ORIGINAL ARTICLE
Multipole Expansion of Relativistic Electric Potential

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Abstract
The concept of relativistic multipole expansion is applied to determine the multipole moments of the electric potential of a charged system moving at constant velocity with respect to a stationary observer. A system of discrete point charges and a charged plate of uniform surface charge density were considered for analysis. For relativistic considerations, two reference frames: namely S and S′ were chosen. S′, which contains the charged system, is moving with constant velocity relative to frame S, to which stationary observer is attached. For easy comparison, both multipole expansions are expressed in terms of the coordinates of the S′ frame. The two observers in S and S′, in general, calculate different expressions of multipole expansion for the same charged system because of Lorentz contraction effect. The multipole terms of the potential evaluated by the observer in S is time dependent. This shows that there are also magnetic effects, which are not observed by an observer in S′.

Keywords: Charge Density; Electric Potential; Magnetic Vector Potential; Multipole Expansion

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INTRODUCTION

The connection between electromagnetism and relativity is not as well understood as we would like to believe. Multipole expansion is one of the most important special techniques for solving electrostatic problems (Griffiths, 1999; Jackson, 1999; Neyfeh and Brussel, 2015). Its importance originates from the fact that it gives more informative non-zero value of electrostatic potential even if the total charge of a given localized charge distribution is zero. Beyond electrostatics, multipole expansion has also a wide range of applications in various aspects of physics and mathematics such as magic cubes (Rogers and Loly, 2005), magnetostatics (Griffiths, 1999; Jackson, 1999; Neyfeh and Brussel, 2015), gravitational field (Blanchet, 1998), radiation (Nieminen et al., 2003), and others (Paolis et al., 1995; Gonzalez, 1998; Esbensen and Bertulani, 2002; Qian and Krimm, 2005; Taylor and Love, 2009; Anandakrishnan et al., 2013; Zhou et al., 2016; Frutos-Alfaro and Soffel, 2018). Therefore, careful understanding of its concept and the possibility of its applications in various fields are crucial for scholars working in the area of electromagnetic theory. Multipole expansion of electric potential has been limited to cases in which the observer is at rest relative to the charged system and relativistic consideration of its multipole expansion has not been reported in literature. However, determination of the potential of a moving charged system with respect to an observer at rest in another frame of reference in a more informative way is quite interesting. Further concepts of multipole expansion such

as relativistic multipole moments of stationary space-times (Frutos-Alfaro and Soffel, 2018) and multipole expansion for relativistic Coulomb excitation (Esbensen and Bertulani, 2002) have been reported in literature. However, the present problem is still not addressed in the reported research works.

The retarded potential of a point charge q that is moving on a specified trajectory (Lienard-Wiechert potentials) has been reported and even included on main reference books for electromagnetic theory (Griffiths, 1999; Jackson, 1999). Even though point charge can be regarded as the limit of an extended charge when the size goes to zero, Lienard-Wiechert potentials cannot be really applied when there is a system of moving point charges or extended charge distribution of significant size (Griffiths, 1999).

Therefore, a simple and reliable technique of determining the relativistic multipole expansion of the electric potential is presented in this work. This gives the most important experience of computing multipole moments of the potential of a system of point charges or a uniformly charged rigid object moving at constant velocity with respect to an observer at rest. It is important to use the term 'retarded potential' if one is dealing with changing charge density, for example, the charge density prevailed at some retarded time t_r. However, throughout this article, non-changing charge density is considered and the term 'retarded potential' is used. First, the theoretical model and method is presented and the mathematical formulation of the problem follows.

THEORETICAL MODEL AND METHOD

Consider two reference frames S and S' with origins at O and O′, respectively, as shown below (Fig.1) and where we have S'moving with constant velocity **v** with respect to S. Suppose that, in the frame S', there is a charge distribution that is localized in a volume V and characterized by a constant density ρ at any time t.

The displacements $\mathbf{r}'$ and $\mathbf{r}''$ locate an element of charge relative to O′ and O, respectively.

