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Stable Numerical Method for Singularly Perturbed Boundary Value Problems with Two Small Parameters

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ABSTRACT

Stable numerical method for singularly perturbed boundary value problem with two small positive parameters is presented. Given problem is converted into asymptotically equivalent boundary value problem. Then, using the finite difference approximations, the obtained differential equation is transformed to a three-term recurrence relation. The stability and convergence of the method have been established. To validate the applicability of the proposed method, three examples have been considered and solved for different values of perturbation parameters. Both theoretical error bounds and numerical rate of convergence have been established for the method. The numerical results have been presented in tables and graphs, as it can be observed from the numerical results, the present method approximates the exact solution very well. Moreover, the present method gives better results than some existing numerical methods reported in the literature.

Keywords: Boundary value problems; Singular perturbation; Stability and convergence analysis; Stable numerical method

INTRODUCTION

The problems in which the highest order derivative term is multiplied by small positive parameters are known to be perturbed problems and the parameter is known as the perturbation parameter (Vasil'eva, 1976). Singularly perturbed problems arise in various branches of applied mathematics and physics such as fluid mechanics, quantum mechanics, elasticity, plasticity, semi-conductor device physics, geophysics, optimal control theory, aerodynamics, oceanography, and mathematical models of chemical reactions (Firdous *et al.*, 2016) and also in engineering, biology and lubrication theory (Kumar, 2012).

A singularly perturbed boundary value problem with two small positive parameters are getting more attention in the more recent years; with the parameters is the coefficient of the second and first order derivatives of the definitional equation. There are various methods proposed for solving such second order singularly perturbed boundary value problems with two small parameters. Some of the methods are B-spline collocation, finite difference, finite element, exponential spline, and Haar wavelet approach and fourth order stable central difference methods; see (Kadalbajoo and Yadaw, 2008, 2011; Kadalbajoo and Kumar, 2010; Zahara and El Mhlawy, 2013; Pandit and Kumar, 2014 and Terefe *et al*., 2016).

Classical numerical methods, which have been known to be effective for solving most problems that arises in application, have failed when applied to singularly perturbed problems. Most of these methods are not effective for solving singularly perturbed boundary value problems, because as the parameter closer and closer to zero the error in the numerical solution increases and often become not comparable in magnitude to the exact solution (Firdous et al., 2016 and Kumar, 2012). To overcome this drawback, we proposed stable numerical method for solving singularly perturbed boundary value problems with two small positive parameters.

MATERIALS AND METHODS

Consider the following two-parameter singularly perturbed boundary value problem.

$$-\varepsilon y''(x)+\mu a(x)y'(x)+b(x)y(x)=f(x)$$
$$x\in\Omega=[0,1] \qquad (1)$$

with boundary conditions

$$y(0)=\alpha \text{ and } y(1)=\beta \qquad (2)$$

where ε and μ are small parameters such that $0<\varepsilon,\mu<<1$ with $a(x),b(x)$ and $f(x)$ are bounded smooth functions in the given domain and α ,β are given constant.

To describe the scheme, we divide the interval $[0,1]$ into N equal subintervals of uniform mesh length h. let $0=x_0, x_1, x_2, ..., x_N=1$ be the mesh points. Then, we have $x_i=x_0+ih$ or $x_{i+1}=x_i+h$ $i=1,2,...,N-1$. For the sake of simplicity, let us denote $a(x_i)=a_i$,$b(x_i)=b_i$, $p(x_i)=p_i$,$q(x_i)=q_i$, $f(x_i)=f_i$, $y(x_i)=y_i$ $y'(x_i)=y_i'$,$y''(x_i)=y_i''$ and $y^{(n)}(x_i)=y_i^{(n)}$.

Now, Eq. (4.1) can be re-written as.

$$-y_i''+p_iy_i'+q_iy_i=r_i$$
$$i=0,1,\ 2,\ldots,\ N. \qquad (3)$$

where $p_i=\dfrac{\mu a_i}{\varepsilon}$, $q_i=\dfrac{b_i}{\varepsilon}$ and $r_i=\dfrac{f_i}{\varepsilon}$.

Assume that the solution $y(x)$ has continuous higher order derivatives on $[0,1]$, continuously differentiable on $(0,1)$ and satisfies Eqs. (1) and (2). Using Taylor series expansion, we obtain central difference approximation for y_i' and y_i'' as.

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + \tau_1, \tag{4}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + \tau_2, \tag{5}$$

where $\tau_1 = -\frac{h^4 y_i^{(5)}}{120}$ and $\tau_2 = -\frac{h^4}{360} y_i^{(6)}$.

