ORIGINAL ARTICLE

Numerical Solution of Second Order One Dimensional Linear Hyperbolic Telegraph Equation

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Abstract

In this paper, the numerical solution of second order one dimensional linear hyperbolic telegraph equation using crank Nicholson and fourth order stable centeral difference methods have been presented. First, the given domain is discretized and the derivatives of the differential equation were replaced by finite difference approximations and then, transformed to system of equations that can be solved by matrix inverse method. The stability and consistency of the method are established. To validate the applicability of the method, model examples have been considered and solved for different mesh sizes. As it can be observed from the numerical results presented in Tables and graphs, the present method approximates the exact solution very well.

***Key words**: Hyperbolic Telegraph equation, numerical solution and convergence of method.*

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INTRODUCTION

Partial differential equations have enormous applications compared to ordinary differential equations, to mention some of these: dynamics, electricity, heat transfer, electromagnetic theory, quantum mechanics and so on (Erwin, 2006). Telegraph equations are pairs of coupled, linear differential equations that describe the voltage and current on an electrical transmission line with distance and time. The telegraph equation is one of the important equations of mathematical physics with applications in many different fields such as transmission and propagation of electrical signals (Kajiwara et al., 2010), vibration systems, random walk theory and mechanical systems (Chakraverty and Behera, 2013), etc. The heat diffusion and wave propagation equations are particular cases of the telegraph equation. The telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion (Dosti and Nazemi, 2012).

Biologists encounter these equations in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge (Eftimie,2012). Also the propagation of acoustic waves in Darcy-type porous media (Heider et al., 2012), and parallel floes of viscous Maxwell fluids (Liu et al., 2011) are just some of the phenomena modeled by the telegraph equation.

In recent years, different methods have been applied to find the numerical solution of the second order one dimensional linear hyperbolic telegraph equation. To mention some: Radial basis function approximation (Saadatmandi and Dehghan, 2010), He's variational iteration method (Dehghan et al., 2011), Laguerre-Legendre spectral collocation method (Tatari and Haghighi, 2014), differential quadrature method (Jiwari et al., 2014), differential transform method (Srivastava et al., 2014), method of weighted residuals (Odejide and Binuyo, 2014), Fibonacci polynomials (Kurt and Yalcinbas, 2016) and meshless local radial point interpolation (Elyas and Hamid, 2015).

However, it is necessary to present the accurate and convergent numerical method for solving the second order one dimensional linear hyperbolic telegraph equation. The fourth order stable central difference method to find the numerical solution of the second order self-adjoint singularly perturbed ordinary differential equation subject to certain types of boundary conditions is presented by Terefe *et al.* (2016). In this paper, our aim is to apply the amalgamation of stable central difference method and the Crank Nicholson method to find the accurate numerical solution of the second order one dimensional linear hyperbolic telegraph equation.

Description of the method

Consider the second order one dimensional linear hyperbolic telegraph equation of the form:

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}+\beta u=\frac{\partial^2 u}{\partial x^2}+f(x,t),\quad 0\le x\le b,\ 0\le t\le T \tag{1}$$

subject to the initial conditions:

$$\begin{aligned} u(x,0)&=f_0(x)\\ \frac{\partial u}{\partial t}(x,0)&=f_1(x)\end{aligned} \tag{2}$$

and with boundary conditions:

$$\begin{aligned} u(0,t)&=g_0(t)\\ u(b,t)&=g_1(t)\end{aligned} \tag{3}$$

where α and β are given positive constants and we assume that $f_0(x)$, $f_1(x)$, $g_0(t)$ and $g_1(t)$ are continuous functions.

To describe the scheme, we divide the interval $[0,b]$ and $[0,T]$ into N and M equal subintervals of mesh length h and k respectively. Let $0=x_0<x_1<x_2<\ldots<x_N=b,$ and $0=t_0<t_1<t_2<\ldots<t_N=T$ be the mesh points with $x_i=x_0+ih$ and $t_j=t_0+jk,$ for $i=1,2,\ldots,N$ and $j=0,1,\ldots,M$.

