FULL-LENGTH ARTICLE

Block Procedure for Solving Stiff First Order Initial Value Problems Using Chebyshev Polynomials

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ABSTRACT

In this paper, discrete implicit linear multistep methods in block form for the solution of initial value problems was presented using the Chebyshev polynomials. The method is based on collocation of the differential equation and interpolation of the approximate solution of power series at the grid points. The procedure yields four consistent implicit linear multistep schemes which are combined as simultaneous numerical integrators to form block method. The basic properties of the method such as order, error constant, zero stability, consistency and accuracy are investigated. The accuracy of the method was tested with two stiff first order initial value problems. The results were compared with a method reported in the literature. All numerical examples were solved with the aid of MATLAB software after the schemes are developed using MAPLE software and the results showed that our proposed method produces better results.

Keywords: Accuracy; Block procedure; Chebyshev polynomials; Consistency; Zero stability
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INTRODUCTION

In this paper, we consider the following general form of initial value problem (IVP) for first order stiff ODEs:

$$y'(x) = f(x, y(x)), \; y(x_0) = y_0 \tag{1}$$

where x_0 and y_0 are finite.

In mathematics, a differential equation of the form Eq. (1) is said to be a stiff differential equation if for solving a differential equation certain numerical methods are numerically unstable, unless the step size is taken to be extremely small (Suli and Mayers, 2003).

The developments of numerical methods for the solution of Eq. (1) have given rise to two major discrete methods (Anake, 2011). One of those discrete numerical methods used to solve stiff initial value problems (IVPs) is multistep methods especially linear multistep methods (LMMs). The general k-step LMM for the solution of Eq. (1) is given by Lambert (1991) as follows:

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k} \beta_j \, f_{n+j} \tag{2}$$

where $\alpha_j \text{'s}$ and $\beta_j \text{'s}$ are constants, $\alpha_k = 0$ and at least one of the coefficients α_0 and β_0 is non-zero.

Several continuous LMMs have been derived using interpolation and collocation points for the solution of Eq. (1) with constant step size using Chebyshev polynomials as a basis (Adeniyi et al., 2006; Adeniyi and Alabi, 2007). The development of block methods for solving Eq. (1) has been studied by various researchers (Mohammed and Yahaya, 2010; Akinfenwa et al., 2013; James et al., 2013; Suleiman et al., 2015; Sunday et al., 2015; Yakusak and Adeniyi, 2015; Okedayo et al., 2018; Nweze et al., 2018). Abualnaja (2015) constructed a block procedure with LMMs ($k = 1, 2, \text{and} \, 3$) using

Legendre polynomial as a basis function for solving first order ODEs. She gave discrete methods and implemented it for solving non-stiff initial value problems. The results are compared with RK4 but not compared with other existing methods.

Recently, Berhan et al. (2019) extended and modified the work of Abualnaja (2015). They constructed a block procedure with implicit LMMs for some k-step $(k = 1, 2, 3, \text{and} 4)$ using Legendre polynomials for solving ODEs. It is an implicit method and used for solving stiff initial value problems. Their method depends on the perturbed collocation approximation with shifted Legendre polynomials as perturbation term. Thus, the aim of this paper is to construct block method for some k-step LMMs $(k = 1, 2, 3, \text{and} 4)$ for the solutions of a general first order IVPs in stiff ODEs based on collocation of the differential equation and interpolation of the approximate solution using Chebyshev polynomial of first kind as a perturbed term, which is the extension and modification of the work by Nweze et al. (2018). The method is an implicit fourth order block method which is self-starting and solves stiff ODEs. It improves the accuracy of sixth order LMM developed by Berhan et al. (2019) for solving stiff IVPs.

MATERIALS AND METHODS

MAPLE and MATLAB software are used in this study. First, numerical schemes are derived using MAPLE software as the study involves solving a matrix with variable entries which is very complicated to solve manually. Next, the problems were solved using MATLAB software after the numerical schemes are coded.

Important data for the study were collected by the researcher using documentary analysis. In addition, the required numerical data was collected by using MATLAB software. The study procedures we followed are as follows:

1. Power series solution of Eq. (1) was defined.
2. The obtained power series solution was truncated to get an approximated solution to eq. (1).
3. A perturbed term was added to the approximated solution.
4. The given interval $\left[x_n, x_{n+k} \right]$ was transformed into the interval $\left[-1, 1 \right]$.
5. System of equations was formulated using the techniques of interpolation and collocation at different grid points.
6. The resulting system of equations was solved using MAPLE software to derive the required schemes.
7. MAT LAB code for the schemes was written.
8. The derived schemes were validated using numerical examples.