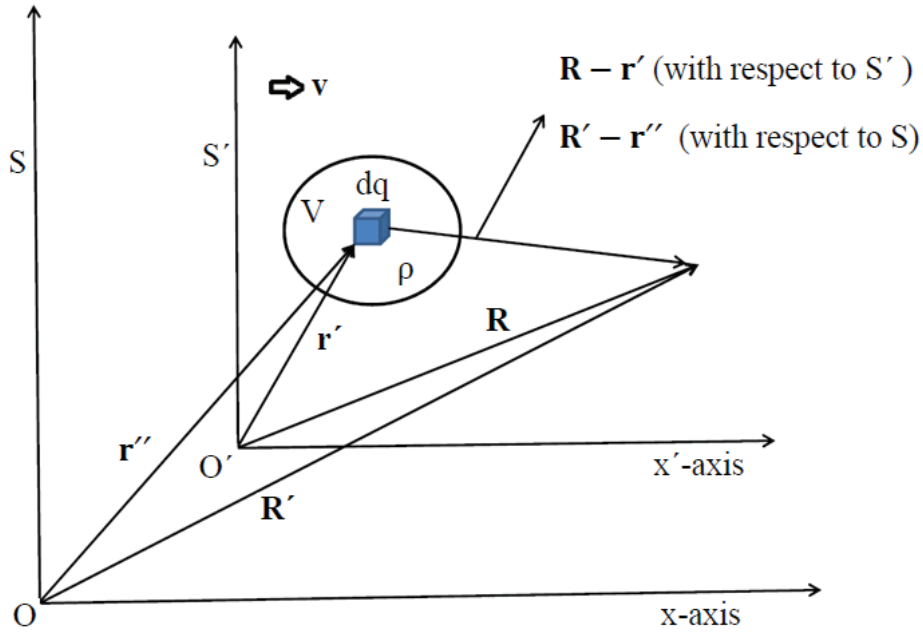

Fig. 1: Two reference frames S and S' with origins at O and O′, respectively. S', containing a charge distribution that is localized in a volume V and characterized by a density ρ, is moving with constant velocity **v** with respect to S.

The displacements **R** and **R′** locate a point in space outside the charge distribution, where we wish to determine the scalar potential ϕ. If one attaches two observers at points O and O′, then the observer at point O′ is stationary relative to the charge distribution. Therefore, the multipole expansion of the potential evaluated by this observer is the same as that reported in literature (Griffiths, 1999; Jackson, 1999; Neyfeh and Brussel, 2015). However, one may ask the question that: what is the multipole expansion of the potential as evaluated by and observer at point O? This is particularly interesting if **v** approaches the speed of light and hence the name relativistic multipole expansion is used in this context. Answering this question really advances electromagnetic theory one step forward and has a high contribution in broadening the conceptual understanding of both undergraduate and postgraduate students. Just to follow common approaches of solving problems in electromagnetic theory, both discrete and continuous charge distributions are used for analysis of the problem.

RESULTS

Consider Fig. 1, let us evaluate the multipole expansion of the potential with respect to point O′ for the arbitrary charge distribution localized in a rather small region of space in frame S'. This problem is discussed in detail (Neyfeh and Brussel, 2015). In this case, the potential $\phi(R)$ is given by:

$$\phi(R) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|\mathbf{R} - \mathbf{r}'|}\,. \qquad (1)$$

Assuming $\frac{|\mathbf{r}'|}{|\boldsymbol{R}|} \ll 1$, the term in the integrand can be expanded as follows.

$$\frac{1}{|\mathbf{R}-\mathbf{r}'|} = \frac{1}{|\mathbf{R}|\left\{1-\frac{2\mathbf{R}.\mathbf{r}'}{|\mathbf{R}|^2}+\left[\frac{\mathbf{r}'}{\mathbf{R}}\right]^2\right\}^{\frac{1}{2}}}. \tag{2}$$