For the sake of simplicity, use $u(x_i,t_j)=u_i^j$, $\frac{\partial^n u}{\partial x^n}(x_i,t_j)=\frac{\partial^n u_i^j}{\partial x^n}$, $\frac{\partial^n u}{\partial t^n}(x_i,t_j)=\frac{\partial^n u_i^j}{\partial t^n}$ $n\ge 1$ and $f(x_i,t_j)=f_i^{\,j}$. Eq. (1) can be re-written at discretized points as:

$$\frac{\partial^2 u_i^j}{\partial t^2}=-\alpha\frac{\partial u_i^j}{\partial t}-\beta u_i^j+\frac{\partial^2 u_i^j}{\partial x^2}+f(x_i,t_j) \tag{4}$$

Assume that $u(x,t)$ has continuous higher order partial derivatives on the region $[0,b]\times[0,T]$. Using Taylor's series expansion for any point $u(x_i,t_j)$ with uniform step mesh sizes h and k in the direction of x and for fixed t, we have:

$$u_{i+1}^j=u_i^j+h\frac{\partial u_i^j}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_i^j}{\partial x^2}+\frac{h^3}{3!}\frac{\partial^3 u_i^j}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_i^j}{\partial x^4}+\frac{h^5}{5!}\frac{\partial^5 u_i^j}{\partial x^5}+\ldots \tag{5}$$

$$u_{i-1}^j=u_i^j-h\frac{\partial u_i^j}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_i^j}{\partial x^2}-\frac{h^3}{3!}\frac{\partial^3 u_i^j}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_i^j}{\partial x^4}-\frac{h^5}{5!}\frac{\partial^5 u_i^j}{\partial x^5}+\ldots \tag{6}$$

In the same way, using Taylor's series expansion in the direction of *t*, for a fixed *x*, we have:

$$u_i^{j+1}=u_i^j+k\frac{\partial u_i^j}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 u_i^j}{\partial t^2}+\frac{k^3}{3!}\frac{\partial^3 u_i^j}{\partial t^3}+\frac{k^4}{4!}\frac{\partial^4 u_i^j}{\partial t^4}+\frac{k^5}{5!}\frac{\partial^5 u_i^j}{\partial t^5}+\ldots \tag{7}$$

$$u_i^{j-1}=u_i^j-k\frac{\partial u_i^j}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 u_i^j}{\partial t^2}-\frac{k^3}{3!}\frac{\partial^3 u_i^j}{\partial t^3}+\frac{k^4}{4!}\frac{\partial^4 u_i^j}{\partial t^4}-\frac{k^5}{5!}\frac{\partial^5 u_i^j}{\partial t^5}+\ldots \tag{8}$$

Adding Eqs. (5) with (6), and subtracting Eq. (8) from Eq. (7), gives:

$$\frac{\partial^2 u_i^j}{\partial x^2} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \tau_1 \qquad (9)$$

$$\frac{\partial u_i^j}{\partial t} = \frac{u_i^{j+1} - u_i^{j-1}}{2k} - \frac{k^2}{6}\frac{\partial^3 u_i^j}{\partial t^3} + \tau_2 \qquad (10)$$

$$\frac{\partial^2 u_i^j}{\partial t^2} = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u_i^j}{\partial t^4} + \tau_3 \qquad (11)$$

where: $\tau_1 = \frac{-h^2}{12}\frac{\partial^4 u_i^j}{\partial x^4}$, $\tau_2 = \frac{-k^4}{120}\frac{\partial^5 u_i^j}{\partial t^5}$ and $\tau_3 = \frac{-k^4}{360}\frac{\partial^6 u_i^j}{\partial t^6}$

From Crank Nicholson finite difference method, average values u_i^j and $\frac{\partial^2 u_i^j}{\partial x^2}$ are:

$$u_i^j = \frac{1}{3}\left(u_i^{j+1} + u_i^j + u_i^{j-1}\right) \qquad (12)$$

$$\frac{\partial^2 u_i^j}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 u_i^{j+1}}{\partial x^2} + \frac{\partial^2 u_i^j}{\partial x^2} + \frac{\partial^2 u_i^{j-1}}{\partial x^2}\right) \qquad (13)$$

Now, substituting Eqs. (10) – (13) into Eq. (4) yields:

$$\begin{aligned} &\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \alpha\left(\frac{u_i^{j+1} - u_i^{j-1}}{2k}\right) - \frac{\alpha k^2}{6}\frac{\partial^3 u_i^j}{\partial t^3} - \frac{k^2}{12}\frac{\partial^4 u_i^j}{\partial t^4} \\ &+ \frac{\beta}{3}\left(u_i^{j+1} + u_i^j + u_i^{j-1}\right) = \frac{1}{3}\left(\frac{\partial^2 u_i^{j+1}}{\partial x^2} + \frac{\partial^2 u_i^j}{\partial x^2} + \frac{\partial^2 u_i^{j-1}}{\partial x^2}\right) + f_i^j + \tau_4 \end{aligned} \qquad (14)$$

where: $\tau_4 = -\tau_3 - \alpha\tau_2$

Differentiating Eq. (1) successively with respect to *t*, and evaluated at (x_i, t_j) we obtain:

$$\frac{\partial^3 u_i^j}{\partial t^3} = -\alpha\frac{\partial^2 u_i^j}{\partial t^2} - \beta\frac{\partial u_i^j}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) + \frac{\partial}{\partial t} f_i^j \qquad (15)$$

$$\frac{\partial^4 u_i^j}{\partial t^4} = (\alpha^2 - \beta)\frac{\partial^2 u_i^j}{\partial t^2} + \alpha\beta\frac{\partial u_i^j}{\partial t} - \alpha\frac{\partial}{\partial t}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) - \alpha\frac{\partial}{\partial t} f_i^j + \frac{\partial^2}{\partial t^2} f_i^j \qquad (16)$$