RESULTS

In this section, we derive the discrete method to solve Eq. (1) at a sequence of nodal points $x_n = x_0 + nh$ where h is the step length and defined by $h = x_{n+j} - x_{n+j-1}$ for $j = 0, 1, 2, \cdots, k$ and n is the number of steps which is a positive integer.

Let the power series solutions of Eq. (1) be

$$y(x) = \sum_{j=0}^{\infty} a_j x^j \tag{3}$$

From Eq. (3), it follows that the approximate solution is

$$y(x) = \sum_{j=0}^{k} a_j x^j, \, x_n \leq x \leq x_{n+k} \tag{4}$$

The first derivative of Eq. (4) is given by

$$y'(x) = \sum_{j=1}^{k} j a_j x^{j-1} \tag{5}$$

Substituting Eq. (5) into Eq. (1), we obtain

$$y'(x) = \sum_{j=1}^{k} j a_j x^{j-1} \approx f(x, y) \tag{6}$$

Now, by adding the perturbed term $\tau T_k \left(x_{n+j} \right)$ for $j = 0(1)k$ to Eq. (6), we obtain

$$\sum_{j=1}^{k} j a_j x^{j-1} = f(x, y) + \tau L_k \left(x_{n+j} \right). \tag{7}$$

where τ is a perturbed parameter (determined by the values of f_{n+k}) and $\tau T_k \left(x_{n+j} \right)$ is the kth Chebyshev polynomial obtained by the recursive formula

$$T_0(x) = 1, T_1(x) = x \text{ and } T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x) = 0. \tag{8}$$

We can deduce from Eq. (8) that the next three Chebyshev polynomials of the first kind are

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x \tag{9}$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Remark: The Chebyshev polynomials of first kind are orthogonal polynomials in the interval $\left[-1,1\right]$.

If a function is defined on $\left[a,b\right]$, it is sometimes necessary in the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$x(t) = \frac{2}{b-a} \left[t - \frac{b+a}{2} \right], a < b. \tag{10}$$

transforms the interval $\left[a,b\right]$ of the t- axis in to the interval $x \in \left[-1,1\right]$ of the x- axis (Suli and Mayers, 2003).

Note that Eq. (10) is the same as

$$x(t) = \frac{2t - (x_{n+k} + x_n)}{x_{n+k} - x_n}, k = 1, 2, 3, \cdots. \tag{11}$$

with the substitution of $a = x_n$ and $b = x_{n+k}$.

To derive the proposed method for each $k = 1, 2, 3, \text{and } 4$, we should follow the following steps: first, we take the Chebyshev polynomials in Eq. (9) and use Eq. (11) to convert in to the range $\left[-1,1\right]$ by collocating each $T_k(x)$ at $x_{n+j}, j = 0(1)k$ to obtain $T_k \left(x_{n+j} \right)$, where $T_k \left(x_{n+j} \right)$ is the Chebyshev polynomial at x_{n+j} such that $-1 \leq T_k \left(x_{n+j} \right) \leq 1$.

Interpolating Eq. (4) at x_{n+j}, collocating Eq. (7) at the collocating points x_{n+j} for $j = 0(1)k$, and substituting the relation $x_{n+k} = x_n + kh$, we get a system of $(k+2)$ equations with $(k+2)$ parameters as shown below.

$$\begin{cases} a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + ... + a_k x_n^k = y_n \\ a_1 + 2a_2 x_n + 3a_3 x_n^2 + ... + ka_k x_n^{k-1} - \tau T_k(x_n) = f_n \\ a_1 + 2a_2(x_n+h) + 3a_3(x_n+h)^2 + ... + ka_k(x_n+h)^{k-1} - \tau T_k(x_{n+1}) = f_{n+1} \\ \quad . \qquad . \qquad\quad . \qquad\qquad . \qquad . \qquad\quad . \\ \quad . \qquad . \qquad\quad . \qquad\qquad . \qquad . \qquad\quad . \\ \quad . \qquad . \qquad\quad . \qquad\qquad . \qquad . \qquad\quad . \\ a_1 + 2a_2(x_n+kh) + 3a_3(x_n+kh)^2 + ... + ka_{n+k}(x_n+kh)^{k-1} - \tau T_k(x_{n+k}) = f_{n+k} \end{cases} \tag{12}$$

Now, the required numerical scheme will be obtained, if we interpolate Eq. (4) at x_{n+k} to get

$$y_{n+k} = a_0 + a_1 x_{n+k} + a_2 x_{n+k}^2 + ... + a_k x_{n+k}^k \tag{13}$$

and substitute the values of $a_0, a_1, a_2, \cdots,$ and a_k.