Substituting Eqs. (4) and (5) into Eq. (3), we obtain.

$$\left(-\frac{1}{h^2} - \frac{p_i}{2h}\right) y_{i-1} + \left(q_i + \frac{2}{h^2}\right) y_i$$

$$+\left(-\frac{1}{h^2} + \frac{p_i}{2h}\right) y_{i+1} - \frac{h^2}{6} p_i y_i'''$$

$$+\frac{h^2}{12} y_i^{(4)} = r_i - \tau_3, \tag{6}$$

where $\tau_3 = -\frac{p_i h^4 y_i^{(5)}}{120} + \frac{h^4}{360} y_i^{(6)}$.

Solving Eq. (3) for y_i'' and differentiating successively yield.

$$y_i''' = p_i y_i'' + (p_i' + q_i) y_i' + q_i' y_i - r_i'. \tag{7}$$

$$y_i^{(4)} = \left(p_i^2 + 2p_i' + q_i\right) y_i'' +$$
$$\left(p_i p_i' + p_i q_i + p_i'' + 2q_i'\right) y_i' +$$
$$\left(q_i'' + p_i q_i'\right) y_i - p_i r_i' - r_i''. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) taking account the finite difference approximations in Eqs. (4) and (5) we get the following equality.

$$-(\frac{1}{h^2}+\frac{p_i}{2h}+\frac{p_i^{\,2}}{6}-\frac{1}{12}\left(p_i^{\,2}+2p_i'+q_i\right)+$$

$$+\frac{h}{24}\left(p_i''+2q_i'+p_i\left(p_i'+q_i\right)\right))y_{i-1}$$

$$+(\frac{2}{h^2}+q_i+\frac{p_i^{\,2}}{3}-\frac{1}{6}\left(p_i^{\,2}+2p_i'+q_i\right)$$

$$-\frac{p_iq_i'h^2}{6}+\frac{h^2}{12}\left(q_i''+p_iq_i'\right)y_i$$

$$-(\frac{1}{h^2}-\frac{p_i}{2h}+\frac{p_i^{\,2}}{6}-\frac{1}{12}\left(p_i^{\,2}+2p_i'+q_i\right)$$

$$-\frac{h}{24}\left(p_i''+2q_i'+p_i\left(p_i'+q_i\right)\right)y_{i+1}$$

$$=\frac{h^2r_i''}{12}-\frac{p_ih^2}{12}r_i'+r_i+\tau \qquad (9)$$

where τ is called local truncation error.

$$\tau=\left(p_i''-2q_i'+p_i'p_i+p_iq_i\right)\frac{h^4y_i'''}{72}+$$

$$\left(-p_i^{\,2}+2p_i'+q_i\right)\frac{h^4y_i^{(4)}}{144}+\frac{h}{360}^{4}y_i^{(6)}$$

$$-\frac{p_ih^4y_i^{(5)}}{120}.$$

Re-writing Eq. (9) in the three-term recurrence relation form:

$$L^Ny\left(x\right)\equiv -E_iy_{i-1}+F_iy_i-G_iy_{i+1}=H_i \qquad (10)$$

for $i=1,2,\ldots,N-1,$ where

$$E_i=\frac{1}{h^2}+\frac{p_i}{2h}+\frac{p_i^{\,2}}{6}-\frac{1}{12}\left(p_i^{\,2}+2p_i'+q_i\right)+$$

$$\frac{h}{24}\left(p_i''+2q_i+p_i\left(p_i'+q_i\right)\right)$$

$$F_i = \frac{2}{h^2} + q_i + \frac{p_i^2}{3} - \frac{1}{6}\left(p_i^2 + 2p_i' + q_i\right)$$
$$- \frac{p_i q_i' h^2}{6} + \frac{h^2}{12}\left(q_i'' + p_i q_i'\right)$$
$$G_i = \frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{1}{12}\left(p_i^2 + 2p_i' + q_i\right)$$
$$- \frac{h}{24}\left(p_i'' + 2q_i' + p_i\left(p_i' + q_i\right)\right)$$
$$H_i = \frac{h^2}{12} r_i'' - \frac{p_i h^2}{12} r_i' + r_i .$$

The system in Eq. (10) gives $N-1$ by $N-1$ tri-diagonal systems, which can easily be solved by Thomas algorithms (Terefe *et al.,* 2016).
Stability and Convergence Analysis

Definition: in mathematics **t**he class of L-matrices are those matrices whose off-diagonal entries are less than or equal to zero and whose diagonal entries are positive; that is, an L-matrix satisfies **(**Varga, 1962**)**

$$L = \left(l_{i,j}\right);\ l_{i,i} > 0;\quad l_{i,j} \le 0,\ i \ne j,$$
for $i, j = 1, 2, 3, \ldots, N-1$

To show the stability and convergence of the method we considered the following two theorems without proofs.