Substituting Eqs. (15) and (16) into Eq. (14), gives:

$$\begin{aligned} &\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \frac{\alpha}{2k}(u_i^{j+1} - u_i^{j-1}) + \frac{\beta}{3}\left(u_i^{j+1} + u_i^j + u_i^{j-1}\right) + \frac{k^2}{12}(\alpha^2 + \beta)\frac{\partial^2 u_i^j}{\partial t^2} \\ &+ \frac{\alpha\beta k^2}{12}\frac{\partial u_i^j}{\partial t} - \frac{\alpha k^2}{12}\frac{\partial}{\partial t}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) - \frac{k^2}{12}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) = \frac{1}{3}\left(\frac{\partial^2 u_i^{j+1}}{\partial x^2} + \frac{\partial^2 u_i^j}{\partial x^2} + \frac{\partial^2 u_i^{j-1}}{\partial x^2}\right) \\ &+ f_i^j + \frac{\alpha k^2}{12}\frac{\partial}{\partial t} f_i^j + \frac{k^2}{12}\frac{\partial^2}{\partial t^2} f_i^j + \tau_4 \end{aligned} \qquad (17)$$

Using the finite difference approximation of Eqs. (9) – (11), we have:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_i^j}{\partial x^2}\right) = \frac{1}{2kh^2}\left(u_{i+1}^{j+1} - u_{i+1}^{j-1} - 2u_i^{j+1} + 2u_i^{j-1} + u_{i-1}^{j+1} - u_{i-1}^{j-1}\right) + \tau_5 \qquad (18)$$

$$\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u_i^j}{\partial x^2}) = \frac{1}{k^2h^2}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} - 2u_{i+1}^j + 4u_i^j - 2u_{i-1}^j + u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + \tau_6 \tag{19}$$

where: $\tau_5 = \frac{-h^2}{12}\frac{\partial}{\partial t}(\frac{\partial^4 u_i^j}{\partial x^4})$ and $\tau_6 = \frac{-h^2}{12}\frac{\partial^2}{\partial t^2}(\frac{\partial^4 u_i^j}{\partial x^4})$

Putting Eqs. (18), (19) and the central finite difference approximation for $\frac{\partial u_i^j}{\partial t}$, $\frac{\partial^2 u_i^j}{\partial t^2}$ and $\frac{\partial^2 u_i^j}{\partial x^2}$ into Eq. (17), we get:

$$\begin{aligned}
&\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \frac{\alpha}{2k}(u_i^{j+1} - u_i^{j-1}) + \frac{\beta}{3}(u_i^{j+1} + u_i^j + u_i^{j-1}) \\
&- \frac{1}{3h^2}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} + u_{i+1}^j - 2u_i^j + u_{i-1}^j + u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + \\
&\frac{(\alpha^2 + \beta)}{12}(u_i^{j+1} - 2u_i^j + u_i^{j-1}) + \frac{\alpha\beta k}{24}(u_i^{j+1} - u_i^{j-1}) - \frac{\alpha k}{24h^2}(u_{i+1}^{j+1} - u_{i+1}^{j-1} \\
&- 2u_i^{j+1} + 2u_i^{j-1} + u_{i-1}^{j+1} - u_{i-1}^{j-1}) - \frac{1}{12h^2}(u_{i+1}^{j+1} - 2u_{i+1}^j + u_{i+1}^{j-1} - 2u_i^{j+1} + 4u_i^j \\
&- 2u_i^{j-1} + u_{i-1}^{j+1} - 2u_{i-1}^j + u_{i-1}^{j-1}) = f_i^j + \frac{\alpha k^2}{12}\frac{\partial}{\partial t} f_i^j + \frac{k^2}{12}\frac{\partial^2}{\partial t^2} f_i^j + \tau_7
\end{aligned} \tag{20}$$

where:

$$\tau_7 = k^4(\frac{1}{360}\frac{\partial^6 u_i^j}{\partial t^6} + \frac{\alpha}{120}\frac{\partial^5 u_i^j}{\partial t^5} + \frac{(\alpha^2+\beta)}{144}\frac{\partial^4 u_i^j}{\partial t^4} + \frac{\alpha\beta}{72}\frac{\partial^3 u_i^j}{\partial t^3}) - \frac{h^2k^2}{144}(\frac{\partial}{\partial t}(\frac{\partial^4 u_i^j}{\partial x^4}) + \frac{\partial^2}{\partial t^2}(\frac{\partial^4 u_i^j}{\partial x^4})) - h^2(\frac{1}{36}\frac{\partial^4 u_i^j}{\partial x^4})$$

