ORIGINAL ARTICLE

Theory of Quantum Entanglement of Intra-Cavity Photons Produced by a Non-Degenerate Parametric Oscillator (Ndpo): Semi-Classical Approach

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ABSTRACT

The von-Neumann entropy (VNE), usually expressed in terms of the density operator, is the mathematical tool to measure the degree of entanglement of a bi-partite system. Applying the solutions of the quantum Langevin equations, the anti-normally ordered characteristic function of the intra-cavity photons produced by a non-degenerate parametric oscillator could be calculated. With the help of the resulting characteristic function, the Q- function which is then used to calculate the entanglement of the intra-cavity photons using VNE is determined. Moreover, the photon number distribution, the mean photon number, the normalized second order correlation function, the intensity difference, and quadrature variance for the intra-cavity photons produced by a non-degenerate parametric oscillator coupled to a two-mode squeezed vacuum reservoir is determined. It was found that when the squeeze parameter increases the entanglement also increases. These show that the entropy entanglement has a direct relation with squeezing.

Keywords: Entanglement, Mean photon number, Parametric down conversion, Q-function

INTRODUCTION

The quantum properties of a system coupled to a squeezed vacuum reservoir could be analyzed by employing a function such as the Q function or the positive P function or the Wigner (W). Generally such a distribution function is obtained by using Fokker-Planck equation, which is determined from the equation of the time evolution for the reduced density operator. However, undertaking the master equation (Louisell, 1973; Scully and Zubairy, 1997) and resolving the associated Fokker-Planck equation (Anwar and Zubairy, 1992) involves a great deal of mathematical operations. On the other hand, the derivation of the quantum Langevin equations (Meystre and Sargent, 1991) helps to minimize the mathematical task.

Optical parametric oscillators (OPO) and amplifiers (OPA) (Yariv, 1989) are

nonlinear optical devices that produces twin photons by spontaneous down conversion of a pump photon interacting with nonlinear crystal in and out of an optical cavity, respectively. It has been studied nearly for a half century due to its twin photons produced by these nonlinear optical devices (Einstein et al., 1935) and used in testing the basic quantum mechanics and in technological applications in areas such as frequency con-version, low noise optical measurement, squeezed light sources, quantum information, teleportation, and cryptography. The type-II OPO (NDPO) used to experimentally demonstrate the implementation of the Einstein-Podolsky-Rosen (EPR) paradox (Ou, et al., 1992). This has made NDPO increasingly important as entangled continuous variables have been proposed and used to implement quantum information protocols such as quantum teleportation (Furusawa et al., 1998).

In this study, with the aid of solutions of quantum Langevin equations, we obtain the anti-normally ordered characteristic function produced by non-degenerate parametric oscillator (NDPO). With the help of this characteristic function, the Q-function is determined which in turn is used to calculate the entanglement of the intra-cavity photons in a sub-threshold NDPO coupled with a two-mode squeezed vacuum by calculating the Von-Neumann Entropy (VNE). In addition, the photon number distribution with the help of the Q-function is calculated. Furthermore, with the aid of solutions of quantum Langevin equations, we also determine, at steady-state, the mean photon number of the two mode cavity light, the quadrature squeezing, and the photon number difference (Intensity difference) of the two-mode intra cavity light.

MATERIAL AND METHODS

Here we consider NDPO coupled with a two-mode squeezed vacuum reservoir, the model described by Fesseha (1998). Applying the two-mode quantum Langevin equations, first we obtain the noise force correlations. Moreover, with the help of these results, we determine the Q-function of the two-mode cavity light. These method help us to calculate the entropy Entanglement and we also determine, at steady-state, the mean photon number of the two mode cavity light, the quadrature squeezing, the intensity difference of the two-mode intra cavity light.

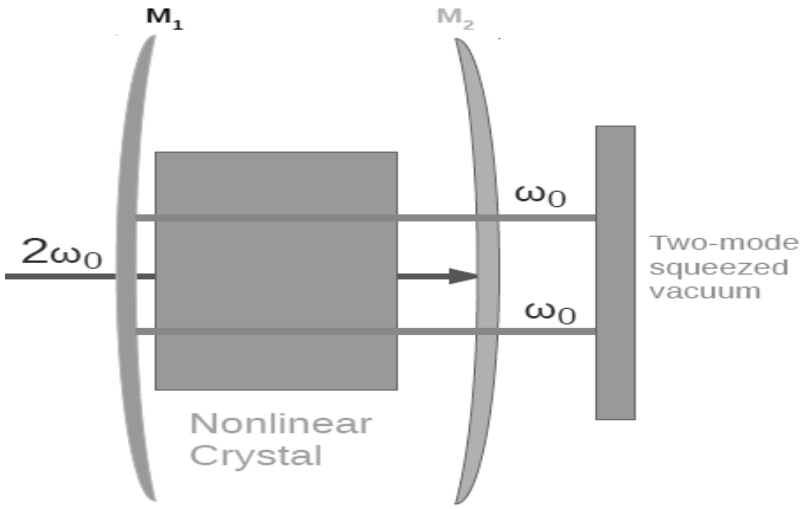

Figure 1: Schematic representation of the model.

The Quantum Langevin equations

In this section, we tried to obtain the quantum Langevin equations for the two- modes. With the pump mode treated classically, this system can be described by the interaction Hamiltonian

$$H = i\varepsilon(\hat{a}_1^{\dagger}\hat{a}_2^{\dagger} - \hat{a}_1\hat{a}_2) + \hat{a}_1\Gamma_1^{\dagger} + \hat{a}_1^{\dagger}\Gamma_1 + \hat{a}_2\Gamma_2^{\dagger} + \hat{a}_2^{\dagger}\Gamma_2 \quad (1)$$

Where ε is a constant proportional to the amplitude of the pump mode, $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operator for the signal and idler modes, and $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are reservoir operators. Following the procedure described by Fesseha (1998) and with the aid of the interaction picture the Langevin equations are found to be:

$$\frac{d\hat{a}_1}{dt} = -\frac{\kappa}{2}\hat{a}_1 + \varepsilon\hat{a}_2^{\dagger} + \hat{F}_1 \quad (2)$$

$$\frac{d\hat{a}_2}{dt} = -\frac{\kappa}{2}\hat{a}_2 + \varepsilon\hat{a} + \hat{F}_2 \quad (3)$$

Where κ is the cavity damping rate for the signal as well as the idler mode, $\hat{F}_1$ and $\hat{F}_2$ are noise operators with zero mean and satisfying the correlation functions (Fesseha, 1998).