In this paper, we derive an implicit block LMM for $k = 1, 2, 3,$ and 4.

Derivation of the Method for $k = 1$

Using Eq. (9) the Chebyshev polynomial is $T_1(x)$ and applying Eq. (11) at collocating points x_n and

x_{n+j}, we get

$$T_1\left(x(x_n)\right) = T_1\left(\frac{2x_n - \left(x_{n+1}+x_n\right)}{x_{n+1}-x_n}\right) = T_1(-1) = -1$$

and

$$T_1\left(x(x_{n+1})\right) = T_1\left(\frac{2x_{n+1} - \left(x_{n+1}+x_n\right)}{x_{n+1}-x_n}\right) = T_1(1) = 1.$$

Thus, Eq. (12) becomes

$$\begin{cases} a_0 + a_1 x_n = y_n \\ a_1 + \tau = f_n \\ a_1 - \tau = f_{n+1} \end{cases}$$

which gives the matrix form

$$\begin{pmatrix} 1 & x_n & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \end{pmatrix}.$$

Solving the above matrix gives the values

$$\begin{cases} \tau = \dfrac{1}{2}\left(f_n - f_{n+1}\right) \\ a_0 = y_n - \dfrac{1}{2}\left(f_n + f_{n+1}\right)x_n \\ a_1 = \dfrac{1}{2}\left(f_n + f_{n+1}\right) \end{cases} \qquad .$$

Substituting these values in Eq. (13), we obtain

$$y_{n+1} = y_n + \frac{h}{2}\left(f_n + f_{n+1}\right). \tag{14}$$

Therefore, Eq. (14) is the numerical scheme when $k = 1$, which is the well-known trapezoidal rule.

Derivation of the Method for $k = 2$

Using Eq. (9) the Chebyshev polynomial for $k = 2$ is $T_2(x) = 2x^2 - 1$ and applying Eq. (11) at collocating points $x_n, x_{n+1},$ and x_{n+2}, we get

$$T_2\left(x\left(x_n\right)\right) = T_2\left(\frac{2x_n - \left(x_{n+2}+x_n\right)}{x_{n+2} - x_n}\right) = T_2(-1) = 1,$$

$$T_2\left(x\left(x_{n+1}\right)\right) = T_2\left(\frac{2x_{n+1} - \left(x_{n+2}+x_n\right)}{x_{n+2} - x_n}\right) = T_2(0) = -1, \text{and}$$

$$T_2\left(x\left(x_{n+2}\right)\right) = T_2\left(\frac{2x_{n+2} - \left(x_{n+2}+x_n\right)}{x_{n+2} - x_n}\right) = T_2(1) = 1.$$

Thus, Eq. (12) becomes

$$\begin{cases} a_0 + a_1 x_n + a_2 x_n^2 = y_n \\ a_1 + 2a_2 x_n - \tau = f_n \\ a_1 + 2a_2 x_{n+1} + \tau = f_{n+1} \\ a_1 + 2a_2 x_{n+2} - \tau = f_{n+2} \end{cases}$$

which gives the matrix form

$$\begin{pmatrix} 1 & x_n & x_n^2 & 0 \\ 0 & 1 & 2x_n & -1 \\ 0 & 1 & 2x_{n+1} & 1 \\ 0 & 1 & 2x_{n+2} & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix}.$$

Solving the above matrix gives the values

$$\begin{cases} \tau = \frac{1}{4}\left(-f_n + 2f_{n+1} - f_{n+2}\right) \\ a_0 = -\frac{1}{4h}\left(3hf_n x_n + 2hf_{n+1}x_n - hf_{n+2}x_n + f_n x_n^2 - f_{n+2}x_n^2 - 4hy_n\right) y_n \\ a_1 = \frac{1}{4h}\left(3hf_n + 2hf_{n+1} - hf_{n+2} + 2f_n x_n - 2f_{n+2}x_n\right) \\ a_2 = -\frac{1}{4h}\left(f_n - f_{n+2}\right) \end{cases}$$

Substituting these values in Eq. (13), we get

$$y_{n+2} = y_{n+1} + \frac{h}{2}(f_{n+1} + f_{n+2}).$$ (15)

Therefore, Eq. (15) is the numerical scheme when $k = 2$.

