method using the properties of "refinement Jacobi method" and "generalized Jacobi method" applied successfully to get the third refinement generalized Jacobi method. Convergence of the method is verified with the help of successive iterations and spectral radius. The validity of the result is also verified by comparing them with previous results using the rate of convergence of stationary iterative process depends on spectral radius of the iterative matrix, any reasonable modification of iterative matrix that will reduce the spectral radius and increases the rate of convergence of that method. We can give the general conclusion by using table:

Table 10. Summary for Examples 1 to 5

	1		2		3		4		5	
Methods	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius
J	15	0.5146	37	0.7937	-	1.4900	23	0.6227	19	0.6036
RJ	8	0.2649	18	0.6294	-	2.2202	12	0.3879	10	0.3643
SRJ	5	0.1362	12	0.5000	-	3.3082	8	0.2415	7	0.2199
GJ	8	0.2972	14	0.4844	-	12.874	8	0.2939	11	0.3867
RGJ	4	0.0884	7	0.2346	-	165.74	4	0.0863	6	0.1496
SRGJ	3	0.0263	5	0.1136	-	2133.7	3	0.0254	4	0.0578
3rdRGJ	2	0.0078	4	0.050	-	27400	2	0.0175	3	0.0224

In this paper, we found for m = 1 that third-refinement of generalized Jacobi iterative method for solving linear system of equations which uses to minimize the number of iteration almost by half as compared to refinement generalized Jacobi iterative method and the rate of convergence of third-refinement of generalized Jacobi method is more better than the others method and it has smallest spectral radius. This means that the new method that we found is much fastest than Jacobi, generalized Jacobi and second refinement generalized Jacobi method. More over one can find for m = 2, 3, ...

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