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Fitted Fourth Order Scheme for Singularly Perturbed Delay Convection-Diffusion Equations

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ABSTRACT

This paper presents a fitted fourth order numerical scheme for solving singularly perturbed convection-diffusion equations. The obtained scheme is transformed into a three-term recurrence relation and solved by Thomas algorithm. The stability and convergence of the present method have been investigated. The numerical results are presented by tables and graphs. The present method helps us to get good results and also to know the behavior of the solution in the boundary layer for perturbation parameter ε is less than mesh size h. Moreover, the present method improves the findings of some existing numerical methods reported in the literature.

Keywords: Boundary layer; Convection-diffusion equation; Singularly perturbed; Stability

INTRODUCTION

The classification of singularly perturbed higher order problems depends on how the order of the original equation is affected when small positive parameter ε is multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection-diffusion type and reaction-diffusion type if the order is reduced by two. Singularly perturbed delay differential equations are special cases of functional differential equations, where the evolution of a system at a certain time, depends on the present state of the system as well as the state of the system at an earlier time. For example, in the predator-prey model (Martin and Raun, 2001), the birth of predators is affected by prior levels of the predator-prey model along with its recent levels. In general, a singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Delay differential equations arise frequently in the mathematical modeling of various occurrences such as reaction-diffusion equations (Bestehorn and

Grigorieva, 2004), control systems (Mackey and Glass, 1997), neuron variability (Lange and Miura, 1994), thermo-elasticity (Ezzat et al., 2002), micro-scale heat transfer (Tzou, 1997), etc. According to Doolan et al. (1980) still there is a lack of accuracy and convergence of numerical methods because of the treatment of singular perturbation problems is not trivial, and the solution depends on perturbation parameter and mesh size. Due to this, the numerical treatment of singularly perturbed delay differential equations needs improvement. In recent years, many researchers have tried to develop different numerical methods for solving singularly perturbed delay differential equations. For examples, finite difference method of various orders and approaches (Awoke and Reddy, 2013; Phaneendra and Soujanya, 2014; Gemechis et al., 2017; Gashu et al., 2018), Spline method (Kanth and Kumar, 2017), Numerical integration method (Reddy et al., 2012; Sirisha and Redddy, 2017), hybrid initial value method (Subburayan, 2016), Galerkin method (Swamy et al., 2016), and Differential quadrature method (Prasad and Reddy, 2012) are presented for solving singularly perturbed delay differential equations. However, the issue of accuracy and convergence of the scheme still needs attention and improvement. In this paper, we present a stable and convergent method and more accurate than the stated methods for solving singularly perturbed delay convection-diffusion equations.

MATERIALS AND METHODS

Description of the Method

Consider a singularly perturbed delay convection-diffusion equation of the form:

$$\varepsilon y''(x)+a(x)y'(x-\delta)+b(x)y(x)=f(x), \quad \text{for } x\in[0,\ 1] \tag{1}$$

with the interval and boundary conditions,

$$y(x)=\alpha(x),\ -\delta\le x\le 0 \text{ and } y(1)=\beta \tag{2}$$

where ε is a perturbation parameter, $0<\varepsilon<<1$ and δ is a delay parameter, $0<\delta<<1$; $a(x), b(x), f(x)$ and $\alpha(x)$ are bounded smooth functions in $(0,1)$ and β is a given constant. For a function $y(x)$ to be a smooth solution of the Eq. (1), it must satisfy the boundary conditions Eq. (2), be continuous on [0,1] and be continuously differentiable on (0,1). It is also assumed that $b(x)\le-\theta<0,\ \forall x\in[0,1]$, where θ is a positive constant. Further, when $a(x)>0$ Eqs. (1) and (2) have boundary layer on left end of the interval and when $a(x)<0$ it has boundary layer on right end of the interval. The layer is maintained at the same end for sufficiently small δ, *i.e.*, when $\delta=o(\varepsilon)$. The layer behavior can change its character and even be destroyed as the delay increase, *i.e.*, when $\delta=O(\varepsilon)$, Lange and Miura, (1994).