Derivation of the Method for $k = 3$

Using Eq. (9) the Chebyshev polynomial for $k = 3$ is $T_3(x) = 4x^3 - 3x$ and applying Eq. (11) at collocating points x_n, x_{n+1}, x_{n+2}, and x_{n+3}, we get

$$T_3\left(x(x_n)\right) = T_2\left(\frac{2x_n - (x_{n+3}+x_n)}{x_{n+3} - x_n}\right) = T_3(-1) = -1,$$

$$T_3\left(x(x_{n+1})\right) = T_3\left(\frac{2x_{n+1} - (x_{n+3}+x_n)}{x_{n+3} - x_n}\right) = T_3\left(-\frac{1}{3}\right) = \frac{23}{27},$$

$$T_3\left(x(x_{n+2})\right) = T_3\left(\frac{2x_{n+2} - (x_{n+3}+x_n)}{x_{n+3} - x_n}\right) = T_3\left(\frac{1}{3}\right) = -\frac{23}{27},\text{and}$$

$$T_3\left(x(x_{n+3})\right) = T_3\left(\frac{2x_{n+3} - (x_{n+3}+x_n)}{x_{n+3} - x_n}\right) = T_3(1) = 1.$$

Thus, Eq. (12) becomes:

$$\begin{cases} a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 = y_n \\ a_1 + 2a_2 x_n + 3a_3 x_n^2 + \tau = f_n \\ a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 - \frac{23}{27}\tau = f_{n+1} \\ a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + \frac{23}{27}\tau = f_{n+2} \\ a_1 + 2a_2 x_{n+3} + 3a_3 x_{n+3}^2 - \tau = f_{n+3} \end{cases}.$$

which gives the matrix form:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & 0 \\ 0 & 1 & 2x_n & 3x_n^2 & 1 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & -\frac{23}{27} \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & \frac{23}{27} \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}.$$

Solving the matrix above and substituting the values of the unknown parameters in Eq. (13), we get:

$$y_{n+3} = y_{n+2} + \frac{h}{96}(-3f_n + f_{n+1} + 55f_{n+2} + 43f_{n+3}).$$ (16)

Therefore, Eq. (16) is the numerical scheme when $k = 3$.

Derivation of the Method for $k = 4$

Using Eq. (9) the Chebyshev polynomial for $k = 4$ is $T_4(x) = 8x^4 - 8x^2 + 1$ and applying Eq. (11) at collocating points $x_n, x_{n+1}, x_{n+2}, x_{n+3},$ and x_{n+4} , we get

$$T_4\left(x\left(x_n\right)\right) = T_4\left(\frac{2x_n - \left(x_{n+4} + x_n\right)}{x_{n+4} - x_n}\right) = T_4(-1) = 1,$$

$$T_4\left(x\left(x_{n+1}\right)\right) = T_4\left(\frac{2x_{n+1} - \left(x_{n+4} + x_n\right)}{x_{n+4} - x_n}\right) = T_4\left(-\frac{1}{2}\right) = \frac{1}{2},$$

$$T_4\left(x\left(x_{n+2}\right)\right) = T_4\left(\frac{2x_{n+2} - \left(x_{n+4} + x_n\right)}{x_{n+4} - x_n}\right) = T_4\left(0\right) = 1,$$

$$T_4\left(x\left(x_{n+3}\right)\right) = T_4\left(\frac{2x_{n+3} - \left(x_{n+4} + x_n\right)}{x_{n+4} - x_n}\right) = T_4\left(\frac{1}{2}\right) = -\frac{1}{2},\text{and}$$

$$T_4\left(x\left(x_{n+4}\right)\right) = T_4\left(\frac{2x_{n+4} - \left(x_{n+4} + x_n\right)}{x_{n+4} - x_n}\right) = T_4\left(1\right) = 1.$$

Thus, Eq. (12) becomes

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & 0 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & -1 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \frac{1}{2} \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & -1 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & \frac{1}{2} \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & -1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix}$$

Solving the above matrix and substituting the values of the unknown parameters in Eq. (13), we get

$$y_{n+4} = y_{n+3} + \frac{h}{48}(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4}). \tag{17}$$

Therefore, Eq. (17) is the numerical scheme when $k = 4$.