Rearranging Eq. (20), gives the recurrence relation:

$$\begin{aligned}
&-\frac{10+\alpha k}{24h^2}u_{i-1}^{j+1} + (\frac{1}{k^2} + \frac{\alpha}{2k} + \frac{5\beta}{12} + \frac{5}{6h^2} + \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{\alpha k}{12h^2})u_i^{j+1} \\
&-\frac{10+\alpha k}{24h^2}u_{i+1}^{j+1} = \frac{1}{6h^2}u_{i+1}^j + (\frac{2}{k^2} - \frac{\beta}{6} - \frac{1}{3h^2} + \frac{\alpha^2}{6})u_i^j + \frac{1}{6h^2}u_{i-1}^j + \\
&\frac{10-\alpha k}{24h^2}u_{i+1}^{j-1} + (\frac{-1}{k^2} + \frac{\alpha}{2k} - \frac{5\beta}{12} - \frac{5}{6h^2} - \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{\alpha k}{12h^2})u_i^{j-1} + \\
&\frac{10-\alpha k}{24h^2}u_{i-1}^{j-1} + f_i^j + \frac{\alpha k^2}{12}\frac{\partial}{\partial t} f_i^j + \frac{k^2}{12}\frac{\partial^2}{\partial t^2} f_i^j + T_i^j
\end{aligned} \tag{21}$$

where: $T_i^j = \tau_7$.

Eq. (21) can be re-written as:

$$Au_{i-1}^{j+1} + Bu_i^{j+1} + Au_{i+1}^{j+1} = Cu_{i-1}^j + Du_i^j + Cu_{i+1}^j + Eu_{i+1}^{j-1} + Fu_i^{j-1} + Eu_{i-1}^{j-1} + H_i^j + T_i^j \tag{22}$$

for $i = 1, 2, 3, ..., N-1$ and $j = 0, 1, 2, ..., M-1$

where: $A = -\dfrac{10+\alpha k}{24h^2}$, $B = \dfrac{1}{k^2} + \dfrac{\alpha}{2k} + \dfrac{5\beta}{12} + \dfrac{5}{6h^2} + \dfrac{\alpha^2}{12} + \dfrac{\alpha\beta k}{24} + \dfrac{\alpha k}{12h^2}$,

$C = \dfrac{1}{6h^2}$, $D = \dfrac{2}{k^2} - \dfrac{\beta}{6} - \dfrac{1}{3h^2} + \dfrac{\alpha^2}{6}$,

$E = \dfrac{10-\alpha k}{24h^2}$, $F = \dfrac{-1}{k^2} + \dfrac{\alpha}{2k} - \dfrac{5\beta}{12} - \dfrac{5}{6h^2} - \dfrac{\alpha^2}{12} + \dfrac{\alpha\beta k}{24} + \dfrac{\alpha k}{12h^2}$

and $H_i^j = f_i^j + \dfrac{\alpha k^2}{12}\dfrac{\partial}{\partial t} f_i^j + \dfrac{k^2}{12}\dfrac{\partial^2}{\partial t^2} f_i^j$

But, for $j = 0$, from Eq. (22), we get:

$$Au_{i-1}^1 + Bu_i^1 + Au_{i+1}^1 = Cu_{i+1}^0 + Du_i^0 + Cu_{i-1}^0 + Eu_{i+1}^{-1} + Fu_i^{-1} + Eu_{i-1}^{-1} + H_i^0 \qquad (23)$$

Using the initial condition given in Eq. (2) and the relations with Eq. (10) at $j = 0$ we have:

$$\frac{\partial u}{\partial t}(x,0) = \frac{u_i^1 - u_i^{-1}}{2k} = f_1(x) \qquad (24)$$

From Eq. (24), we get the value for u_{i-1}^{-1}, u_i^{-1} and u_{i+1}^{-1}, and then putting these values into Eq. (23) and then, rearranging, yields:

$$(A-E)u_{i-1}^1 + (B-F)u_i^1 + (A-E)u_{i+1}^1 = Cu_{i+1}^0 + Du_i^0 + Cu_{i-1}^0 - 2k\Big(E\frac{\partial u_{i-1}^0}{\partial t} + F\frac{\partial u_i^0}{\partial t} + E\frac{\partial u_{i+1}^0}{\partial t}\Big) + H_i^0 \qquad (25)$$

For $i = 1, 2, 3, ..., N-1$.