When $\delta=o(\varepsilon)$, the use of Taylor's series expansion for the term containing delay is valid (Tian, 2000). Thus, by using Taylor series expansion, we have:

$$y'(x-\delta) \approx y'(x) - \delta y''(x) + O(\delta^2) \quad (3)$$

Substituting Eq. (3) into Eq. (1), we obtain an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\gamma y''(x) + a(x) y'(x) + b(x) y(x) = f(x), \text{ for } [0,1], \quad (4)$$

where $\gamma = \varepsilon - \delta a(x) > 0$, and $0 < \gamma << 1$, under the boundary conditions,

$$y(0) = \alpha_0 \text{ and } y(1) = \beta. \quad (5)$$

Discretizing the given interval $[0,1]$ into N equal parts with constant mesh size h, we have $x_i = x_0 + ih, \ i = 0,1,2,\ldots,N$.

Using the Taylor's series expansions of y_{i+1} and y_{i-1} up to $O(h^5)$, we get the finite difference approximations for y_i' and y_i'' as:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + \tau_1 \quad (6)$$

where, $\tau_1 = -\dfrac{h^4}{120} y^{(5)}(\xi_1)$, for $\xi_1 \in [x_{i-1}, x_i]$, and

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + \tau_2 \quad (7)$$

Where $\tau_2 = -\dfrac{h^4}{360} y^{(6)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$.

Substituting Eqs. (6) and (7) into Eq. (4) and simplifying, we obtain:

$$\begin{aligned} &\frac{\gamma}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + \frac{a_i}{2h}\left(y_{i+1} - y_{i-1}\right) \\ &- \frac{h^2}{6} a_i y_i''' - \frac{\gamma h^2}{12} y_i^{(4)} + b_i y_i = f_i + \tau_i \end{aligned} \quad (8)$$

where, $\tau_i = h^4 \left(\dfrac{a_i}{120} y^{(5)}(\xi_1) + \dfrac{\gamma}{360} y^{(6)}(\xi_2) \right) + O(h^5)$ is the local truncation error and $a(x_i) = a_i$, $b(x_i) = b_i$, $f(x_i) = f_i$ and $y_i \approx y(x_i)$.

By successively differentiating both sides of Eq. (4) and evaluating at x_i, and using into Eq. (8), we obtain:

$$\frac{\gamma}{h^2}\left(y_{i+1}-2y_i+y_{i-1}\right)+\frac{a_i}{2h}\left(y_{i+1}-y_{i-1}\right)$$
$$+\gamma\left\{\frac{h^2a_i^2}{6\gamma}+\frac{h^2}{12}\left(2a_i'+b_i-\frac{a_i^2}{\gamma}\right)\right\}y_i'' \qquad (9)$$
$$+A_iy_i'+\left(B_i+b_i\right)y_i=f_i+C_i$$

where,

$$A_i=\frac{h^2}{6\gamma}a_i\left(a_i'+b_i\right)-\frac{h^2}{12}\left(\frac{a_i}{\gamma}\left(a_i'+b_i\right)-a_i''-2b_i'\right)$$

$$B_i=\frac{h^2}{6\gamma}a_ib_i'-\frac{h^2}{12}\left(\frac{a_ib_i'}{\gamma}-b_i''\right)$$

$$C_i=f_i+\frac{a_ih^2}{12\gamma}f_i'+\frac{h^2}{12}f_i''+\tau_i$$

Now, introducing a fitting parameter σ and using central difference approximation for y_i'' and y_i' in Eq. (9), we have:

$$\frac{\sigma\gamma}{h^2}\left\{1+\frac{h^2a_i^2}{12\gamma}+\frac{h^2}{12}\left(2a_i'+b_i\right)\right\}\left(y_{i+1}-2y_i+y_{i-1}\right)$$
$$+\frac{1}{2h}\left(a_i+A_i\right)\left(y_{i+1}-y_{i-1}\right)$$
$$+\left(B_i+b_i\right)y_i=f_i+C_i \qquad (10)$$

Multiplying both sides of Eq. (10) by h and taking the limit as $h\rightarrow 0$, we obtain:

$$\frac{\sigma}{12\rho}\left(12+\rho^2a_i^2\right)\lim_{h\rightarrow 0}\left\{y_{i+1}-2y_i+y_{i-1}\right\}$$
$$+\frac{a_i}{2}\lim_{h\rightarrow 0}(y_{i+1}-y_{i-1})=0 \qquad (11)$$

where $\rho=\dfrac{h}{\gamma}$.