$$\left\langle \hat{F}_1(t)\hat{F}_1^{\dagger}(t') \right\rangle = \left\langle \hat{F}_2(t)\hat{F}_2^{\dagger}(t') \right\rangle = \kappa(N+1)\delta(t-t'), \quad (4)$$

$$\left\langle \hat{F}_1^{\dagger}(t)\hat{F}_1(t') \right\rangle = \left\langle \hat{F}_2^{\dagger}(t)\hat{F}_2(t') \right\rangle = \kappa N\delta(t-t'), \quad (5)$$

$$\left\langle \hat{F}_1(t)\hat{F}_2(t')\right\rangle = \left\langle \hat{F}_1^{\dagger}(t)\hat{F}_2^{\dagger}(t')\right\rangle = \kappa M\delta(t-t'), \qquad (6)$$

$$\left\langle \hat{F}_1(t)\hat{F}_1(t')\right\rangle = \left\langle \hat{F}_1^{\dagger}(t)\hat{F}_1^{\dagger}(t')\right\rangle = 0, \qquad (7)$$

$$\left\langle \hat{F}_2(t)\hat{F}_2(t')\right\rangle = \left\langle \hat{F}_2^{\dagger}(t)\hat{F}_2^{\dagger}(t')\right\rangle = 0, \qquad (8)$$

$$\left\langle \hat{F}_1(t)\hat{F}_2^{\dagger}(t')\right\rangle = \left\langle \hat{F}_1^{\dagger}(t)\hat{F}_2(t')\right\rangle = 0, \qquad (9)$$

Where for a given reservoir, $N = \sinh^2 r$, $M = \cosh r \sinh r$, and r is the squeeze parameter. In order to decouple the differential equations (2) and (3), it is necessary to introduce new operators defined earlier by Fesseha (1998)

$$\hat{A}_1 = \hat{a}_1 + \hat{a}_2^{\dagger} \qquad (10)$$

and

$$\hat{A}_2 = \hat{a}_1 - \hat{a}_2^{\dagger} \qquad (11)$$

so that combination of Eqs. (2), (3), (10) and (11), it yields:

$$\frac{d\hat{A}_1}{dt} = -\frac{1}{2}\mu_{-}\hat{A}_1 + \hat{F}_1 + \hat{F}_2^{\dagger} \qquad (12)$$

and

$$\frac{d\hat{A}_2}{dt} = -\frac{1}{2}\mu_{+}\hat{A}_2 + \hat{F}_1 - \hat{F}_2^{\dagger}, \qquad (13)$$

where

$$\mu_{\pm} = \kappa \pm 2\varepsilon. \qquad (14)$$

The solutions of equations (12) and (13) turn out to be:

$$\hat{A}_1(t) = \hat{A}_1(0)e^{-\mu_{-}t/2} + \int_0^t e^{-\mu_{-}(t-t')/2}(\hat{F}_1(t') + \hat{F}_2^{\dagger}(t'))dt' \qquad (15)$$

and

$$\hat{A}_2(t) = \hat{A}_2(0)e^{-\mu_{+}t/2} + \int_0^t e^{-\mu_{+}(t-t')/2}(\hat{F}_1(t') - \hat{F}_2^{\dagger}(t'))dt'. \qquad (16)$$

The Q function

The time dependent Q-function describing the intra-cavity idler (α_1) and signal (α_2) modes given by

$$Q(\alpha_1,\alpha_2,t) = Tr(\hat{\rho}\left|\alpha_1,\alpha_2\right\rangle\left\langle\alpha_2,\alpha_1\right|) \qquad (17)$$

Or the Q-function is expressible in terms of the anti-normally ordered characteristic function defined by

$$Q(\alpha_1,\alpha_2,t) = \frac{1}{\pi^4}\int d^2z d^2\eta\phi(z,\eta,t)\exp(z^*\alpha_1 - z\alpha_1^* + \eta^*\alpha_2 - \eta\alpha_2^*), \qquad (18)$$

Where the anti-normally ordered characteristic function $\phi(z,\eta,t)$ defined by

$$\phi(z,\eta,t) = Tr(\hat{\rho}\,\mathrm{e}^{-z^*a_1(t)}\, e^{-\eta^*\hat{a}_2(t)} e^{z\hat{a}_1^\dagger} e^{\eta\hat{a}_2^\dagger}) \qquad (19)$$

On account of equations (10) and (11) along with (15) and (16), one gets

$$\hat{a}_1(t) = \hat{A}_1(t)\hat{a}_1(0) + \hat{A}_2(t)\hat{a}_2^\dagger(0) + \hat{F}(t) + \hat{G}(t)\,, \qquad (20)$$

$$\hat{a}_2(t) = \hat{A}_1(t)\hat{a}_2(0) + \hat{A}_2(t)\hat{a}_1^\dagger(0) + \hat{F}^\dagger(t) - \hat{G}^\dagger(t)\,, \qquad (21)$$

in which

$$\hat{A}_1(t) = \frac{1}{2}[e^{-\mu_- t/2} + e^{-\mu_+ t/2}]\,, \qquad (22)$$

$$\hat{A}_2(t) = \frac{1}{2}[e^{-\mu_- t/2} - e^{-\mu_+ t/2}]\,, \qquad (23)$$

$$\hat{F}(t) = \frac{1}{2}\int_0^t e^{-\mu_-(t-t')/2}[\hat{F}_1(t') + \hat{F}_2^\dagger(t')]dt'\,, \qquad (24)$$

$$\hat{G}(t) = \frac{1}{2}\int_0^t e^{-\mu_+(t-t')/2}[\hat{F}_1(t') - \hat{F}_2^\dagger(t')]dt'\,. \qquad (25)$$

Taking into account equations (20) and (21) and noting that the boson operators $\hat{a}_1(0)$ and $\hat{a}_1^\dagger(0)$ commute with the noise operators $\hat{F}_1$ and $\hat{F}_1^\dagger$, $\hat{a}_2(0)$ and $\hat{a}_2^\dagger(0)$ commute with the noise operators $\hat{F}_2$ and $\hat{F}_2^\dagger$, the characteristic function can be put in the form

$$\begin{aligned}\phi(z,\eta,t) = Tr_S\{\hat{\rho}_S(0) & e^{-z^*\hat{A}_1(t)\hat{a}_1(0)} e^{-z^*\hat{A}_2(t)\hat{a}_2^\dagger(0)} e^{([z\hat{A}_1(t)-\eta^*\hat{A}_2(t)]\hat{a}_1^\dagger(0))} \\ & \times e^{\eta\hat{A}_1(t)\hat{a}_2^\dagger(0)} e^{([z\hat{A}_2(t)-\eta^*\hat{A}_1(t)]\hat{a}_1(0))} e^{\eta\hat{A}_2(t)\hat{a}_1(0)}\} \\ & \times Tr_R\{\hat{\rho}_R(0) e^{-z^*(\hat{F}(t)+\hat{G}(t))} e^{-\eta^*(\hat{F}^\dagger(t)-\hat{G}^\dagger(t))} e^{\eta(\hat{F}(t)-\hat{G}(t))} e^{z(\hat{F}^\dagger(t)+\hat{G}^\dagger(t))}\}.\end{aligned} \qquad (26)$$

Denoting the trace involving the system density operator by U and using the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]}\,, \qquad (27)$$

one easily finds

$$\begin{aligned}U = Tr_S\{ & e^{[\eta\hat{A}_1(t)-z^*\hat{A}_2(t)]\hat{a}_2^\dagger(0)}\hat{\rho}_S(0) \\ & \times e^{[z\hat{A}_1(t)-\eta^*\hat{A}_2(t)]\hat{a}_1^\dagger(0)} e^{[z\hat{A}_2(t)-\eta^*\hat{A}_1(t)]\hat{a}_2(0)} \\ & \times e^{[\eta\hat{A}_2(t)-z^*\hat{A}_1(t)]\hat{a}_1(0)} e^{[\hat{A}_2^2(t)-\hat{A}_1^2(t)]z^*z}\}\end{aligned} \qquad (28)$$