Hence, Eqs. (22) and (25) gives system of equations which can be solved by matrix inverse method.

To apply the matrix inverse method, we considered the schemes given in Eqs. (22) and (25) which can be re-written as a matrix vector form of:

$$M\,x^{j+1} = r^j \qquad (26)$$

where: $M = [m_{ij}]$ a square matrix of order $(N-1)\times(N-1)$, with x^{j+1} and r^j are column matrices and it can be expressed for both cases as:

Case-I: using Eq. (25), for $j = 0,\ i = 1, 2, \ldots, N-1$, we have:

$$M = \begin{bmatrix} B-F & A-E & 0 & 0 & \cdots & 0 \\ A-E & B-F & A-E & 0 & \cdots & 0 \\ 0 & A-E & B-F & A-E & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A-E & B-F & A-E \\ 0 & \ddots & \ddots & 0 & A-E & B-F \end{bmatrix}, x^1 = \begin{bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ \vdots \\ u_{N-2}^1 \\ u_{N-1}^1 \end{bmatrix} \text{ and } r^0 = \begin{bmatrix} \eta_1^0 \\ \eta_2^0 \\ \vdots \\ \vdots \\ \eta_{N-2}^0 \\ \eta_{N-1}^0 \end{bmatrix}$$

where: For $i = 1$,

$$\eta_1^0 = Cu_2^0 + Du_1^0 + Cu_0^0 - 2k(E\frac{\partial u_0^0}{\partial t} + F\frac{\partial u_1^0}{\partial t} + E\frac{\partial u_2^0}{\partial t}) + H_1^0 - (A-E)u_0^1,$$

For $i = 2, 3, \ldots, N-2$, $\eta_i^0 = Cu_{i+1}^0 + Du_i^0 + Cu_{i-1}^0 - 2k(E\frac{\partial u_{i-1}^0}{\partial t} + F\frac{\partial u_i^0}{\partial t} + E\frac{\partial u_{i+1}^0}{\partial t}) + H_i^0$

and for $i = N-1$,

$$\eta_{N-1}^0 = Cu_N^0 + Du_{N-1}^0 + Cu_{N-2}^0 - 2k(E\frac{\partial u_{N-2}^0}{\partial t} + F\frac{\partial u_{N-1}^0}{\partial t} + E\frac{\partial u_N^0}{\partial t}) + H_{N-1}^0 - (A-E)u_N^1$$

Case-II: using Eq. (22), for $j = 1, 2, \ldots M-1$ and $i = 1, 2, \ldots, N-1$, we have:

$$M = \begin{bmatrix} B & A & 0 & 0 & \ldots & 0 \\ A & B & A & 0 & \ldots & 0 \\ 0 & A & B & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A & B & A \\ 0 & \ddots & \ddots & 0 & A & B \end{bmatrix}, \quad x^{j+1} = \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ \vdots \\ u_{N-2}^{j+1} \\ u_{N-1}^{j+1} \end{bmatrix} \text{ and } r^j = \begin{bmatrix} \eta_1^j \\ \eta_2^j \\ \vdots \\ \vdots \\ \eta_{N-2}^j \\ \eta_{N-1}^j \end{bmatrix}$$

where: $\eta_1^j = Cu_0^j + Du_1^j + Cu_2^j + Eu_2^{j-1} + Fu_1^{j-1} + Eu_0^{j-1} + H_1^j - Au_0^{j+1}$, for $i = 1$

$\eta_i^j = Cu_{i-1}^j + Du_i^j + Cu_{i+1}^j + Eu_{i+1}^{j-1} + Fu_i^{j-1} + Eu_{i-1}^{j-1} + H_i^j$, for $i = 2, 3, \ldots, N-2$

$\eta_{N-1}^j = Cu_{N-2}^j + Du_{N-1}^j + Cu_N^j + Eu_N^{j-1} + Fu_{N-1}^{j-1} + Eu_{N-2}^{j-1} + H_{N-1}^j - Au_N^{j+1}$, for $i = N-1$

A square matrix $M = [m_{ij}]$ to be strictly diagonally dominant if for every row, the magnitude of the diagonal entry in a row is larger than the sum of the magnitude of all the non-diagonal entries in that row, that is: $|m_{ii}| > \sum_{j \neq i} |m_{ij}|$ for all $1 \le i \le N-1$, where m_{ij} denotes the entry in the i^{th} row and j^{th} column. And we know that if the matrix $M = [m_{ij}]$ is strictly diagonally dominant matrix, then M is invertible.