From the theory of singular perturbations and (O'Malley, 1974) we have two cases for $a(x)>0$ and $a(x)<0$.

Case 1: For $a(x)<0$ (right-end boundary layer), we have:

$$\lim_{h\to 0}\left(y_{i-1}-2y_i+y_{i+1}\right)$$
$$=\left(\alpha_0-y_0(0)\right)e^{-a(0)\left(\frac{-1}{\gamma}+i\rho\right)}\left(e^{-a(0)\rho}+e^{a(0)\rho}-2\right)$$
$$\lim_{h\to 0}\left(y_{i+1}-y_{i-1}\right)$$
$$=\left(\alpha_0-y_0(0)\right)e^{-a(0)\left(\frac{-1}{\gamma}+i\rho\right)}\left(e^{-a(0)\rho}-e^{a(0)\rho}\right)$$

Thus, from Eq. (11), we get:

$$\sigma(0)=\frac{6\rho a(0)}{\left(12+\rho^2a^2(0)\right)}\coth\left(\frac{a(0)\rho}{2}\right)$$

Case 2: For $a(x)>0$ (left-end boundary layer), we have:

$$\lim_{h\to 0}\left(y_{i-1}-2y_i+y_{i+1}\right)$$
$$=\left(\beta-y_0(1)\right)e^{-a(1)\left(\frac{-1}{\gamma}+i\rho\right)}\left(e^{-a(1)\rho}+e^{a(1)\rho}-2\right)$$
$$\lim_{h\to 0}\left(y_{i+1}-y_{i-1}\right)$$
$$=\left(\beta-y_0(1)\right)e^{-a(1)\left(\frac{-1}{\gamma}+i\rho\right)}\left(e^{-a(1)\rho}-e^{a(1)\rho}\right)$$

Thus, from Eq. (11), we get:

$$\sigma(1)=\frac{6\rho a(1)}{\left(12+\rho^2a^2(1)\right)}\coth\left(\frac{a(1)\rho}{2}\right)$$

In general, for discretization, we take a variable fitting parameter as:

$$\sigma_i=\frac{6\rho_i a_i}{\left(12+(\rho_i a_i)^2\right)}\coth\left(\frac{a_i\rho_i}{2}\right) \qquad (12)$$

where, $\rho_i=\dfrac{h}{\varepsilon-\delta a_i}$.

Simplifying Eq. (10), we get the tri-diagonal system of the equation of the form:

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i\,, \qquad (13)$$

for $i=1,2,\ldots,N-1$.

where,

$$E_i=\frac{\sigma_i\gamma}{h^2}+\frac{\sigma_i a_i^2}{12}+\frac{\sigma_i\gamma}{12}\left(2a_i'+b_i\right)-\frac{1}{2h}\left(a_i+A_i\right)$$

$$F_i = \frac{2\sigma_i\gamma}{h^2} + \frac{\sigma_i a_i^2}{6} + \frac{\sigma_i\gamma}{6}\left(2a_i' + b_i\right) - b_i - B_i$$

$$G_i = \frac{\sigma_i\gamma}{h^2} + \frac{\sigma_i a_i^2}{12} + \frac{\sigma_i\gamma}{12}\left(2a_i' + b_i\right) + \frac{1}{2h}\left(a_i + A_i\right)$$

$$H_i = f_i + C_i$$

The tri-diagonal system in Eq. (13) can be easily solved by Thomas algorithm with the help of Matlab 2013.

Stability and Convergence Analysis

Eqs. (4) and (5) can be written as:

$$L_\gamma y(x) \equiv \gamma y''(x) + a(x)y'(x)$$
$$+b(x)y(x) = f(x)$$

under the boundary conditions, $y(0) = \alpha_0$ and $y(1) = \beta$.

Lemma 1 (Minimum Principle): Suppose $\psi(x)$ is a smooth function satisfying $\psi(0) \ge 0,\ \psi(1) \ge 0$. Then $L_\gamma\psi(x) \le 0,\ \forall x \in (0,1)$ implies $\psi(x) \ge 0,\ \forall x \in [0,1]$ (Gashu et al., 2018).