Theorem 1: For any partition JUK of the index set $\{1, 2, 3, \ldots., n\}$ of an $n \times n$ matrix A, if there exists $j \in J$ and $k \in K$ such that $a_{jk} \ne 0$ then A is an irreducible matrix (Young, 1971).

Theorem 2: If A is an L-matrix which is symmetric, irreducible and has weak diagonal dominance, then A is a monotone matrix (Young, 1971).

Theorem 3: Let A be a coefficient matrix of the discretized problem of Eq. (10). Then, for all $\varepsilon, \mu > 0$ and sufficiently small h, the matrix A is an irreducible and diagonally dominant matrix.

Proof: Writing Eq. (10) in matrix vector form we obtain.
$$AY = B$$
Where, A is a coefficient matrix, $Y = (y_{1,}\ y_{2,} \cdots y_{N-1})^T$ and
$$B = (H_1 + E_1\alpha, H_2, \ldots, H_{N-2}, H_{N-1} + G_{N-1}\beta)^T .$$

Clearly, A is a tri-diagonal matrix. Matrix A is irreducible if its co-diagonals E_i and G_i contain non-zero elements only.

Multiplying both sides of Eq. (10) by h , we get the equivalent tri-diagonal scheme:

$$
\begin{aligned}
&-\left(\frac{1}{h}+\frac{p_i}{2}+\frac{hp_i^2}{6}-\frac{h}{12}\left(p_i^2+2p_i'+q_i\right)+\right.\\
&\qquad\left.\frac{h^2}{24}\left(p_i''+2q_i'+p_i(p_i'+q_i)\right)\right)y_{i-1}\\
&+\left(\frac{2}{h}+hq_i+\frac{hp_i^2}{3}-\frac{h}{6}\left(p_i^2+2p_i'+q_i\right)\right.\\
&\qquad\left.-\frac{h^3p_iq_i'}{6}+\frac{h^3}{12}\left(q_i''+p_iq_i'\right)\right)y_i\\
&-\left(\frac{1}{h}-\frac{p_i}{2}+\frac{hp_i^2}{6}-\frac{h}{12}\left(p_i^2+2p_i'+q_i\right)\right.\\
&\qquad\left.-\frac{h^2}{24}\left(p_i''+2q_i'+p_i(p_i'+q_i)\right)\right)y_{i+1}\\
&\qquad=h\left(r_i-\frac{p_ih^2}{12}r_i'+\frac{h^2}{12}r_i''\right)+T(h),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
T(h)=&(p_i''-2q_i'+p_i'p_i+p_iq_i)\frac{h^5y_i'''}{72}+\\
&(-p_i^2-2p_i'+q_i)\frac{h^5y_i^{(4)}}{144}+\frac{h^5y_i^{(6)}}{360}\\
&-\frac{p_ih^5y_i^{(5)}}{120}+O\left(h^6\right).
\end{aligned}
$$

It is easily seen that, for sufficiently small h, we have: $E_i \neq 0$ and $G_i \neq 0$, $\forall i = 1,2,\ldots,N-1$. Hence A is irreducible (Terefe *et al.*, 2016).

Again one can observe that $E_i,\ G_i\ > 0,\ \ \|F_i\| > 0$ and the sum of the two off diagonal elements is less than or equal to the modulus of the diagonal element. This proves the diagonal dominance of A. Hence, A is diagonally dominant.

Under these conditions, the Thomas Algorithm is stable (Kadalbajoo and Reddy, 1989).

Theorem 4: Let $y(x)$ be the analytical solution of the problem in Eq. (1) and (2) and $y^N(x)$ be the numerical solution of the discretized problem of Eq. (10). Then, $\left\|y-y^N\right\| \le Ch^4$ for sufficiently small h and C is positive constant.

Simplifying Eq. (11), we get the equivalent tri-diagonal scheme.

$$(-1+u_i)y_{i-1}+(2+v_i)y_i+$$
$$(-1+w_i)y_{i+1}+g_i+T_i=0, \qquad (12)$$

Where

$$u_i=-\frac{hp_i}{2}-\frac{h^2p_i^2}{6}+\frac{h^2}{12}\left(p_i^2+2p_i'+q_i\right)$$
$$-\frac{h^3}{24}\left(p_i''+2q_i'+p_i(p_i'+q_i)\right).$$
$$v_i=h^2q_i+\frac{h^2p_i^2}{3}-\frac{h^2}{6}\left(p_i^2+2p_i'+q_i\right)$$
$$-\frac{h^4p_iq_i'}{6}+\frac{h^4}{12}\left(q_i''+p_iq_i'\right).$$
$$w_i=\frac{hp_i}{2}-\frac{h^2p_i^2}{6}+\frac{h^2}{12}(p_i^2+2p_i'+q_i)+$$
$$\frac{h^3}{24}(p_i''+2q_i'+p_i(p_i'+q_i))$$
$$g_i=h^2r_i-\frac{p_ih^4}{12}r_i'+\frac{h^4}{12}r_i''$$
$$T_i(h)=(p_i''-2q_i'+p_i'p_i+p_iq_i)\frac{h^6y_i'''}{72}+$$
$$(-p_i^2-2p_i'+q_i)\frac{h^6y_i^{(4)}}{144}+\frac{h^6y_i^{(6)}}{360}$$
$$-\frac{p_ih^6y_i^{(5)}}{120}+O(h^7).$$