For $j = 0$, $|B-F| = \left|\frac{2}{k^2} + \frac{5\beta}{6} + \frac{5}{3h^2} + \frac{\alpha^2}{6}\right|$ and $|A-E| = \left|-\frac{5}{6h^2}\right|$,

this implies $|B-F| > |A-E|$; for $i = 1$ and $i = N-1$.

and $|B-F| = \left|\frac{2}{k^2} + \frac{5\beta}{6} + \frac{5}{3h^2} + \frac{\alpha^2}{6}\right|$ and $2|A-E| = \left|-\frac{5}{3h^2}\right|$.

Thus, $|B-F| > 2|A-E|$; for $i = 2, 3, \ldots, N-2$.

For $j = 1, 2, \ldots, M$; $|B| = \left|\frac{1}{k^2} + \frac{\alpha}{2k} + \frac{5\beta}{12} + \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{10+\alpha k}{12h^2}\right|$ and $|A| = \left|-\frac{10+\alpha k}{24h^2}\right|$

which implies $|B| > |A|$; for $i = 1$ and $i = N-1$,

$$|B| = \left|\frac{1}{k^2} + \frac{\alpha}{2k} + \frac{5\beta}{12} + \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{10+\alpha k}{12h^2}\right| \text{ and } |2A| = \left|-\frac{10+\alpha k}{12h^2}\right|,$$

which shows that $|B| > |2A|$, for $i = 2, 3, \ldots, N-2$

Thus, matrix M is strictly diagonally dominant matrix. Thus, matrix M is invertible.

Stability Analysis and Consistency of the method

The Von Neumann stability technique is applied to investigate the stability of the proposed method. Such an approach has been used by many researchers like (Rashidinia et al., 2013, Gemechis et al., 2016 and Shokofeh and Rashidinia, 2016). We assume that the solution of Eq. (4.22) at the grid point $u(x_i, t_j)$ is given by:

$$u_i^{\,j} = \zeta^j e^{ip\theta} \tag{27}$$

where $p = \sqrt{-1}$, θ is the real number and ζ is the complex number.

Now, putting Eq. (27) into the homogenous part of Eq. (22), gives:

$$A\zeta^{j+1}e^{(i-1)p\theta} + B\zeta^{j+1}e^{ip\theta} + A\zeta^{j+1}e^{(i+1)p\theta} = C\zeta^j e^{(i-1)p\theta} + D\zeta^j e^{ip\theta} + C\zeta^j e^{ip\theta} + E\zeta^{j-1}e^{(i+1)p\theta} + F\zeta^{j-1}e^{ip\theta} + E\zeta^{j-1}e^{(i-1)p\theta}$$

This implies:

$$\zeta^{j+1}e^{ip\theta}(Ae^{-p\theta} + B + Ae^{p\theta}) + \zeta^j e^{ip\theta}(-Ce^{-p\theta} - D - Ce^{p\theta}) + \zeta^{j-1}e^{ip\theta}(-Ee^{p\theta} - F - Ee^{-p\theta}) = 0$$

Since, the value of $p = \sqrt{-1}$ and $e^{\pm p\theta} = \cos\theta \pm p\sin\theta$, the above equation can be written:

$$\zeta^{j+1}e^{ip\theta}(2A\cos\theta + B) + \zeta^j e^{ip\theta}(-2C\cos\theta - D) + \zeta^{j-1}e^{ip\theta}(-2E\cos\theta - F) = 0 \tag{28}$$

Dividing both sides Eq. (28) by $\zeta^{j-1}e^{ip\theta}$, we obtain:

$$\zeta^2(2A\cos\theta + B) + \zeta(-2C\cos\theta - D) + (-2E\cos\theta - F) = 0 \tag{29}$$

Since, $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$, Eq. (29) is written in the form of:

$$P\zeta^2 + Q\zeta + R = 0 \tag{30}$$

where:

$$P = 2A + B - 4A\sin^2(\frac{\theta}{2}), \qquad Q = -2C - D + 4C\sin^2(\frac{\theta}{2}) \text{ and}$$

$$R = -2E - F + 4E\sin^2(\frac{\theta}{2})$$

Using Routh-Hurwitz criterion and the transformation $\zeta = \dfrac{1+z}{1-z}$ into Eq. (30), we have:

$$P\left(\frac{1+z}{1-z}\right)^2 + Q\left(\frac{1+z}{1-z}\right) + R = 0 \text{, which is reduced to:}$$

$$(P - Q + R)z^2 + 2(P - R)z + (P + Q + R) = 0 \tag{31}$$

The necessary and sufficient condition for $|\zeta| < 1$, from Eq. (31) is:

$$P - Q + R > 0, \quad P - R > 0 \text{ and } P + Q + R > 0. \tag{32}$$

From Eq. (30), we have:

$$\begin{aligned} P - Q + R &= \frac{8}{3h^2}\sin^2\left(\frac{\theta}{2}\right) + \frac{\alpha^2 + 2\beta}{3} + \frac{4}{k^2} \\ P - R &= \alpha\left(\frac{k}{3h^2}\sin^2\left(\frac{\theta}{2}\right) + \frac{1}{k} + \frac{\beta k}{12}\right) \quad \text{and} \\ P + Q + R &= \frac{4}{h^2}\sin^2\left(\frac{\theta}{2}\right) + \beta \end{aligned} \tag{33}$$