Lemma 2: (Boundedness of the Solution)

Let $\hat{y}(x)$ be the solution of the Eqs. (4) and (5), then we have:

$$\|\hat{y}\| \le \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\},$$

where $\|.\|$ is the L_∞ norm given by $\|\hat{y}\| = \max_{0\le x\le 1}|\hat{y}(x)|$.

Proof: Let us construct the two barrier functions $\psi^\pm$ defined by:

$$\psi^\pm(x) = \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\} \pm \hat{y}(x)$$

Then we have:

$$\psi^\pm(0) = \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\} \pm \hat{y}(0)$$
$$= \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\} \pm \alpha_0 \ge 0$$
$$\psi^\pm(1) = \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\} \pm \hat{y}(1)$$
$$= \theta^{-1}\|f\| + \max\{|\alpha_0|, |\beta|\} \pm \beta \ge 0$$

Thus, $$L_\gamma\psi^\pm(x) = \gamma(\psi^\pm(x))'' + a(x)(\psi^\pm(x))' + b(x)\psi^\pm(x)$$

$$= b(x)\left\{\theta^{-1}\|f\| + \max\left\{|\alpha_0|,|\beta|\right\}\right\} \pm L_\gamma \hat{y}(x)$$

$$= b(x)\left\{\theta^{-1}\|f\| + \max\left\{|\alpha_0|,|\beta|\right\}\right\} \pm f(x)$$

As $b(x) \le -\theta < 0$ implies $b(x)\theta^{-1} \le -1$ and since $\|f\| \ge f(x)$, we get:

$$L_\gamma \psi^{\pm}(x) \le \left(-\|f\| \pm f(x)\right)$$
$$+ b(x)\max\left\{|\alpha_0|,|\beta|\right\} \le 0,\ \forall x \in (0,1).$$

Therefore by Lemma 1, we obtain $\psi^{\pm}(x) \ge 0,\ \forall x \in [0,1]$, which gives the required estimate.

Lemma 3: (Stability)

Let D is a coefficient matrix of the tri-diagonal of Eq. (13). Then, for all $\gamma > 0$, the matrix D is an irreducible and diagonally dominant matrix.

Proof: Writing Eq. (13), in matrix-vector form, we obtain:

$$DY = L$$

where, D is a tri-diagonal coefficient matrix,

$$Y = \left(y_1, y_2, \cdots, y_{N-1}\right)^T \text{ and } L = \left(H_1 - E_1\alpha_0,\ H_2, \cdots,\ H_{N-1} - G_{N-1}\beta\right)^T.$$

The co-diagonals of matrix D are E_i and G_i.

It is easily seen that, $E_i \ne 0$ and $G_i \ne 0$, $\forall i = 1,2,\ldots,N-1$. Hence, D is irreducible (Varga, 1962). By the assumption $b_i < 0$, so $|E_i + G_i| < |F_i|$. Thus, D is diagonally dominant. Hence, the scheme in Eq. (13) is stable (Gemechis et al., 2017).

Definition 1 (Consistency): The method is consistent if the local truncation error $\tau_i(h) \to 0$ as $h \to 0$ (Richard and Douglas, 2011).

The local truncation error in Eq. (8) is $\tau_i(h) \to 0$ as $h \to 0$, for $i = 3(1)N-1$. Thus, the present method is consistent by Definition 1. Therefore, it is convergent of order four. Since, stability + consistency ⇔ convergence.

RESULTS

To demonstrate the applicability of the method, three model examples having constant and variable coefficients with left-end and right-end boundary layers have been carried out. For the variable coefficients, the maximum absolute errors are computed using the double mesh principle (Doolan et al., 1980).