So, that brings together the completeness relation:

$$I = \int \frac{d^2\alpha_1}{\pi}\frac{d^2\alpha_2}{\pi}\left|\alpha_1,\alpha_2\right\rangle\left\langle\alpha_2,\alpha_1\right|, \tag{29}$$

and assuming the signal-idler modes and the squeezed vacuum reservoir are not correlated at the initial time or initially in the vacuum state (Gardiner, 1991), one finds

$$U = e^{[\hat{A}_2^2(t)-\hat{A}_1^2(t)]z^*z}\int \frac{d^2\alpha_1}{\pi}\frac{d^2\alpha_2}{\pi}\exp[(z\hat{A}_1(t)-\eta^*\hat{A}_2(t))\alpha_2^* + (z\hat{A}_2(t)-\eta^*\hat{A}_1(t))\alpha_2 + (\eta\hat{A}_2(t)-z^*\hat{A}_1(t))\alpha_1 - |\alpha_1|^2 - |\alpha_2|^2]. \tag{30}$$

Hence on performing the integration over α_1 and α_2, there follows

$$U = \exp[-\hat{A}_2^2(t)(z^*z+\eta^*\eta) + \hat{A}_1(t)\hat{A}_2(t)(z\eta+z^*\eta^*)]. \tag{31}$$

Furthermore, on account of Eq. (24), one can readily verify that

$$[\hat{F}(t),\hat{F}^\dagger(t)] = \frac{1}{4}\int_0^t \exp(-\frac{1}{2}\mu_-[2t-t'-t''])([\hat{F}_1(t'),\hat{F}_1^\dagger(t'')]+[\hat{F}_1(t'),\hat{F}_2^\dagger(t'')] + [\hat{F}_2^\dagger(t'),\hat{F}_1^\dagger(t'')]+[\hat{F}_2^\dagger(t'),\hat{F}_2(t'')])dt'dt'' \tag{32}$$

and on with the aid of the commutation relations

$$[\hat{F}_1(t'),\hat{F}_1^\dagger(t'')] = \kappa\delta(t'-t''), \tag{33}$$

$$[\hat{F}_2^\dagger(t'),\hat{F}_2(t'')] = -\kappa\delta(t'-t''), \tag{34}$$

$$[\hat{F}_1(t'),\hat{F}_2^\dagger(t'')] = [\hat{F}_2^\dagger(t'),\hat{F}_1^\dagger(t'')] = 0, \tag{35}$$

one gets the result

$$[\hat{F}(t),\hat{F}^\dagger(t)] = 0. \tag{36}$$

Following a similar procedure, one can also establish that

$$[\hat{G}(t),\hat{G}^\dagger(t)] = 0, \tag{37}$$

$$[\hat{G}^\dagger(t),\hat{F}(t)] = -\frac{1}{2}(1-e^{-\kappa t}), \tag{38}$$

$$[\hat{G}(t),\hat{F}^\dagger(t)] = \frac{1}{2}(1-e^{-\kappa t}). \tag{39}$$

With the help of the commutation relations described by equations (36)-(39), one can readily verify that

$$[\hat{F}(t)+\hat{G}(t),\hat{F}^\dagger(t)-\hat{G}^\dagger(t)] = [\hat{F}^\dagger(t)+\hat{G}^\dagger(t),\hat{F}(t)-\hat{G}(t)] = 0. \tag{40}$$

Thus denoting the trace involving the reservoir density operator that appears in equation (26) by V and applying the Backer-Hausdorff relation together with (40), one can easily see that

$$V = \left\langle \exp[-z^*(\hat{F}(t)+\hat{G}(t))-\eta^*(\hat{F}^\dagger(t)-\hat{G}^\dagger(t))]\exp[z(\hat{F}^\dagger(t)+\hat{G}^\dagger(t))+\eta(\hat{F}(t)-\hat{G}(t))]\right\rangle_R, \tag{41}$$

so that applying again the Baker-Hausdorff relation, we see that

$$V=\left\langle \begin{array}{l}\exp[-z^{*}(\hat{F}(t)+\hat{G}(t))-\eta^{*}(\hat{F}^{\dagger}(t)-\hat{G}^{\dagger}(t))+z(\hat{F}^{\dagger}(t)+\hat{G}^{\dagger}(t))+\eta(\hat{F}(t)-\hat{G}(t))] \\ \times\exp(\frac{1}{2}[-z^{*}(\hat{F}(t)+\hat{G}(t))-\eta^{*}(\hat{F}^{\dagger}(t)-\hat{G}^{\dagger}(t)),z(\hat{F}^{\dagger}(t)+\hat{G}^{\dagger}(t))+\eta(\hat{F}(t)-\hat{G}(t))])\end{array}\right\rangle_{R}. \quad (42)$$

Now using equations (36)-(39) along with the fact that

$$[\hat{F}(t),\hat{G}(t)]=[\hat{F}^{\dagger}(t),\hat{G}^{\dagger}(t)]=0, \quad (43)$$

one can easily obtain

$$\begin{aligned}&\exp([-z^{*}(\hat{F}(t)+\hat{G}(t))-\eta^{*}(\hat{F}^{\dagger}(t)-\hat{G}^{\dagger}(t)),z(\hat{F}^{\dagger}(t)+\hat{G}^{\dagger}(t))+\eta(\hat{F}(t)-\hat{G}(t))]) \\ &=\exp[(z^{*}z+\eta^{*}\eta)(1-e^{\kappa t})]\end{aligned} \quad (44)$$

In view of this result, expression (42) reduces to

$$V=\exp[-\frac{1}{2}(z^{*}z+\eta^{*}\eta)(1-e^{\kappa t})]\left\langle \begin{array}{l}\exp[-z^{*}(\hat{F}(t)+\hat{G}(t))-\eta^{*}(\hat{F}^{\dagger}(t)-\hat{G}^{\dagger}(t)) \\ +z(\hat{F}^{\dagger}(t)+\hat{G}^{\dagger}(t))+\eta(\hat{F}(t)-\hat{G}(t))]\end{array}\right\rangle_{R}. \quad (45)$$

Since the operators $\hat{F}$ and $\hat{G}$ represent random Gaussian processes, one can express equation (45) in the form (Chow et al., 1994)

$$V=\exp[-\frac{1}{2}(z^{*}z+\eta^{*}\eta)(1-e^{\kappa t})]\exp[\frac{1}{2}\left\langle \hat{H}\right\rangle_{R}], \quad (46)$$

in which

$$\hat{H}=[-z^{*}(\hat{F}(t)+\hat{G}(t))-\eta^{*}(\hat{F}^{\dagger}(t)-\hat{G}^{\dagger}(t))+z(\hat{F}^{\dagger}(t)+\hat{G}^{\dagger}(t))+\eta(\hat{F}(t)-\hat{G}(t))]^{2} \quad (47)$$