Incorporating the boundary conditions $y_0=y(0)=\alpha$ and $y_N=y(1)=\beta$ in Eq. (12), we obtain.

$$\begin{bmatrix} (2+v_1) & (-1+w_1) & 0 & \ldots & 0 \\ (-1+u_2) & (2+v_2) & \ddots & . & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & (-1+w_{N-2}) \\ 0 & \cdots & 0 & (-1+u_{N-1}) & (2+v_{N-1}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \end{bmatrix}$$

$$+\begin{bmatrix} g_1+(-1+u_1)\alpha \\ g_2 \\ g_3 \\ \vdots \\ g_{N-1}+(-1+w_{N-1}) \end{bmatrix} + \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-1} \end{bmatrix} = \overline{0}$$

This implies

$$(D+P)y+M+T(h)=\overline{0}, \qquad (13)$$

where

$$D=\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \quad P=\begin{bmatrix} v_1 & w_1 & 0 & \ldots & 0 \\ u_2 & v_2 & w_2 & \ldots & 0 \\ 0 & u_3 & v_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & w_{N-2} \\ 0 & \cdots & 0 & u_{N-1} & v_{N-1} \end{bmatrix}$$

are tri-diagonal matrices of order $N-1$, and

$M=\left[(g_1+(-1+u_1)\alpha), g_{2,} g_3, \ldots, (g_{N-1}+(-1+w_{N-1})\beta)\right]^T$,

$T(h)=O(h^6)$ and $y=\left[y_1, y_2, y_3, \ldots, y_{N-1}\right]^T$, $T(h)=\left[T_1, T_2, T_3, \ldots, T_{N-1}\right]^T$,

$\overline{0}=\left[0,0,0,\ldots,0\right]^T$ are the associated column vectors of Eq.(13).

Let $y^N=\left[y_1^N, y_2^N, y_3^N, \ldots, y_{N-1}^N\right]^T \cong y$ be the solution, which satisfies Eq. (14), and then we have.

$$(D+P)y^N+M=\overline{0} \qquad (14)$$

Let $e_i = y_i - y_i^N$, for $i = 1, 2, ..., N-1$ be the discretization error, then

$$y - y^N = [e_1, e_2, e_3 ..., e_{N-1}]^T .$$

Subtracting Eq. (13) from Eq. (14) we get.

$$(D+P)(y^N - y) = T(h) \tag{15}$$

Let $|p_i| \le c_1$, $|p_i'| \le c_2$, $|p_i''| \le c_3$, $|q_i| \le k_1$, $|q_i'| \le k_2$, $|q_i''| \le k_3$ and t_{ij} be the $(i, j)^{th}$ element of the matrix P. Then, for, $i = 1, 2, ..., N-2$ and for sufficiently small h the $(i, i+1)^{th}$ of the matrix D is -1. Hence, the matrix $(D+P)$ is an irreducible, (Verga, 1962). Let S_i be the sum of the elements of the i^{th} row of the matrix $(D+P)$, then.

For, $i = 1$

$$S_i = 1 + \frac{hp_i}{2} + \frac{h^2}{12}(11q_i + p_i^2 - 2p_i' +$$
$$\frac{h^3}{24}(p_i'' + 2q_i' + p_i(p_i' + q_i)) + O(h^4).$$

For, $i = 2, 3, ...N-2$, $S_i = h^2 q_i + O(h^4).$

For, $i = N-1$

$$S_i = 1 - \frac{hp_i}{2} + \frac{h^2}{12}\left(p_i^2 - 2p_i' + 11q_i\right)$$
$$-\frac{h^3}{24}\left(p_i'' + 2q_i' + p_i(p_i' + q_i)\right) + O(h^4).$$

For sufficiently small h, $(D+P)$ and $(D+P)^{-1}$ is monotone, since, $(D+P) \to D$ which is symmetric and has weak diagonal dominance, (Verga, 1962 and Young, 1971).

Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1} \ge 0$.