Example 1. Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \text{ and } y(1) = 1.$$

The analytical solution of this equation is given by:

$$y(x) = \frac{(1-e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}} \quad \text{Where,}$$

$$m_1 = \frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \quad \text{and} \quad m_1 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}.$$

Table 1. The maximum absolute errors of Example 1, for different values of δ with $\varepsilon = 0.01$

$\delta \downarrow$	$N=10^2$	$N=10^3$	$N=10^4$
Present Method			
0.1ε	1.9498e-05	1.7660e-09	3.2174e-11
0.3ε	4.0241e-05	3.7673e-09	2.9427e-11
0.6ε	1.5860e-04	2.0195e-08	1.7700e-11
0.8ε	4.8525e-04	1.6543e-07	1.4487e-11
Sirisha and Reddy (2017)			
0.1ε	0.090733	0.012286	0.001279
0.3ε	0.108033	0.015622	0.001644
0.6ε	0.127778	0.026309	0.002870
0.8ε	0.100404	0.048338	0.005688
Phaneendra and Soujanya (2014)			
0.1ε	5.4191e−04	5.0759e−08	3.2564e−11
0.3ε	1.4385e−03	1.3787e−07	2.9030e−11
0.6ε	9.9988e−03	1.2835e−06	1.1959e−10
0.8ε	6.1971e−02	2.0628e−05	2.0281e−09
Reddy et al. (2012)			
0.1ε	0.09073	0.01228	0.00127
0.3ε	0.10803	0.01562	0.00164
0.6ε	0.12777	0.02630	0.00287
0.8ε	0.10040	0.04833	0.00568

Example 2. Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) + e^{-0.25x} y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1,\ -\delta \le x \le 0 \text{ and } y(1) = 1.$$

Table 2. The maximum absolute errors of Example 2, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N=10^2$	$N=10^3$	$N=10^4$
Present Method			
0.1ε	4.6555e-07	4.5285e-09	4.5816e-11
0.3ε	2.3007e-06	2.2768e-08	2.2038e-10
0.6ε	1.4360e-05	1.4228e-07	1.4228e-09
0.8ε	7.7836e-05	7.6779e-07	7.6790e-09
Sirisha and Reddy, (2017)			
0.1ε	6.2687e – 003	6.6646e − 004	6.7072e − 005
0.3ε	8.0060e – 003	8.6458e − 004	8.7156e − 005
0.6ε	1.342e – 002	1.5282e − 003	1.5493e − 004
0.8ε	2.3860e – 002	3.0459e − 003	3.1280e − 004
Reddy et al., (2012)			
0.1ε	0.00632996	0.000674268	6.7871251e−005
0.3ε	0.00815917	0.000882563	8.8986856e−005
0.6ε	0.01384760	0.001579726	1.6020004e−004
0.8ε	0.02477158	0.003173235	3.2602775e−004

Example 3. Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1,\ -\delta \le x \le 0 \text{ and } y(1) = -1.$$

The analytical solution of this equation is given by:

$$y(x) = \frac{(1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x}}{e^{m_2} - e^{m_1}}$$

where $m_1 = \dfrac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$ and $m_1 = \dfrac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$.

Table 3. The maximum absolute errors of Example 3, for different values of δ with $\varepsilon = 0.01$

$\delta \downarrow$	$N=10^2$	$N=10^3$	$N=10^4$
Present Method			
0.1ε	2.9926e-05	2.8375e-09	3.8704e-11
0.15ε	2.5884e-05	2.4837e-09	4.0326e-11
0.25ε	1.9653e-05	1.9319e-09	3.3466e-11
Sirisha and Reddy (2017)			
0.1ε	0.165949	0.022109	0.002285
0.15ε	0.158945	0.021173	0.002186
0.25ε	0.146034	0.019539	0.002013
Phaneendra and Soujanya (2014)			
0.07ε	$1.8573e{-}01$	$8.4819e{-}05$	$8.4629e{-}09$
0.15ε	$8.0711e{-}02$	$1.7927e{-}05$	$1.8318e{-}09$
0.25ε	$3.2547e{-}02$	$4.7036e{-}06$	$4.9565e{-}10$

Example 4. Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1,\ -\delta \le x \le 0 \text{ and } y(1) = 1.$$

Table 4. The maximum absolute errors of Example 4, for different values of δ with $\varepsilon = 0.1$