It is then easy to check that

$$\begin{aligned}\hat{H}&=-z^{*}z[\hat{F}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{F}+\hat{G}\hat{G}^{\dagger}+\hat{G}^{\dagger}\hat{G}+\hat{F}\hat{G}^{\dagger}+\hat{G}^{\dagger}\hat{F}+\hat{G}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{G}]+z^{2}[\hat{F}^{2}+\hat{G}^{2}+\hat{F}\hat{G}+\hat{G}\hat{F}] \\ &+z^{*2}[\hat{F}^{\dagger 2}+\hat{G}^{\dagger 2}+\hat{F}^{\dagger}\hat{G}^{\dagger}+\hat{G}^{\dagger}\hat{F}^{\dagger}]-\eta^{*}\eta[\hat{F}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{F}+\hat{G}\hat{G}^{\dagger}+\hat{G}^{\dagger}\hat{G}-\hat{F}\hat{G}^{\dagger}-\hat{G}^{\dagger}\hat{F}-\hat{G}\hat{F}^{\dagger}-\hat{F}^{\dagger}\hat{G}] \\ &+\eta^{*2}[\hat{F}^{\dagger 2}+\hat{G}^{\dagger 2}-\hat{F}^{\dagger}\hat{G}^{\dagger}-\hat{G}^{\dagger}\hat{F}^{\dagger}]+\eta^{2}[\hat{F}^{2}+\hat{G}^{2}-\hat{F}\hat{G}-\hat{G}\hat{F}]+z^{*}\eta^{*}[\hat{F}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{F}-\hat{G}\hat{G}^{\dagger}-\hat{G}^{\dagger}\hat{G} \\ &-\hat{F}\hat{G}^{\dagger}-\hat{G}^{\dagger}\hat{F}+\hat{G}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{G}]+z\eta[\hat{F}\hat{F}^{\dagger}+\hat{F}^{\dagger}\hat{F}-\hat{G}\hat{G}^{\dagger}-\hat{G}^{\dagger}\hat{G}+\hat{F}\hat{G}^{\dagger}+\hat{G}^{\dagger}\hat{F}-\hat{G}\hat{F}^{\dagger}-\hat{F}^{\dagger}\hat{G}] \\ &-2z^{*}\eta[\hat{F}^{2}-\hat{G}^{2}]-2z\eta^{*}[\hat{F}^{\dagger 2}-\hat{G}^{\dagger 2}].\end{aligned} \quad (48)$$

Employing equations (24) and (25) along with the correlation functions in (4)-(9), it can be readily established that

$$\left\langle \hat{F}\hat{F}^{\dagger}\right\rangle_{R}=\left\langle \hat{F}^{\dagger}\hat{F}\right\rangle_{R}=\frac{\kappa}{4\mu_{-}}(2N+2M+1)(1-e^{-\mu_{-}t}), \quad (49)$$

$$\left\langle \hat{G}\hat{G}^{\dagger}\right\rangle_{R}=\left\langle \hat{G}^{\dagger}\hat{G}\right\rangle_{R}=\frac{\kappa}{4\mu_{+}}(2N-2M+1)(1-e^{-\mu_{+}t}), \quad (50)$$

$$\left\langle \hat{F}\hat{G}\right\rangle_{R}=\left\langle \hat{G}\hat{F}\right\rangle_{R}=\left\langle \hat{F}^{\dagger}\hat{G}^{\dagger}\right\rangle_{R}=\left\langle \hat{G}^{\dagger}\hat{F}^{\dagger}\right\rangle_{R}=0, \quad (51)$$

$$\left\langle \hat{F}\hat{G}^{\dagger}\right\rangle_R = \left\langle \hat{G}\hat{F}^{\dagger}\right\rangle_R = -\left\langle \hat{G}^{\dagger}\hat{F}\right\rangle_R = -\left\langle \hat{F}^{\dagger}\hat{G}\right\rangle_R = \frac{1}{4}(1-e^{-\kappa t}), \tag{52}$$

$$\left\langle \hat{F}^2\right\rangle_R = \left\langle \hat{G}^2\right\rangle_R = \left\langle \hat{F}^{\dagger 2}\right\rangle_R = \left\langle \hat{G}^{\dagger 2}\right\rangle_R = 0. \tag{53}$$

Hence taking into account equations (46), (48)-(53), one gets

$$V = \exp(-\frac{1}{2}(z^*z+\eta^*\eta)[1-e^{\kappa t}+\frac{\kappa}{4\mu_-}(2N+2M+1)(1-e^{-\mu_- t})+\frac{\kappa}{4\mu_+}(2N-2M+1)(1-e^{-\mu_+ t})]$$
$$+\frac{1}{2}(z\eta+z^*\eta^*)[1-e^{\kappa t}+\frac{\kappa}{4\mu_-}(2N+2M+1)(1-e^{-\mu_- t})-\frac{\kappa}{4\mu_+}(2N-2M+1)(1-e^{-\mu_+ t})]). \tag{54}$$

Now in view of equations (31) and (54) together with (26), the characteristic function turns out to be

$$\phi(z,\eta,t) = \exp[-b_+(z^*z+\eta^*\eta)+b_-(z\eta+z^*\eta^*)], \tag{55}$$

Where in view of Eqs. (10) and (11), we have:

$$b_{\pm} = e^{-\mu_- t}[\cosh(\mu_- t)+e^{2r}\sinh(\mu_- t)/2\mu_-]\pm e^{-\mu_+ t}[\cosh(\mu_+ t)+e^{-2r}\sinh(\mu_+ t)/2\mu_+]. \tag{56}$$

Finally, on introducing equation (55) into (18) and carrying out the integration over z and η the Q-function is found to be of the form

$$Q(\alpha_1,\alpha_2,t) = \frac{4}{\pi^2(b_+^2-b_-^2)}\exp[\frac{-b_+(\alpha_1^*\alpha_1+\alpha_2^*\alpha_2)+b_-(\alpha_1^*\alpha_2^*+\alpha_1\alpha_2)}{(b_+^2-b_-^2)}]. \tag{57}$$

This is the Q-function for the signal and idler modes for the given specified system.

RESULT AND DISCUSSION

Next we proceed to obtain the entanglement, quadrature variance, mean photon number, the second order correlation function, and intensity of photon number difference for the signal-idler modes produced by a non-degenerate parametric oscillator coupled to a two-mode squeezed vacuum.