From the error Eq. (16) we have.

$$\|y - y^N\| \le \|(D+P)^{-1}\| \|T(h)\| \tag{16}$$

For sufficiently small h, we have:

For, $i = 1$: $S_i > \frac{11}{12} h^2 k_{1^*}$

For $i = 2, 3, ..., N-2$: $S_i > h^2 k_{1^*}$

For $i = N-1$: $S_i > \frac{11}{12}h^2 k_{1^*}$

where

$$k_{1^*} = \min_{1\le i\le N-1}\left| q_i + \frac{1}{12}\left(q_i{}'' - p_i q_i{}'\right)\right|$$
$$= \min_{1\le i\le N-1}\left| q_i + O(h^4)\right| = \min_{1\le i\le N-1}\left| q_i \right|.$$

Let $(D+P)^{-1}{}_{i,k}$ be the $(i,j)^{th}$ elements of $(D+P)^{-1}$ and we define

$$\left\|(D+P)^{-1}\right\| = \max_{1\le i\le N-1}\sum_{k=1}^{N-1}\left(D+P\right)^{-1}{}_{i,k} \text{and}$$

$$\left\|T\left(h\right)\right\| = \max_{1\le i\le N-1}\left|T_i\right| \tag{17}$$

Since $(D+P)^{-1}{}_{i,k} \ge 0$, then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1}\left(D+P\right)^{-1}{}_{i,k} .S_k = 1, \tag{18}$$

for $i = 1, 2, .., N-1$

Hence, for $k = 1,$

$$(D+P)^{-1}{}_{i,1} \le \frac{1}{s_1} < \frac{12}{11}\left(\frac{1}{h^2 k_{1^*}}\right).$$

For, $k = N-1$

$$(D+P)^{-1}{}_{i,N-1} \le \frac{1}{s_{N-1}} < \frac{12}{11}\left(\frac{1}{h^2 k_{1^*}}\right) \tag{19}$$

Further, for $k = 2, 3, ...N-2$

$$\sum_{k=2}^{N-2}\left(D+P\right)^{-1}{}_{i,k} \le \frac{1}{\min\limits_{2\le k\le N-2} S_k} \le \frac{1}{h^2 k_{1^*}} \tag{20}$$

Now, from Eqs. (16)-(20) we get:

$$\left\|y - y^N\right\| \le Ch^4$$

Where,

$$C=\frac{35}{11k_{1^*}}\left|\left(-p_i''-2q_i'+p_i'p_i+p_iq_i\right)\frac{y_i'''}{72}+\right.$$
$$\left.\left(-p_i^2+2p_i'+q_i\right)\frac{y_i^{(4)}}{144}-\frac{p_iy_i^{(5)}}{120}+\frac{y_i^{(6)}}{360}\right|$$

Which is independent of the mesh sizes h .

Then, $\left\|y-y^N\right\|\le Ch^4$. This establishes that the method is fourth order convergent.

RESULTS

To demonstrate the applicability of the method, it is applied on three model numerical examples. The maximum absolute errors, at the nodal points are evaluated by the formula $\|E\|=\max|y(x_i)-y_i|$, for $i=0,1,2,...N$.

Where, $y(x_i)$ and y_i are the exact and computed solution of the given problem respectively. For those examples, which have no exact solutions, the maximum absolute errors are computed by using double mesh principle given by $Z_h=\max\limits_i\left|y_i^h-y_i^{\frac{h}{2}}\right|$ for $i=1,2,\ ...,\ N-1$, where y_i^h is the numerical solution on the mesh $\{x_i\}_{i=1}^{N-1}$ at the nodal point x_i and $x_i=x_0+ih$ for $i=1,2,...,N-1$ and $y_i^{\frac{h}{2}}$ is the numerical solution at the nodal point x_i on the mesh $\{x_i\}_{i=1}^{2N-1}$ where $x_i=x_0+\dfrac{ih}{2}$ for $i=1,2,...,2N-1$ (*i.e.*, the numerical solution on a mesh, obtained by bisecting the original mesh with number of mesh intervals, [12]. In the same way one can define $Z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$, that is,

$$Z_{h/2}=\max_i\left|y_i^{h/2}-y_i^{h/4}\right|,$$

for $i\ =1,\ 2,\ ...,\ 2N-1.$

The computed rate of convergence is given by $\rho=\dfrac{\log(Z_h)-\log(Z_{h/2})}{\log 2}$.