$\delta \downarrow$	$N = 10^2$	$N = 10^3$	$N = 10^4$
Present Method			
0.1ε	2.8367e-06	2.8375e-08	9.2817e-10
0.3ε	4.3993e-06	4.3986e-08	6.8711e-10
0.6ε	4.5006e-06	4.4996e-08	6.6965e-10
0.8ε	4.2986e-06	4.2982e-08	5.8605e-10
Sirisha and Reddy (2017)			
0.1ε	7.7065e − 003	8.5743e − 004	8.6724e − 005
0.3ε	5.5572e − 003	6.0006e − 004	6.0487e − 005
0.6ε	3.8911e − 003	4.1085e − 004	4.1314e − 005
0.8ε	3.2241e − 003	3.3750e − 004	3.3908e − 005
Reddy et al. (2012)			
0.1ε	0.00575975	0.00050842	5.02478e-005
0.3ε	0.003932768	0.00036132	3.58384e-005
0.6ε	0.002702569	0.00025507	2.53643e-005
0.8ε	0.00224689	0.00021413	2.13134e-005

Table 5. The maximum absolute errors for different values of ε with $\delta = 0.1\varepsilon$.

$\varepsilon \downarrow$ $\underset{\rightarrow}{h}$	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Example 1					
2^{-5}	3.3421e-06	2.1111e-07	1.3110e-08	8.1806e-10	5.1093e-11
2^{-6}	3.0230e-05	1.7100e-06	1.0764e-07	6.6856e-09	4.1734e-10
2^{-7}	1.9052e-04	1.5280e-05	8.6506e-07	5.4360e-08	3.3764e-09
2^{-8}	6.5639e-04	9.6012e-05	7.6816e-06	4.3509e-07	2.7317e-08
2^{-9}	2.0070e-03	3.2994e-04	4.8197e-05	3.8514e-06	2.1819e-07
2^{-10}	5.9446e-03	1.0094e-03	1.6541e-04	2.4146e-05	1.9283e-06
Example 2					
2^{-5}	6.7763e-06	1.1087e-06	2.4256e-07	5.8380e-08	1.4454e-08
2^{-6}	3.7179e-05	3.4920e-06	5.6227e-07	1.2275e-07	2.9501e-08
2^{-7}	2.2014e-04	1.8974e-05	1.7725e-06	2.8339e-07	6.1742e-08
2^{-8}	9.0978e-04	1.1163e-04	9.5843e-06	8.9292e-07	1.4239e-07
2^{-9}	2.1924e-03	4.5874e-04	5.6208e-05	4.8167e-06	4.4814e-07

2^{-10}	5.6334e-03	1.0991e-03	2.3033e-04	2.8202e-05	2.4145e-06
Example 3					
2^{-5}	5.4692e-06	3.3774e-07	2.1167e-08	1.3220e-09	8.2685e-11
2^{-6}	4.6516e-05	2.7755e-06	1.7201e-07	1.0766e-08	6.7245e-10
2^{-7}	3.7262e-04	2.3428e-05	1.3986e-06	8.6851e-08	5.4309e-09
2^{-8}	1.9573e-03	1.8759e-04	1.1758e-05	7.0206e-07	4.3643e-08
2^{-9}	6.6024e-03	9.8475e-04	9.4116e-05	5.8900e-06	3.5173e-07
2^{-10}	1.0734e-02	1.8986e-03	2.2852e-04	1.7447e-05	9.9921e-07
Example 4					
2^{-5}	1.7254e-05	4.9015e-06	1.2492e-06	3.1359e-07	7.8612e-08
2^{-6}	1.4820e-05	8.0941e-06	2.3694e-06	6.0717e-07	1.5261e-07
2^{-7}	3.4529e-04	8.7906e-06	3.9110e-06	1.1637e-06	2.9906e-07
2^{-8}	1.5400e-03	1.7630e-04	4.7413e-06	1.9211e-06	5.7651e-07
2^{-9}	4.2463e-03	7.7697e-04	8.9065e-05	2.4573e-06	9.5192e-07
2^{-10}	9.7408e-03	2.1328e-03	3.9021e-04	4.4760e-05	1.2503e-06

Table 6. Rate of convergence ρ for different values of δ with $\varepsilon = 0.01$ at $x = 0.5$.