Let us now write the binomial expansion,

$$(1+\mathrm{x})^{-\frac{1}{2}} = \left[1-\frac{1}{2}\mathrm{x}-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{\mathrm{x}^2}{2!}+\cdots\right], |\mathrm{x}|<1. \tag{3}$$

In equation (2), using $\mathrm{x} = \frac{2\mathbf{R}.\mathbf{r}'}{|\mathbf{R}|^2}+\left[\frac{\mathbf{r}'}{\mathbf{R}}\right]^2$ and grouping the powers of $\frac{|\mathbf{r}'|}{|\boldsymbol{R}|}$ in ascending order,

$$\frac{1}{|\mathbf{R}-\mathbf{r}'|} = \frac{1}{|\mathbf{R}|}\left\{1+\frac{\hat{\mathbf{R}}.\mathbf{r}'}{|\mathbf{R}|}+\frac{1}{2}\left[3\left(\frac{\hat{\mathbf{R}}.\mathbf{r}'}{|\mathbf{R}|}\right)^2-\left[\frac{\mathbf{r}'}{\mathbf{R}}\right]^2\right]+\cdots\right\}, \tag{4}$$

Therefore, the potential becomes,

$$\phi(\mathbf{R}) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|}\int \mathrm{dq} + \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|^2}\hat{\mathbf{R}}.\int \mathbf{r}'\,\mathrm{dq} + \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|^3}\int\left[\frac{3\left(\hat{\mathbf{R}}-\mathbf{r}'\right)^2-\left(\mathbf{r}'\right)^2}{2}\right]\mathrm{dq}. \tag{5}$$

$$\phi(\mathbf{R}) = \left(\phi^{(0)}\right)' + \left(\phi^{(1)}\right)' + \left(\phi^{(2)}\right)' + \cdots, \tag{6}$$

Where $\left(\phi^{(0)}\right)'$, $\left(\phi^{(1)}\right)'$ and $\left(\phi^{(2)}\right)'$ are the monopole, dipole and quadrupole terms, respectively. This is the multipole expansion of the potential calculated by an observer at point O′ at rest relative to the localized charge distribution and is an obvious result reported in several reference books of electromagnetic theory (Griffiths, 1999; Jackson, 1999; Neyfeh and Brussel, 2015). Now consider an interesting situation in which the observer at point O evaluates the multipole expansion for the same localized charge distribution, which however, is moving with constant velocity **v** along positive x-axis. For the observer attached with S′ (at point O′), the displacement vector between the localized charge dq and a point in space outside the charge distribution where we wish to determine the scalar potential ϕ is $\mathbf{R}-\mathbf{r}'$. But, for an observer attached with S (at point O), this displacement vector measures $\mathbf{R}'-\mathbf{r}''$ and is different from $\mathbf{R}-\mathbf{r}'$ due to length contraction

For an observer at point O, equation (5) becomes,

$$\phi(\mathbf{R}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|}\int dq + \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|^2}\widehat{\mathbf{R}}'.\int \mathbf{r}''dq + \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|^3}\int\left[\frac{3\left(\widehat{\mathbf{R}}' - \mathbf{r}''\right)^2 - \left(\mathbf{r}''\right)^2}{2}\right]dq. \quad (7)$$

$$\phi(\mathbf{R}') = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \cdots. \quad (8)$$

Where $\phi^{(0)}$, $\phi^{(1)}$ and $\phi^{(2)}$ are the monopole, dipole and quadrupole terms, respectively, as evaluated by this observer. This is not much interesting since the effect of the Lorentz contraction on the multipole expansion is not observed here. Moreover, it is difficult to compare the two expansions (5) and (7). Hence, let us take another illustrative view by focusing on the displacement vector $\mathbf{R}' - \mathbf{r}''$. For an observer at point O′,