Since, α and β are positive real constants and from Eq. (33), it is clearly observed that the inequality of Eq. (32) are satisfied for any values of θ. Thus, the proposed method is stable for the second order one dimensional linear hyperbolic telegraph equation.

To show the consistency of the method, expand Eq. (4) in Taylor series and replace the derivatives involving *x* and *t* for the relation:

$$\frac{\partial^2 u_i^j}{\partial t^2} + \alpha\frac{\partial u_i^j}{\partial t} + \beta u_i^j = \frac{\partial^2 u_i^j}{\partial x^2} + f_i^j$$

and then we drive a local truncation error. The truncation errors of the proposed method, using Eqs. (9) – (13) given for the one dimensional linear hyperbolic telegraph equation is:

$$\begin{aligned} T_i^j = {}& k^4\left(\frac{1}{360}\frac{\partial^6 u_i^j}{\partial t^6} + \frac{\alpha}{120}\frac{\partial^5 u_i^j}{\partial t^5} + \frac{(\alpha^2 + \beta)}{144}\frac{\partial^4 u_i^j}{\partial t^4} + \frac{\alpha\beta}{72}\frac{\partial^3 u_i^j}{\partial t^3}\right) \\ & - \frac{h^2k^2}{144}\left(\frac{\partial}{\partial t}\left(\frac{\partial^4 u_i^j}{\partial x^4}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 u_i^j}{\partial x^4}\right)\right) - h^2\left(\frac{1}{36}\frac{\partial^4 u_i^j}{\partial x^4}\right) \end{aligned} \tag{34}$$

Thus, the right hand side of Eq. (34) vanishes as $h \to 0$ and $k \to 0$ and implies $T \to 0$. Hence, the scheme is consistent with the order of $O(k^4 + h^2k^2 + h^2)$. Therefore, the scheme developed in Eq. (22), is convergent.

Numerical Examples and Results

To demonstrate the applicability of the method, two model examples of the one - dimensional linear hyperbolic telegraph equations have been considered.

Example 1: Consider the telegraphic equation of the form:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2e^{-t}$$

subject to the initial conditions: $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$ for $0 \le x \le 1$

and the boundary conditions: $u(0,t) = 0$, $u(1,t) = 0$, $t \ge 0$

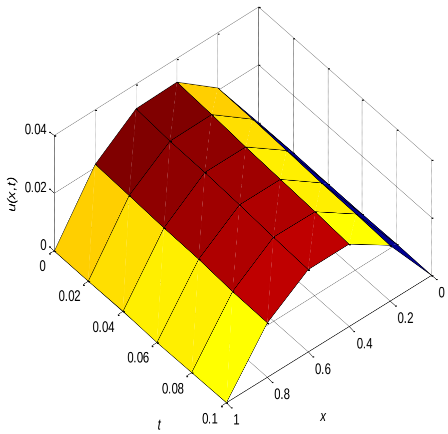
The exact solution is given by $u(x,t) = (x - x^2)t^2e^{-t}$

Table 1: Point wise absolute and root mean square errors for Example 1 at $t = 0.01$

x	**Present method**	**Odejide and Binuyo, (2014)**
0.00	0.0000	0.0000
0.25	1.2454e-09	3.6735633e-07
0.50	1.6758e-09	4.8980846e-07
0.75	1.2454e-09	3.6735635e-07
1.00	0.0000	0.0000
RMS	2.6832e-09	3.193160467e-07

Table 2: Point wise and maximum absolute errors for Example 1, in the region $(x,t) \in [0,1]\times[0,1]$

x_i	t_i	$h = k = 0.25$	$h = k = 0.125$	$h = k = 0.0625$
0.25	0.25	5.0503e-04	9.8387e-05	2.0727e-05
	0.5	8.9854e-04	3.1433e-04	7.9327e-05
	0.75	1.8188e-03	4.8147e-04	1.2420e-04
0.5	0.25	9.5221e-04	2.5029e-04	6.5151e-05
	0.5	7.7531e-04	3.2036e-04	9.5877e-05
	0.75	2.4344e-03	8.1746e-04	2.1688e-04
Max. Abs. errors		2.4344e-03	8.1746e-04	2.1688e-04