The Entanglement

The entanglement (E) of the signal and idler photons is analyzed using the VNE (Vedral et al., 1997),

$$E = -Tr[\hat{\tilde{\rho}}\log_2\hat{\tilde{\rho}}], \tag{58}$$

where $\hat{\tilde{\rho}} = Tr_{s(i)}(\hat{\rho}_{si})$ is the density operator for signal (s) or idler (i) photons obtained from the joint density operator $\hat{\rho}_{si}$ by tracing over the states of the other. For maximal entanglement $E = 1$ and for minimal entanglement $E = 0$, we have derived a relation for E in terms of the none-diagonal Q function

$$Q(\alpha_1^*,\lambda,t) = \left\langle \alpha_1\left|\hat{\tilde{\rho}}\right|\lambda\right\rangle, \tag{59}$$

This is a semi-classical representation of the density operator in terms of the coherent state for normal ordering, for the signal or idler photons given by:

$$E = -\frac{1}{\ln 2}\int \frac{d^2\alpha_1}{\pi}\frac{d^2\lambda}{\pi}\left\langle \alpha_1 \left| \hat{\tilde{\rho}} \right| \lambda \right\rangle \left\langle \lambda \left| \ln \hat{\tilde{\rho}} \right| \alpha_1 \right\rangle . \tag{60}$$

By applying the completeness relation for a coherent state, one can readily established that

$$E = -\sum_{k,n=0}^{\infty} \frac{(-1)^n}{k+1}\binom{k+1}{n}\int \frac{d^2\alpha_1}{\pi^2}\frac{d^2\lambda}{\ln 2}\left\langle \alpha_1 \left| \hat{\tilde{\rho}} \right| \lambda \right\rangle \left\langle \lambda \left| \hat{\tilde{\rho}}^n \right| \alpha_1 \right\rangle \tag{61}$$

and a series expansion for $\left\langle \lambda \left| \ln \hat{\tilde{\rho}} \right| \alpha_1 \right\rangle$ about $\hat{\tilde{\rho}}_0 = 1 - \hat{\tilde{\rho}} \le 1$ followed by a binomial expansion. Suppose $\hat{\tilde{\rho}}$ is obtained by tracing over the idler mode, the diagonal Q-function for the signal mode can be determined using

$$Q(\alpha_1^*, \alpha_1, t) = \left\langle \alpha_1 \left| \hat{\tilde{\rho}} \right| \alpha_1 \right\rangle = \int d^2\alpha_2 Q(\alpha_1^*, \alpha_1, \alpha_2^*, \alpha_2, t) . \tag{62}$$

With the aid of equation (57), one can see that

$$Q(\alpha_1^*, \alpha_1, t) = \frac{1}{\pi b_{\pm}} \exp[-\frac{1}{b_{\pm}}\alpha_1^*\alpha_1] . \tag{63}$$

This result is used to determine the non-diagonal Q-function of

$$Q(\alpha_1^*, \lambda, t) = \frac{1}{\pi b_{\pm}} \exp[-\frac{1}{b_{\pm}}\lambda\alpha_1^*] . \tag{64}$$

Applying this none-diagonal form of the Q-function, after a rigorous mathematical derivation that is based on multiple use of the completeness relation for a coherent state, we found for the VNE that measures the entanglement of the intra-cavity signal and idler photons

$$E = -[b_+(1-\frac{1}{b_+})\log_2(1-\frac{1}{b_+}) - \log_2 b_+] , \tag{65}$$

where $b_{\pm}$ is a time dependent variable that depends on the parameters r, κ, and ε.

We have analyzed the entanglement, *E*, of the intra-cavity signal and idler photons for a steady-state and the results are shown in Figure 2. It displays the entropy as function of the parameter $\varepsilon/\kappa \le 1$ which is less than one for sub threshold NDPO for different squeeze parameter, *r*. In Figure 2, we note that for values of the squeeze parameter, *r*, increases, the entanglement, *E*, increases. This indicates that the semi-classical approaches for calculating quantum entropy are valid only when the NDPO is operating far below threshold. For far below threshold range the results display that the entropy becomes one and therefore shows maximal entanglement. This property becomes stronger as *r* increases up to a certain limit and when *r* exceeds this limit, *E* becomes greater than one and again the semi-classical approach breaks down. The reason for the maximal entanglement in far below threshold range could be the weak coupling constant or/and pumping field amplitude contributing in low ε or high cavity damping rate κ at the port mirror since for sub threshold $\varepsilon = \kappa < 1$.

This potentially make the NDPO to behave like Type-II spontaneous parametric conversion due to insignificant down conversion because of small ε or high cavity escape rate of the down converted photons, κ. Moreover, from Figure 2, in the absence of squeezed light, $r = 0$, the

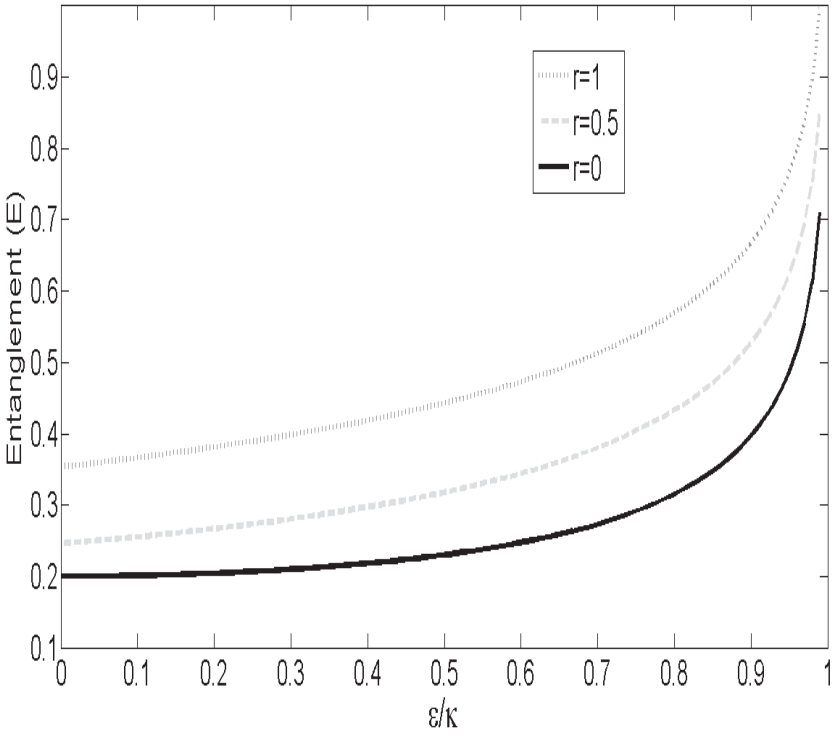

Figure 2: The steady-state quantum entropy of the intra-cavity signal and idler modes; r = 1 (dotted), r = 0.5 (dashed), and r = 0 (solid).

entropy entanglement is about 71%. But as the squeeze parameter, r increases the entanglement increases. For example, as shown on the plots from Figure 2, we observe that when $r = 0.5$, $E = 85.2\%$ and when $r = 1$, $E = 99.6\%$. These show that the entropy entanglement has a direct relation with squeezing.