δ/N	40	60	80	100
Example 1				
0.1ε	3.8056	3.9074	3.9465	3.9653
0.15ε	3.7856	3.8972	3.9405	3.9614
0.25ε	3.7355	3.8710	3.9248	3.9510
Example 2				
0.1ε	3.8523	3.9125	3.9224	3.9134
0.15ε	3.8369	3.8985	3.9046	3.8894
0.25ε	3.8038	3.8734	3.8774	3.8555
Example 3				
0.1ε	3.8631	3.9360	3.9633	3.9763
0.15ε	3.8735	3.9411	3.9663	3.9782
0.25ε	3.8910	3.9496	3.9712	3.9814
Example 4				
0.1ε	3.8257	3.9670	4.0618	4.1560
0.15ε	3.8642	4.0238	4.1586	4.1195
0.25ε	3.9483	4.1752	4.2576	4.2270

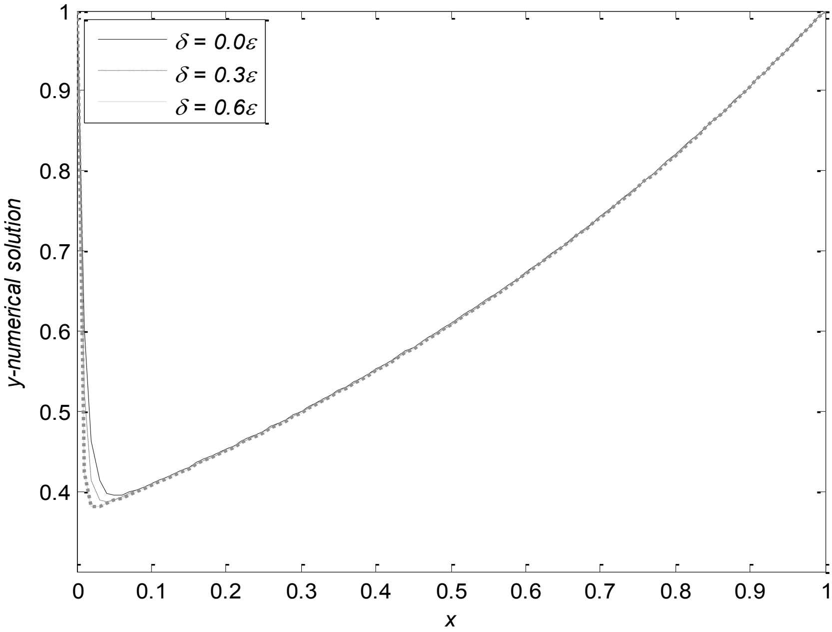

Fig. 1: The numerical solution of Examples 1 with $\varepsilon = 0.01$ and N = 100.

Remark 1: The figure of Example 2 is similar with the figure of Example 1 at the above given parameters ε and δ.

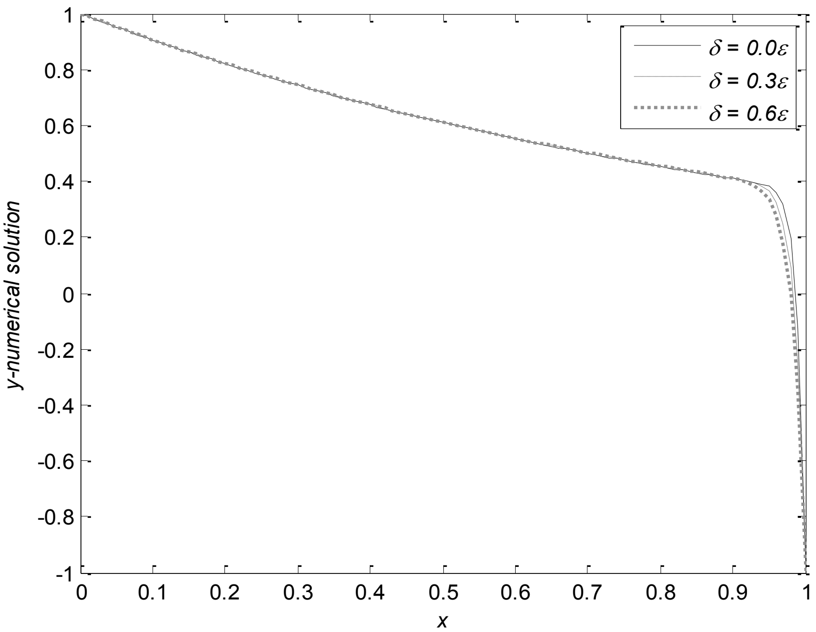

Fig. 2: The numerical solution of Example 3 with $\varepsilon = 0.01$ and N = 100.