$$\mathbf{R} = R_x i + R_y j + R_z k, \quad (9)$$

$$\mathbf{r}' = r'_x\, i + r'_y\, j + r'_z\, k, \quad (10)$$

However, for an observer at point O,

$$\mathbf{R} = \frac{R_x}{\gamma} i + R_y j + R_z k, \quad (11)$$

$$\mathbf{r}' = \frac{r'_x}{\gamma}\, i + r'_y\, j + r'_z\, k, \quad (12)$$

Where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Note that only the components of the vectors parallel to the velocity **v** of the S′ frame are Lorentz contracted. Therefore,

$$\mathbf{R}' - \mathbf{r}'' = \left(\frac{R_x}{\gamma} i + R_y j + R_z k\right) - \left(\frac{r'_x}{\gamma}\, i + r'_y\, j + r'_z\, k\right). \quad (13)$$

Moreover, at any time t,

$$\mathbf{r}''_x = \frac{r'_x}{\gamma} + vt, \quad (14)$$

Therefore, $\mathbf{r}'' = \left(\frac{r'_x}{\gamma} + vt\right) i + r'_y\, j + r'_z\, k$, (15)

Hence $\mathbf{R}'$ can be expressed as

$$\mathbf{R}' = \left[\left(\frac{R_x}{\gamma} i + R_y j + R_z k\right) - \left(\frac{r'_x}{\gamma}\, i + r'_y\, j + r'_z\, k\right)\right] + \mathbf{r}''. \quad (16)$$

$$\mathbf{R}' = \left[\left(\frac{R_x}{\gamma}i + R_y j + R_z k\right) - \left(\frac{r'_x}{\gamma}\, i + r'_y\, j + r'_z\, k\right)\right] + \left(\frac{r'_x}{\gamma} + vt\right)i + r'_y\, j + r'_z\, k. \quad (17)$$

$$\mathbf{R}' = \left(\frac{R_x}{\gamma} + vt\right)i + R_y j + R_z k. \quad (18)$$

Now, with the obtained expressions of $\mathbf{r}''$ and $\mathbf{R}'$, equation (7) can readily be expressed in terms of the coordinates of the S' frame and this allows us easily compare the two expansions. Hence, the monopole, dipole, and quadrupole terms of equation (5) are given by

$$\left(\phi^{(0)}\right)' = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|}\int dq = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|}Q, \quad (19)$$

Where Q is the total net charge of the discrete charge distribution.

$$\left(\phi^{(1)}\right)' = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|^2}\hat{\mathbf{R}}.\int \mathbf{r}'dq. \quad (20)$$

$$\left(\phi^{(2)}\right)' = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|^3}\int\left[\frac{3\left(\hat{\mathbf{R}} - \mathbf{r}'\right)^2 - \left(\mathbf{r}'\right)^2}{2}\right]dq. \quad (21)$$

Similarly, the monopole, dipole, and quadrupole terms of equation (7) are given by,

$$\phi^{(0)} = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|}\int dq = \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{\left(\frac{R_x}{\gamma} + vt\right)^2 + \left(R_y\right)^2 + (R_z)^2}}\int dq. \quad (22)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{\left(\frac{R_x}{\gamma} + vt\right)^2 + \left(R_y\right)^2 + (R_z)^2}}Q, \quad (23)$$

Where Q is also the total net charge of the discrete charge distribution.

$$\phi^{(1)} = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|^2}\hat{\mathbf{R}}'.\int \mathbf{r}''dq. \quad (24)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\left(\frac{R_x}{\gamma} + +vt\right)^2 + \left(R_y\right)^2 + (R_z)^2} \times \frac{\left(\frac{R_x}{\gamma} + vt\right)i + R_y j + R_z k}{\sqrt{\left(\frac{R_x}{\gamma} + vt\right)^2 + \left(R_y\right)^2 + (R_z)^2}}.\int\left(\left(\frac{r'_x}{\gamma} + vt\right)i + r'_y\, j + r'_z\, k\right)dq \quad (25)$$