Example 1: Consider the singularly perturbed boundary valued problem with two small parameters (Pandit and Kumar, 2014):

$$\varepsilon y''+\mu y'+y=\cos(\pi x),\qquad x\in(0,\ 1),$$

With boundary conditions $y(0)=0=y(1).$

Its exact solution is

$$y(x) = a_1 \cos(\pi x) + b_1 \sin(\pi x) + Ae^{(\lambda_1 x)} + Be^{(-\lambda_2 (1-x))}$$

Where,

$$a_1 = \frac{\varepsilon\pi^2 + 1}{\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2},$$

$$b_1 = \frac{\mu\pi}{\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2},$$

$$A = \frac{-a_1(1 + e^{-\lambda_2})}{1 - e^{(\lambda_1 - \lambda_2)}}, \quad B = \frac{b_1(1 + e^{\lambda_1})}{1 - e^{(\lambda_1 - \lambda_2)}}.$$

For λ_1, λ_2 are the roots of the characteristic equation of $-\varepsilon\lambda^2 + \mu\lambda + 1 = 0$. The maximum absolute errors are presented in Table 1 for different values of ε, μ at $N = 128$. The graph of the computed solution for different values of ε, μ and $N = 32$ is also given in Figure 1 below.

Table 1: The comparison of maximum absolute errors for Example 1

	$\varepsilon = 10^{-2}$ and $N = 128$		$\varepsilon = 10^{-4}$ and $N = 128$	
$\mu \downarrow$	Pandit (2014)	Our Method	Pandit (2014)	Our Method
10^{-3}	4.2303 e-5	2.4810e-08	5.1964e-3	2.7138e-04
10^{-4}	4.1318 e-5	2.4656e-08	4.1710e-3	2.7139e-04
10^{-5}	4.1220e-5	2.4641e-08	4.0754e-3	2.7148e-04
10^{-6}	4.1210e-5	2.4640e-08	4.0659e-3	2.7240e-04

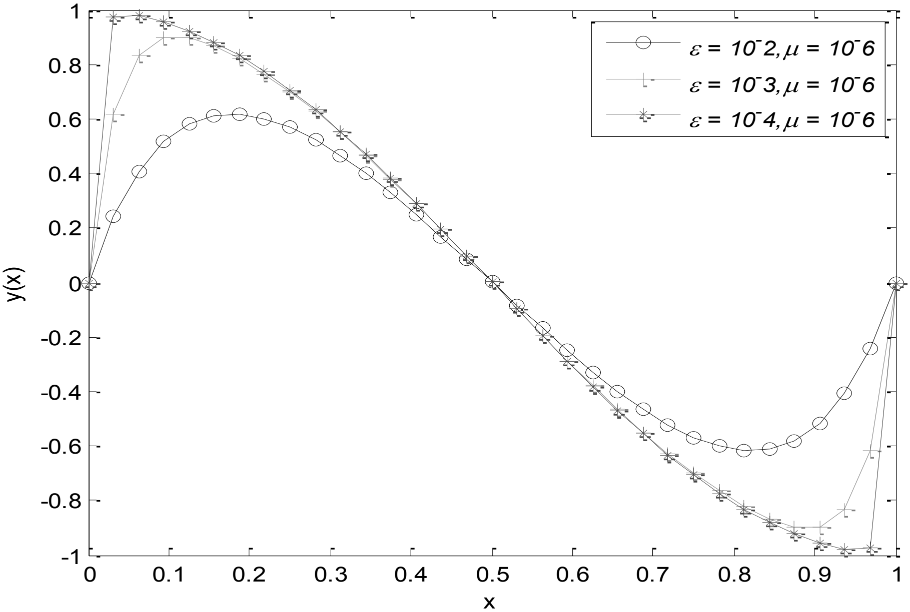

Figure 1. Physical behavior of numerical solution of Example 1 for $\mu = 10^{-6}$, $N = 32$ and different values of ε

Example 2: Consider the singularly perturbed boundary valued problem with two small parameters (Kadalbajoo and Kumar, 2010).

$$-\varepsilon y'' - \mu(1+x)y' + y = x,$$

boundary conditions $y(0) = 1$ and $y(1) = 0.$

The exact solution is not available and the maximum absolute errors are presented for present method, in Tables 2 and 3 for different values of μ, ε and N. The graph for the behavior of the computed solution at different values of ε, μ and $N = 64$ is given in Figure 2.

Table 2: The comparison of maximum absolute errors of Example 2 for $\mu = 10^{-4}$

	$N = 128$		$N = 256$	
$\varepsilon \downarrow$	Kadalbajoo and Kumar(2010)	Our Method	Kadalbajoo and Kumar (2010)	Our Method
10^{-1}	4.0386e-06	1.5028e-10	9.8517e-07	9.3899e-12
10^{-2}	6.9830e-05	2.6712e-08	1.7521e-05	1.6698e-09
10^{-3}	6.9993e-04	2.6776e-06	1.7652e-04	1.6766e-07
10^{-4}	1.3252e-03	2.5620e-04	5.7425e-04	1.6587e-05

Table 3: The comparison of maximum absolute errors of Example 2 for $\varepsilon = 10^{-2}$