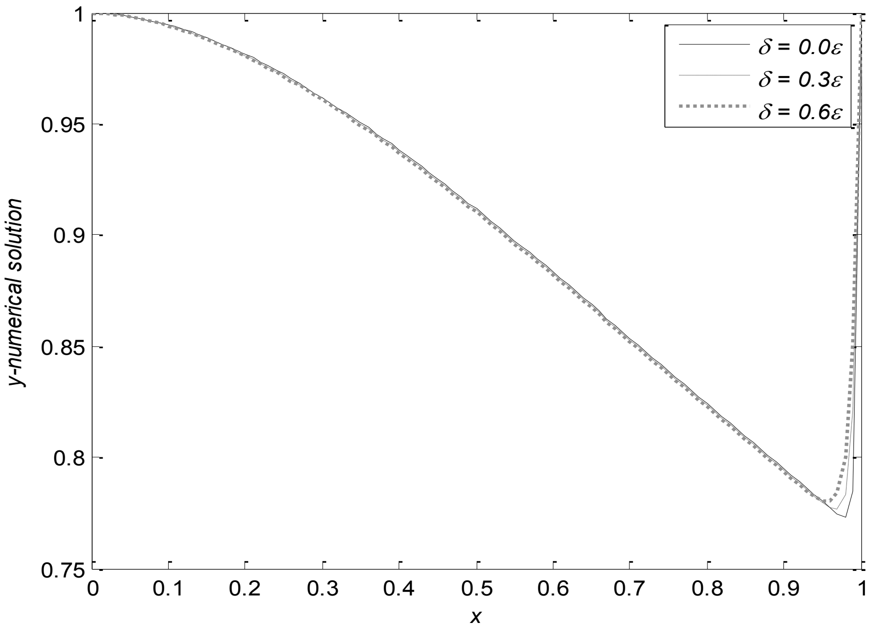

Fig. 3: The numerical solution of Example 4 with $\varepsilon = 0.01$ and N = 100.

DISCUSSION

Fitted fourth order numerical method for solving singularly perturbed delay convection-diffusion equations has been presented. To demonstrate the efficiency of the method, four model examples with constant and variables coefficients have been considered for different values of the perturbation parameter ε, and delay parameter δ. The numerical solutions are tabulated (Tables 1 to 5) in terms of maximum absolute errors and observed that the present method improves the findings of Reddy et al. (2012), Phaneendra and Soujanya (2014) and Sirisha and Reddy (2017). Further, it is significant that all of the maximum absolute errors decrease rapidly as N increases, which in turn shows the convergence of the computed solution. Also, we have presented the numerical results when $\varepsilon \leq h$ and obtain a good results, (see Table 5). The convergence analysis of the present method is investigated. The results presented in Table 6 confirmed that the computational rate of convergence, as well as theoretical estimates, indicates that the present method is of fourth order convergence.

To demonstrate the effect of delay on the left and right boundary layer of the solution, the graphs for different values of delay parameter δ are plotted in Figs. 1-3. Accordingly, depending on the sign of the coefficient of delay term one can see that, from Fig. 1 as a delay parameter δ increases the width of the left boundary layer decreases. When the coefficient of the delay term in the problem is of $o(1)$ and delay increases, the thickness of the right boundary layer decreases (Fig. 2) but when the coefficient of the delay term of $O(1)$ and delay increases, the thickness of the right boundary layer increases (Fig. 3).

CONCLUSION

A fitted fourth order numerical method is developed. To assured the presented scheme is useful, the stability and convergence of the method are established well. To check the validity of the present method, we have considered four model examples of singularly perturbed delay convection-diffusion equations. In general, the present method is stable, convergent and more accurate for solving singularly perturbed convection-diffusion equations.

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