Generally, the proposed block procedure with the implicit LMM is given as follows.

$$\begin{cases} y_{n+1} = y_n + \dfrac{h}{2}\left(f_n + f_{n+1}\right) \\[2mm] y_{n+2} = y_{n+1} + \dfrac{h}{2}\left(f_{n+1} + f_{n+2}\right) \\[2mm] y_{n+3} = y_{n+2} + \dfrac{h}{96}\left(-3f_n + f_{n+1} + 55f_{n+2} + 43f_{n+3}\right) \\[2mm] y_{n+4} = y_{n+3} + \dfrac{h}{48}\left(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4}\right) \end{cases} \tag{18}$$

Order and Error Constant of the Method

By using the substitutions $y_{n+j} = z^j$ and $hf_{n+j} = z^j$ where z is a variable and $j = 0(1)k$ for the LMMs given in Eq. (2), we introduce at this point a polynomial which is known as characteristic polynomials as shown below.

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \text{ and } \sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

The polynomials $\rho(z)$ and $\sigma(z)$ are called the first and second characteristics polynomials respectively. Moreover, following Henrici (1962), the approach developed in Lambert (1991), and Suli and Mayers (2003), the local truncation error associated with Eq. (2) by the difference operator is defined as:

$$L\big[y(x):h\big] = \frac{1}{h\sum_{j=0}^{k}\beta_j}\left(\sum_{j=0}^{k}\Big[\alpha_j y(x_n + jh) - h\beta_j f(x_n + jh)\Big]\right) \qquad (19)$$

Assuming $y(x)$ is smooth and then expanding the function $y(x + jh)$ and its derivative $y'(x + jh)$ in Taylor series about x and substituting them into Eq. (19) after collecting like terms, we obtain

$$L[y(x):h] = \frac{1}{h\sigma(1)}[c_0 y(x_n) + c_1 hy'(x_n) + c_2 h^2 y''(x_n) + ... + c_{p+1}h^{p+1}y^{p+1}(x_n)] \qquad (20)$$

where $\sigma(1)$ is the value of the second characteristic polynomial at $z = 1$ and the values of the coefficients is given as follows:

$$c_0 = \sum_{j=0}^{k} \alpha_j,$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j,$$

$$c_2 = \sum_{j=1}^{k} \frac{j^2}{2!}\alpha_j - \sum_{j=1}^{k} j\beta_j,$$

$$\vdots$$

$$c_p = \sum_{j=1}^{k} \frac{j^p}{p!}\alpha_j - \sum_{j=1}^{k} \frac{j^{p-1}}{(p-1)!}\beta_j.$$

According to Lambert (1991), a given LMM is order p, if in Eq. (20), we get

$c_0 = c_1 = c_2 = \cdots = c_p = 0$ and $c_{p+1} \neq 0.$ In this case, the number $\dfrac{c_{p+1}}{\sigma(1)}$ is called the error constant of the method.

Accordingly, the orders and the error constants of Eq. (18) are shown below (Table 1).

Table 1. Orders and Error Constants of the Schemes

Step	Order	Error Constant
$k = 1$	2	–0.08333
$k = 2$	2	–0.08333
$k = 3$	3	–0.07292
$k = 4$	4	–0.04722

Stability of the Block Method

According to Fatunla (1988), a linear multistep method (LMM) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one and that any root with modulus one is

simple. In other words, a LMM is said to be zero stable if $|z| \leq 1$ and if z is a repeated root, then $|z| < 1$. Based on this definition, the zero stability of the block method given by Eq. (18) is checked using the first characteristic polynomials as follows:

For $k = 1$, we have $\rho(z) = z - 1 = 0$.

For $k = 2$, we have $\rho(z) = z^2 - 1 = z(z - 1) = 0$.

For $k = 3$, we have $\rho(z) = z^3 - z^2 = z^2(z - 1) = 0$.

For $k = 4$, we have $\rho(z) = z^4 - z^3 = z^3(z - 1) = 0$.

Owing to the work of Fatunla (1988), our method is zero stable.

Consistency of the Method
According to Lambert (1991), a LMM is said to be consistent if it has order at least one. Since all the schemes for each $k = 1, 2, 3, \text{and } 4$ are all degree greater than one, the block method given by Eq. (18) is consistent (See Table 1).

Convergence of the Method
According to Dahlquist (1974), consistency and zero stability are the necessary and sufficient conditions for the convergence of any numerical scheme. Since our method is both consistent and zero stable, it is thus convergent.

Numerical Examples
The mode of implementation of our method is by combining the schemes Eq. (18) as a block for solving Eq. (1). It is a simultaneous integrator without requiring the starting values except the initial condition from the problem. To assess the performance of the present method, we considered two stiff first order initial value problems. Its maximum absolute errors were compared with that of the Sixth Order LMM developed by Berhan et al. (2019).