Photon number distribution

The mutual probability to find *n* signal photons and *m* idler photons can be written in terms of the *Q* function as:

$$P(n,m,t) = \frac{\pi^2}{n!m!}\frac{\partial^{2n}}{\partial\alpha_1^{*n}\partial\alpha_1^{n}}\frac{\partial^{2m}}{\partial\alpha_2^{*m}\partial\alpha_2^{m}}[Q(\alpha_1,\alpha_2,t)\exp(\alpha_1^{*}\alpha_1+\alpha_2^{*}\alpha_2)]_{\alpha_1^{*}=\alpha_1=\alpha_2^{*}=\alpha_2=0}, \qquad (66)$$

so that on account of Eq. (57), we have

$$P(n,m,t)=\frac{1}{n!m!(b_+^2-b_-^2)}\frac{\partial^{2n}}{\partial\alpha_1^{*n}\partial\alpha_1^n}\frac{\partial^{2m}}{\partial\alpha_2^{*m}\partial\alpha_2^m}\quad, \tag{67}$$

$$\times[\exp(u(\alpha_1^*\alpha_1+\alpha_2^*\alpha_2)+v(\alpha_1^*\alpha_2^*+\alpha_1\alpha_2))]_{\alpha_1^*=\alpha_1=\alpha_2^*=\alpha_2=0}$$

in which

$$u=1-\frac{b_+}{b_+^2-b_-^2}, \tag{68}$$

$$v=\frac{b_-}{b_+^2-b_-^2}. \tag{69}$$

Now, on expanding the exponential terms in power series, Eq. (67) takes the form:

$$P(n,m,t)=\frac{1}{n!m!(b_+^2-b_-^2)}\sum_{ijkl}\frac{u^{i+j}v^{k+l}}{i!j!k!l!}\frac{\partial^{2n}}{\partial\alpha_1^{*n}\partial\alpha_1^n}\frac{\partial^{2m}}{\partial\alpha_2^{*m}\partial\alpha_2^m}\quad, \tag{70}$$

$$\times[(\alpha_1^*)^{i+k}\alpha_1^{i+l}+(\alpha_2^*)^{j+k}\alpha_2^{j+l}]_{\alpha_1^*=\alpha_1=\alpha_2^*=\alpha_2=0}$$

so that carrying out the differentiation and upon setting $\alpha_1^*=\alpha_1=\alpha_2^*=\alpha_2=0$, one can get

$$P(n,m,t)=\frac{1}{n!m!(b_+^2-b_-^2)}\sum_{ijkl}\frac{u^{i+j}v^{k+l}}{i!j!k!l!}\frac{(i+k)!}{(i+k-n)!}\frac{(i+l)!}{(i+l-n)!}\frac{(j+k)!}{(j+k-m)!}\frac{(j+l)!}{(j+l-m)!} \tag{71}$$

$$\times\delta_{i+k,n}\delta_{i+l,n}\delta_{j+k,m}\delta_{j+l,m}.$$

One can easily note that

$$k=l=n-i=m-j. \tag{72}$$

Moreover, for $m=n$ the photon number distribution turns out to be

$$P(n,n,t)=\frac{1}{(b_+^2-b_-^2)}\sum_{i=0}^{n}\frac{[n!]^2u^{2i}v^{2(n-i)}}{[i!]^2[(n-i)!]^2}. \tag{73}$$

With the aid of Eqs. (14), (56), (68), and (69), one finds for $\varepsilon=0$ and at steady state:

$$b_+^2-b_-^2=\cosh^2 r, \tag{74}$$

$$u=0, \tag{75}$$

$$v=\tanh r. \tag{76}$$

In view of these results, the photon number distribution, Eq. (73), goes over into (Caves et al., 1991)

$$P(n,n)=\frac{\tanh^{2n}r}{\cosh^2 r}. \tag{77}$$

This shows that the photon number distribution for a two-mode squeezed vacuum reservoir produced by the system under consideration.

Quadrature Variance

In this section, the squeezing properties of the two-mode cavity radiation and mean number of photon pairs of the two-photon phase-sensitive three-level laser coupled to a two-mode squeezed vacuum reservoir are investigated. In general, the squeezing properties of a two-mode cavity radiation can be described by the quadrature operators,

$$\hat{c}_+ = \hat{c} + \hat{c}^\dagger \tag{78}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}) \tag{79}$$

in which

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2) \tag{80}$$

On account of Eqs.(10) and (11) together with (15) and (16), these operators take the form

$$\hat{c}_+ = \hat{c}_+(0)e^{-\frac{\mu_- t}{2}} + \frac{1}{2}\int_0^t e^{-\frac{\mu_-(t-t')}{2}}\left[\hat{F}_1(t') + \hat{F}_1^\dagger(t') + \hat{F}_2(t') + \hat{F}_2^\dagger(t')\right]dt' \tag{81}$$

$$\hat{c}_- = \hat{c}_-(0)e^{-\frac{\mu_+ t}{2}} - \frac{i}{2}\int_0^t e^{-\frac{\mu_+(t-t')}{2}}\left[\hat{F}_1(t') - \hat{F}_1^\dagger(t') + \hat{F}_2(t') - \hat{F}_2^\dagger(t')\right]dt' \tag{82}$$

Next employing Eqs. (81) and (82)along with the correlation functions (4)-(9), the variances of the quadrature operators can be shown that

$$(\Delta\hat{c}_+)^2 = (\Delta\hat{c}_+)_0^2 e^{-\mu_- t} + \frac{\kappa}{\mu_-}[2N + 2M + 1](1 - e^{-\mu_- t}) \tag{83}$$

$$(\Delta\hat{c}_-)^2 = (\Delta\hat{c}_-)_0^2 e^{-\mu_+ t} + \frac{\kappa}{\mu_+}[2N - 2M + 1](1 - e^{-\mu_+ t}) \tag{84}$$

It then follows that

$$(\Delta\hat{c}_+)^2 = (\Delta\hat{c}_+)_0^2 e^{-\mu_- t} + \frac{\kappa}{\mu_-}e^{2r}(1 - e^{-\mu_- t}) \tag{85}$$

$$(\Delta\hat{c}_-)^2 = (\Delta\hat{c}_-)_0^2 e^{-\mu_+ t} + \frac{\kappa}{\mu_+}e^{-2r}(1 - e^{-\mu_+ t}) \tag{86}$$

As clearly indicated in Figure 3, the degree of two-mode squeezing decreases with the cavity damping constant for smaller values of squeeze parameter, *r*, but it increases for larger values. It is not difficult to note that a maximum obtainable squeezing is witnessed slightly above the threshold value and it is independent of the squeeze parameter, *r*. It is found that a maximum two-mode squeezing of about 87.8% occurs at various values of$\kappa t = 3.32$. Most recently, the same result has been reported following a different approach and using different parameters (Tesfa, 2008). Moreover, the value of κt for which a maximum squeezing occurs increases with the squeeze parameter, *r*. Therefore, interaction of atom with squeeze vacuum reservoir enhances squeezing.