$$\phi^{(2)} = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}'|^3}\int\left[\frac{3(\hat{\mathbf{R}}\cdot\mathbf{r}'')^2-(\mathbf{r}'')^2}{2}\right]dq. \quad (26)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\left(\left(\frac{R_x}{\gamma}+vt\right)^2+(R_y)^2+(R_z)^2\right)^{\frac{3}{2}}} \times \int\left[\frac{3\left(\frac{\left(\frac{R_x}{\gamma}+vt\right)i+R_y j+R_z k}{\sqrt{\left(\frac{R_x}{\gamma}+vt\right)^2+(R_y)^2+(R_z)^2}}-\left(\left(\frac{r'_x}{\gamma}+vt\right)i+r'_y j+r'_z k\right)\right)^2-\left(\left(\frac{r'_x}{\gamma}+vt\right)i+r'_y j+r'_z k\right)^2}{2}\right]dq \quad (27)$$

DISCUSSION

From the above results, it can be clearly seen that the expressions of the monopole, dipole, and quadrupole terms calculated by the two observers at O and O′ are in general different from one another. The multipole terms are only position dependent for an observer at point O′, but they are both position and time dependent for an observer at O. In both cases, the presence of the monopole term indicates that far enough away the charge distribution in the lowest-order approximation looks like a point charge. Therefore, it is clearly observed that there is time dependence in the potential expansion for one observer and not for the other. This suggests that there are also magnetic effects observed in one frame not observed in the other. In particular, the time dependence of the multipole terms for an observer at O shows the observation of magnetic effects and this is not seen for an observer at O′. For an observer at O, the magnetic field can be evaluated form the expression $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A}$ is the magnetic vector potential which can further be calculated form the following expression (Griffiths, 1999).

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\boldsymbol{r}'')dV}{|\boldsymbol{R}'-\boldsymbol{r}''|}, \quad (28)$$

Where,

$$\mathbf{J} = \rho_0\frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (29)$$

and ρ_0 is the proper charge density (Griffiths, 1999).

Let us consider the case where there is a continuous charge distribution within the volume V. In particular, imagine a thin plate of length L and width W with charge per unit area (surface charge density) σ_0 at rest in frame S' with its length oriented parallel to the velocity **v** of the frame S'. Since charge is invariant, the two observers at points O and O' measure the same charge both for the case of discrete and continuous surface charge distributions within the volume V. This means that the two observers measure the same charge on the thin plate of length L and width W. Wait a minute! Do they still measure the same charge density on the plate? Obviously, no. This is because of the fact that the two observers measure different lengths of the thin plate. For observer at point O, the length L is Lorentz contracted by a factor of:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (30)$$

Therefore, for this observer, the charge per unit area is increased by a factor of γ. The charge density of the plate as viewed from point O is,

$$\sigma = \gamma\sigma_0, \text{or } Q = \sigma A = \gamma\sigma_0 A. \qquad (31)$$

Where A is the area as measured from point O. If Q' is the charge on the plate as measured by observer at O', then $Q' = \sigma_0 A' = \gamma\sigma_0 A$, since $A' = \gamma A$. Therefore, $Q' = Q$ and this confirms that equation (7) holds true for any arbitrary charge distribution.

CONCLUSION

Using relativistic multipole expansion, the multipole moments of the electric potential of a charged system moving at constant velocity with respect to a stationary observer are determined. The two observers attached to points O and O', in general, calculate different expressions of multipole expansion for the same charged system because of Lorentz contraction effect. It is clearly observed that there is time dependence in the potential expansion for one observer and not for the other. This suggests that there are also magnetic effects observed in one frame not observed in the other.

In summary, the manuscript tried to address a very problematic and subtle subject of the multipole moments of the electric potential of a charged system moving at constant velocity with respect to a stationary observer. In particular, different expressions of multipole expansion for the charged system are examined using non-trivia mathematical steps. Therefore, we strongly recommend scholars working in the field of electromagnetic theory to further investigate this subtle issue and include it in their postgraduate lessons.

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