	$N = 128$		$N = 256$	
$\mu \downarrow$	Kadalbajoo and Kumar, (2010)	Our Method	Kadalbajoo and Kumar, (2010)	Our Method
10^{-1}	7.0951e-05	2.6914e-08	1.7731e-05	1.6824e-09
10^{-2}	7.0120e-05	2.6712e-08	1.7521e-05	1.6698e-09
10^{-3}	7.0037e-05	2.6690e-08	1.7500e-05	1.6685e-09
10^{-4}	7.0029e-05	2.6688e-08	1.7498e-05	1.6683e-09

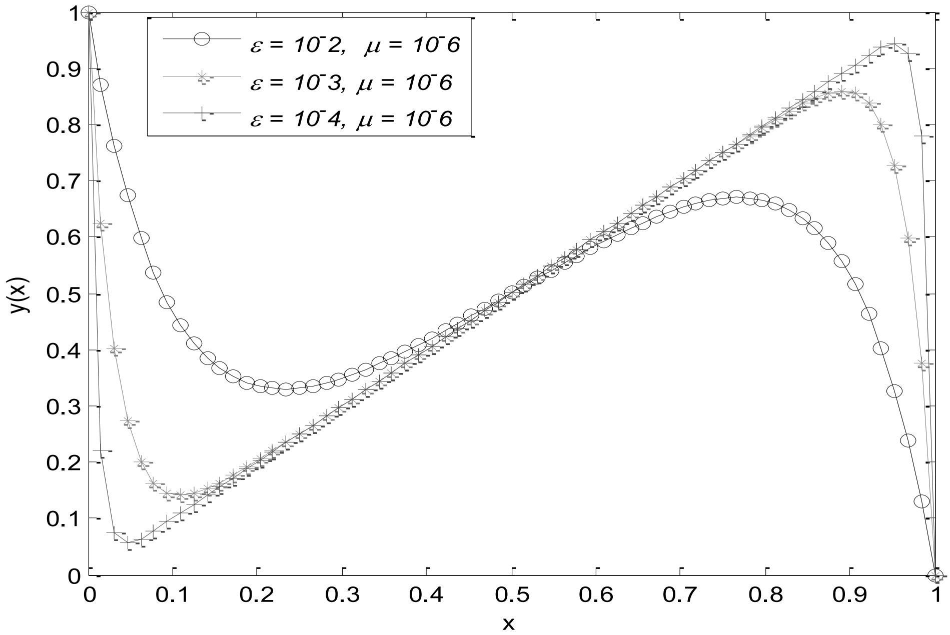

Figure 2. Physical behavior of numerical solution of Example 2 for $\mu = 10^{-6}$, N = 64 and at different values of ε.

Example 3: Consider the singularly perturbed boundary valued problem with two small parameters (Kadalbajoo and Yadaw, 2008).

$$-\varepsilon y'' + \mu(3-2x^2)y' + y = (1+x)^2,$$

with boundary conditions $y(0) = 0 = y(1)$ and its exact solution is not given. The maximum absolute errors are presented in Table 4 for different values of ε, μ at $N = 128$ and, in Table 5 at different values of ε, μ and N. The graph of the computed solution for different values of ε, μ and $N = 32$ is also given in Figure 3.

Table 4: Maximum absolute errors of Example 3 at $N = 128$

$\varepsilon\downarrow$ $\mu\rightarrow$	10^{-3}	10^{-4}	10^{-5}
10^{-1}	3.3121e-10	3.2986e-10	3.2973e-10
10^{-2}	2.7801e-08	2.7343e-08	2.7299e-08
10^{-3}	2.7828e-06	2.6888e-06	2.6765e-06
10^{-4}	2.6305e-04	2.5724e-04	2.5476e-04
10^{-5}	1.6163e-02	1.5337e-02	1.4059e-02

Table 5: Maximum absolute errors of Example 3 at different values of ε, μ and *N*

	$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-4}$	
$\mu \downarrow$	N = 128	N = 256	N = 128	N = 256
10^{-3}	2.7801e-08	1.7379e-09	2.6305e-04	1.7595e-05
10^{-4}	2.7347e-08	1.7095e-09	2.5724e-04	1.6687e-05
10^{-5}	2.7299e-08	1.7065e-09	2.5476e-04	1.6440e-05
10^{-6}	2.7294e-08	1.7062e-09	2.5448e-04	1.6413e-05
10^{-7}	2.7293e-08	1.7062e-09	2.5445e-04	1.6410e-05