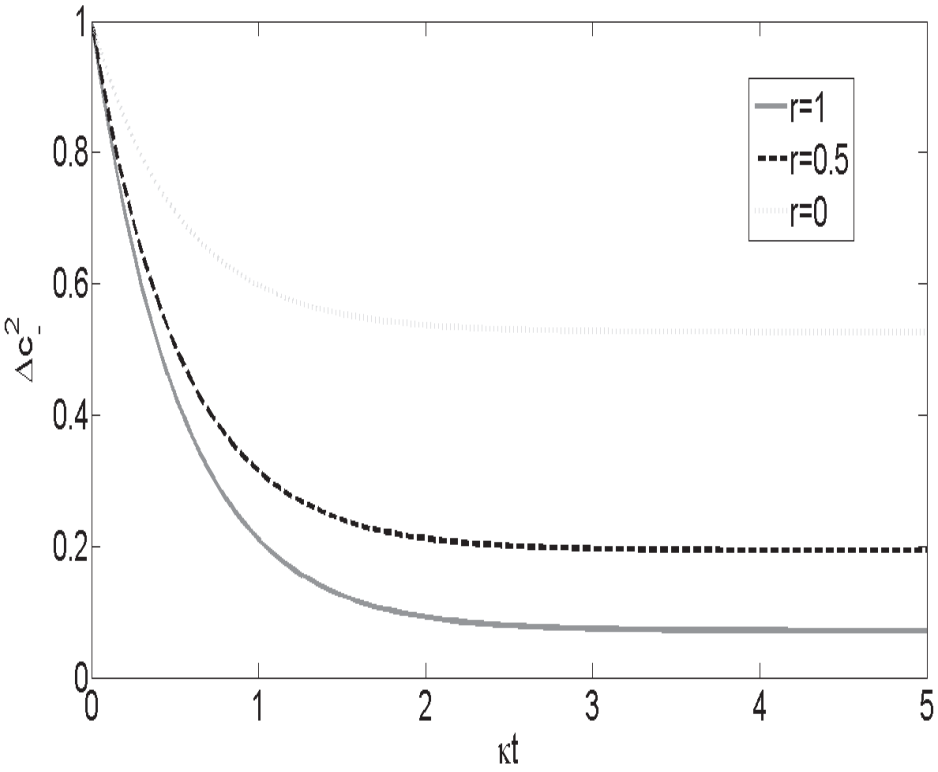

Figure 3: The steady-state quadrature variance of the intra-cavity signal and idler modes versus κt for $\varepsilon/\kappa = 0.45$ and for different values of r.

Mean photon number

The mean number of photon pairs describing the two-mode cavity radiation can be defined as

$$\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle \qquad (87)$$

In view of Eq. (80), one can readily obtain

$$\bar{n} = \frac{1}{2}[\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1 \rangle] \qquad (88)$$

With the aid of (15) and (16) together with (10) and (11), the mean photon number takes the form

$$\bar{n} = \frac{\kappa}{2\mu_+}[2N + 2M + 1](1 - e^{-\mu_+ t}) - \frac{\kappa}{2\mu_-}[2N - 2M + 1](1 - e^{-\mu_- t}) \qquad (89)$$

At steady-state, it turns out to be

$$\bar{n} = \left(\frac{\kappa}{\mu_+} - \frac{\kappa}{\mu_-}\right)\left[N + \frac{1}{2}\right] + M\left(\frac{\kappa}{\mu_+} + \frac{\kappa}{\mu_-}\right) \qquad (89)$$

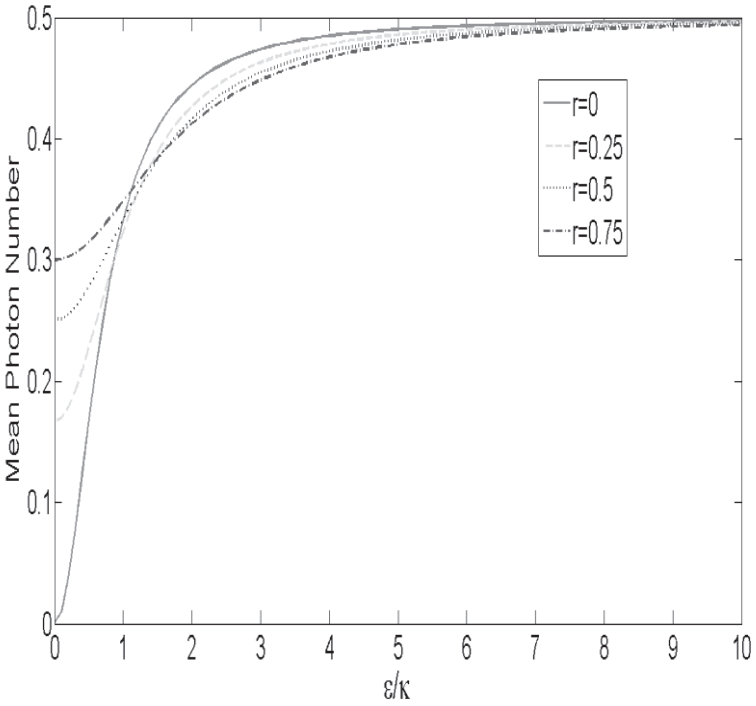

Figure 4: The steady-state mean photon number of the intra-cavity signal and idler modes versusε/κ for different values of *r*.

In Figure 4, we plot the mean photon number of the two-mode light versus g in the absence and presence of the squeezed vacuum reservoir. It can be seen from this figure that the squeezed vacuum increases the mean photon number in region where there is strong squeezing and entanglement. Hence this system generates a bright and highly squeezed as well as entangled light.

Intensity difference

The intensity difference can be defined as

$$\Delta\hat{I} = \hat{a}_1^{\dagger}\hat{a}_1 - \hat{a}_2^{\dagger}\hat{a}_2 \qquad (91)$$

Employing equations (10), (11), (15), and (16), the mean of the intensity difference at steady state turns out to be

$$\Delta\hat{I} = \left(\frac{\kappa}{\mu_+} - \frac{\kappa}{\mu_-}\right)\left[N + \frac{1}{2}\right] - M\left(\frac{\kappa}{\mu_+} + \frac{\kappa}{\mu_-}\right) \qquad (92)$$

In Figure 5, we plot the steady state variance of the normalized photon number difference for the intra-cavity signal and idler modes coupled to squeezed vacuum reservoir. The red dotted curve, representing the steady state normalized photon number difference in the absence of squeeze parameter, i.e., for vacuum reservoir only, shows that the intensity difference has greater value when $\frac{\varepsilon}{\kappa} = 1$. But initially in the absence of parametric

amplifier ($\frac{\varepsilon}{\kappa} = 0$), the normalized intensity difference is 0.75 which is a maximum value. On the other hand, for $0 < \frac{\varepsilon}{\kappa} < 0.3$, as the squeeze parameter, r, increases the intensity difference increases. Furthermore, for $0 < \frac{\varepsilon}{\kappa} < 1$, as r increases normalized photon number difference decreases.