Example 1: (LeVeque, 2007). Consider the first order stiff ODE

$$y'(x) = -2100\big(y - \cos(x)\big) - \sin(x), \, y(0) = 1, \, x \in \big[0,1\big].$$

The exact solution is $y(x) = \cos(x)$.

Table 2. Maximum Absolute errors of Example 1 with different values of mesh size h

h	RK4	Sixth Order LMM	Present Method
10^{-1}	1.22516e+74	1.22516e-5	5.86307e-7
10^{-2}	2.41053e+304	9.67880e-8	5.71593e-9
10^{-3}	1.53563e-7	6.46040e-11	3.33170e-11
10^{-4}	5.09304e-12	3.33844e-13	3.33844e-13
10^{-5}	1.22125e-15	4.10783e-15	4.10783e-15

The present method gives better result than the Sixth Order LMM. Moreover, RK4 method diverges for $h = 10^{-1}$ and $h = 10^{-2}$ (Table 2).

Example 2: (Faul, 2018). Consider the first order stiff ODE

$$y'(x) = -1000000\left(y - \frac{1}{x}\right) - \frac{1}{x^2}, \, y(1) = 1, x \in [1,2].$$

The exact solution is $y(x) = \frac{1}{x}$.

Table 3. Maximum Absolute errors of Example 2 with different values of mesh size h

h	RK4	Present Method
10^{-1}	7.50777e+178	1.26594e-8
10^{-2}	5.11556e+298	1.12913e-10
10^{-3}	4.20759e+298	9.95981e-13
10^{-4}	1.09133e+300	9.76996e-15
10^{-5}	5.89885e+301	2.22044e-16
10^{-6}	3.35287e-14	2.22044e-16

Runge Kutta (RK4) method diverges for various values of step size h which shows that the IVP considered is a stiff differential equation (Table 3).
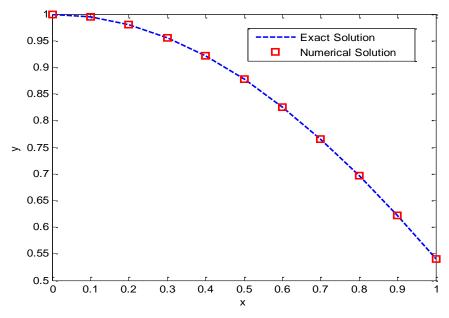

Figure 1: Plot of the numerical solution and exact solution for $h = 0.1$.

The numerical solution is in agreement with the exact solution (Fig. 1).

DISCUSSION

A stiff differential equation is an equation if for solving a differential equation of certain numerical methods are numerically unstable, unless the step size is taken to be extremely small (Suli and Mayers, 2003). Stiff equations are problems for which explicit methods don't work (Higham and Trefethen, 1993). That is why RK4 becomes numerically unstable for example for $h = 10^{-1}$ and $h = 10^{-2}$ as indicated in the current study. Our result is better may be due to the fact that the performance of numerical methods depend on the characteristics of the ODE considered such as stiffness (Hull et al., 1972; Muhammmad and Arshad, 2013). While the central activity of numerical analysts is providing accurate and efficient general purpose numerical methods and algorithms, there has always been a realization that some problem types have distinctive features that they will need their own special theory and techniques (Butcher, 2008).

The other reason may be due to the behavior of the basis function we have used. The Chebyshev and Legendre spectral methods have excellent error properties in the approximation of a globally smooth function (Wang and Xiang, 2012). Chebyshev polynomials have a wide variety of practical uses in

numerical algorithms and are easy to compute and apply. Being orthogonal polynomials, they share many properties with the familiar Legendre polynomials, but are generally much better behaved (Thompson, 1994). He further explained that, as orthogonal basis functions, Chebyshev polynomials are related to Fourier cosine functions and to Fourier series, with the advantage that they are polynomials rather than the infinite series defining the cosine. That may be the reason that our method is better even though it is fourth order compared to the sixth order LMM developed by Berhan et al. (2019).

CONCLUSION

This paper presented a block procedure with the implicit linear multistep method based on Chebyshev polynomials for solving first order IVPs in ODEs. A perturbed collocation approach along with interpolation at some grid points which produces a family block schemes with maximal order four has been proposed for the numerical solution of stiff problems in ODEs. The method is tested and found to be consistent, zero stable and convergent. We implemented the method on two numerical examples, and the numerical results showed that the method is accurate and effective for stiff IVPs.

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