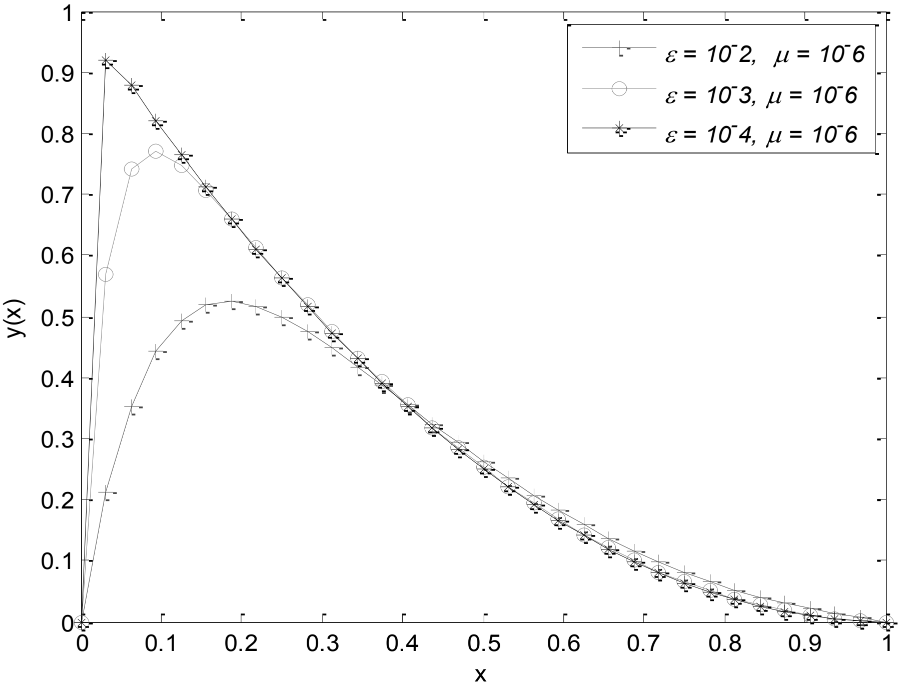

Figure 3: Physical behavior of numerical solution of Example 2 for $\mu = 10^{-6}$, N = 32 and at different values of ε.

Using double mesh principle for those Examples 1- 3, we have the following rate of convergence at different values of ε, μ and N.

Table 6: Rate of convergence for Examples 1–3 with $\varepsilon = 10^{-2}$

	N $\mu \downarrow$	64	128	256
Example 4.1	10^{-3}	3.9988	3.9997	4.0000
	10^{-4}	3.9988	3.9997	4.0000
Example 4.2	10^{-3}	3.9989	3.9997	3.9999
	10^{-4}	3.9989	3.9997	3.9999
Example 4.3	10^{-2}	3.9987	3.9997	3.9999
	10^{-3}	3.9989	3.9997	3.9999
	10^{-4}	3.9989	3.9997	3.9999

DISCUSSION

In this paper, stable central finite difference method is presented for solving singularly perturbed boundary value problems with two small positive parameters. First, the given interval is discretized and then a singularly perturbed boundary value problem is converted into an asymptotically equivalent boundary value problem. Then, the derivative of the given differential equation is replaced by the finite difference approximations which transformed into a three-term recurrence relation, that form system of equations whose solution can be obtained using Thomas algorithms. The stability and convergence of the method have been investigated and the present method is fourth order convergent.

To validate the applicability of the method, the numerical results have been presented on three model examples in Tables 1–5 for different values of the two small positive parameters and different number of mesh sizes. The results obtained by the present method are compared with the methods in (Kadalbajoo and Yadaw, 2011; Zahara and El Mhlawy, 2013 and Pandit and Kumar, 2014). It shows the present method give better results than the findings of the aforementioned scholars and the accuracy of the problem increased by increasing the resolution of the grid. Table 6 depicts that the present methods have the rate of convergence is four, which is in agreement with the theoretical rate of convergence.

The graphs of the considered examples for different values of parameters and step size are plotted in Figures 1 –3, to examine the effect of the sufficiently small positive parameters on the solution of the problem. From Figures 1 and 2, when ε decreases for fixed value of μ the width of boundary layer decreases and became more and more stiff at $x=0$ and $x=1$. Also, in Figure 3 when ε decreases for fixed value of μ the width of boundary layer decreases and became more and more stiff at $x=0$ and we get the boundary layer at the left end point of the interval. The results presented confirmed that computational rate of convergence as well as theoretical estimates indicate that the method is a fourth order convergent.

CONCLUSION

Stable numerical method for solving singularly perturbed boundary value problem with two small positive parameters is presented. Given differential equation is transformed into the finite difference approximations that can be written in a three-term recurrence relation form. The stability and convergence of the method have been established. To validate the applicability of the proposed method, three examples have been considered and solved for different values of small positive parameters ε , μ and mesh sizes h with the given numerical results in tabular and graphics. Both theoretical error bounds and numerical rate of convergence have been established for the method which shows the method is fourth order convergent. Moreover, the present method gives better results than some existing numerical methods reported in the literature.

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