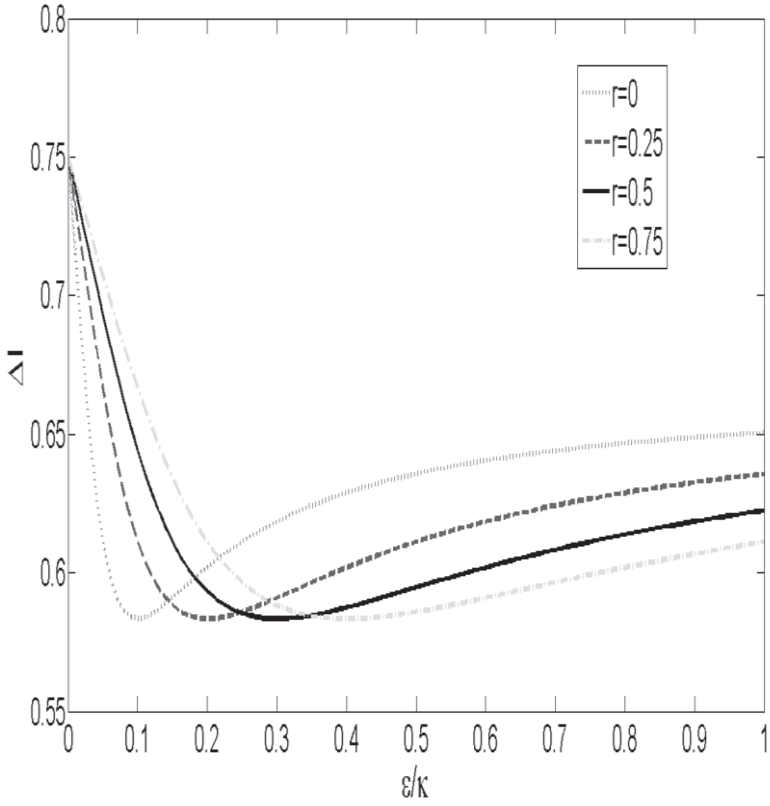

Figure 5: The steady-state photon number difference of the intra-cavity signal and idler modes versus ε/κ for different values of *r*.

Normalized second-order correlation functions

In this section, we analyze the second-order correlation function for the separate mode as well as for the superposition of the two modes. Moreover, we calculate the linear correlation coefficient between the cavity modes. The normalized second-order correlation function for the two-mode light can be expressed as

$$g^{(2)}_{(a_1, a_2)}(0) = \frac{\langle \hat{a}_1^{\dagger}\hat{a}_1\hat{a}_2^{\dagger}\hat{a}_2 \rangle}{\langle \hat{a}_1^{\dagger}\hat{a}_1 \rangle \langle \hat{a}_2^{\dagger}\hat{a}_2 \rangle} \tag{93}$$

Since $\hat{a}_1$and $\hat{a}_2$ are Gaussian variables with vanishing mean, Eq. (93) takes the form

$$g^{(2)}_{(a_1,a_2)}(0) = 1 + \frac{\langle \hat{a}_1\hat{a}_2\rangle\langle \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle + \langle \hat{a}_1\hat{a}_2^{\dagger}\rangle\langle \hat{a}_1^{\dagger}\hat{a}_2\rangle}{\langle \hat{a}_1^{\dagger}\hat{a}_1\rangle\langle \hat{a}_2^{\dagger}\hat{a}_2\rangle} \quad (94)$$

With the aid of Eqs. (10), (11), (15) and (16), the steady-state second-order correlation function takes the form:

$$g^{(2)}_{(a_1,a_2)}(0) = 1 + \left[\frac{\left(\frac{\kappa}{\mu_+}-\frac{\kappa}{\mu_-}\right)\left(\frac{\kappa}{\mu_+}+\frac{\kappa}{\mu_-}\right)M(N+1)+\left(\frac{4\kappa}{\mu_-\mu_+}N\right)\left[N+\frac{1}{2}\right]}{\left(\frac{\kappa}{\mu_+}-\frac{\kappa}{\mu_-}\right)\left(\frac{\kappa}{\mu_+}+\frac{\kappa}{\mu_-}\right)\left[N+\frac{1}{2}\right]M}\right] \quad (95)$$

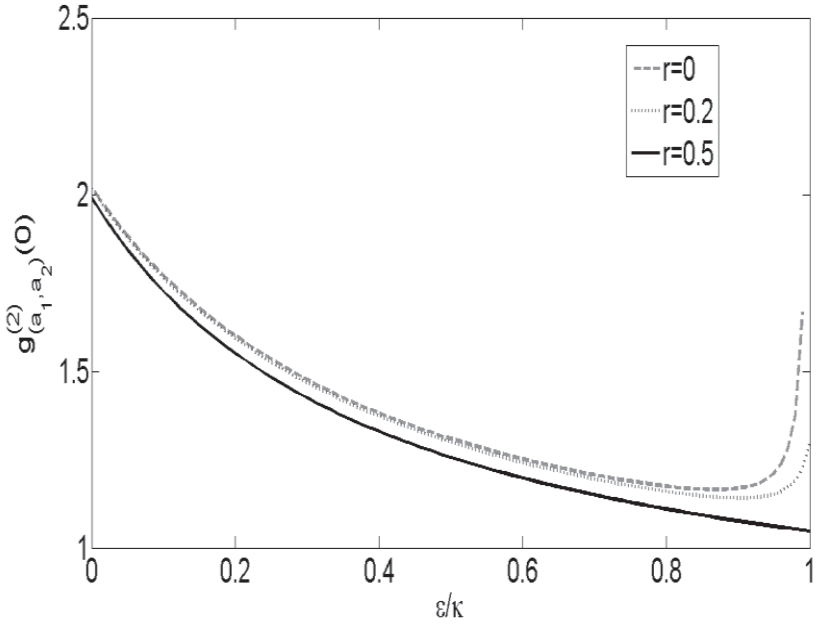

Figure 6: The steady-state photon number correlation of the intra-cavity signal and idler modes versus ε/κ for different values of *r*.

We plot, in Figure 6, the second-order correlation function for the two-mode light coupled with squeeze vacuum reservoir. We easily see from this figure that $g^{(2)}_{(a_1,a_2)}(0)$increases with in both cases. We also see from the same figure that the effect of squeeze parameter, r, is to decrease the second-order correlation function. As this function deals with the correlation between the photon numbers, as already seen in the plots, it would not be a direct measure of the squeezing as well as entanglement in this system. However, it could be an indirect inference for the existence of squeezing and entanglement as it exhibits quantum correlation that violates certain classical inequalities like Hillery-Zubairy and Cauchy-Schwarz inequalities.

CONCLUSION

In this work, we have seen the simplicity with which the entanglement of the von-Neumann entropy generated by coherently driven non-degenerate three level laser

whose cavity contains a non-degenerate parametric oscillator (NDPO) could be analyzed with the aid of the quantum Langevin equations. Applying the solutions of these equations, we obtained the anti-normally ordered characteristic function. With the aid of the resulting characteristic function, the Q-function is determined which in turn used to calculate the entanglement of the twin intra-cavity photons in a sub-threshold NDPO coupled with a two-mode squeezed vacuum by calculating the Von-Neumann Entropy (VNE). We found that the maximum entanglement is 99.6% for maximum squeeze parameter $r = 1$. Here we observe that the entropy entanglement increases with squeeze parameter r.

Furthermore, applying the *Q* function, the photon number distribution for the signal and idler modes are obtained. We have seen that at steady state and in the absence of parametric oscillation, the two-mode photon distribution shrinks to the photon number distribution of the single-mode squeezed vacuum. Moreover, employing the solutions of the quantum Langevin equations, we determined the mean photon number, the quadrature variance, the intensity of the photon number difference, and the normalized second-ordered correlation functions for the signal-idler modes produced by a non-degenerate parametric oscillator coupled to a two-mode squeezed vacuum.

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