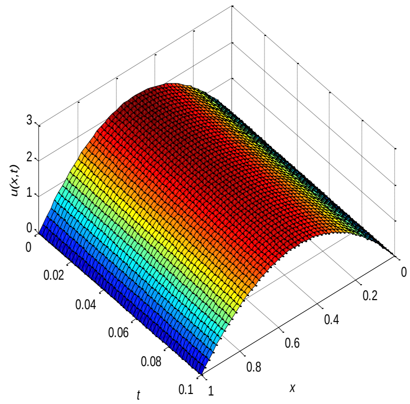


Figure 1: The physical behavior of Example 1 at different mesh sizes.

Example 2: Consider the telegraphic equation of the form:

$$\frac{\partial^2 u}{\partial t^2}+20\frac{\partial u}{\partial t}+25u=\frac{\partial^2 u}{\partial x^2}+(6t+60t^2)(x^2(1-x)^2)-t^3(12x^2-12x+2)$$

subject to the conditions: $\begin{cases} u(x,0)=0 \\ u_t(x,0)=0 \\ u(0,t)=0=u(1,t) \end{cases}$; $\quad 0\le x\le 1,$ and $t>0$

The exact solution is given by $u(x,t)=t^3x^2(1-x)^2$

Table 3: Pointwise, maximum absolute and root mean square errors for Example 2

(x_i, t_j)	$h=k=0.2$	$h=k=0.1$	$h=k=0.05$
(0.2, 0.2)	5.4350e-05	1.2885e-05	3.9343e-06
(0.4, 0.4)	4.2612e-04	1.1950e-04	3.2055e-05
(0.6, 0.6)	8.4846e-04	2.3539e-04	6.1170e-05
(0.8, 0.8)	3.0432e-04	7.8122e-05	1.9923e-05
Max. Absolute errors	1.6476e-03	4.9860e-04	1.2690e-04
RMS	1.5581e-03	6.7579e-04	2.5208e-04

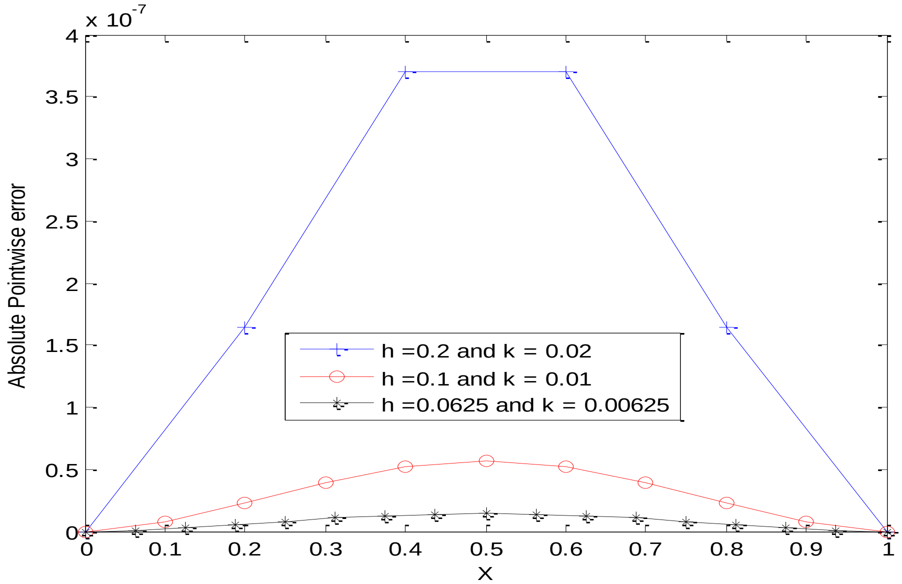


Figure 2: Absolute pointwise errors decreases as the number of mesh sizes decreases for Example 2.

DISCUSSION AND CONCLUSION

In this paper, Crank Nicholson and fourth order stable central finite difference methods are used to obtain the scheme for solving the second order one-dimensional linear hyperbolic telegraph equation. First, the given domain is discritized and the derivatives of the partial differential equation are replaced by finite difference approximations and then, transformed to system of equations which can be solved by matrix inverse method. The stability and consistency of the method is well established. To validate the applicability of the method, two model examples have been considered and solved at different mesh sizes of h and k.

As it can be observed from the numerical results presented in Table 1, the present method approximates the exact solution very well. From Tables (2) and (3) as the values of h and k decreases, the accuracy of the method increases. Figure 1 shows the physical behavior of telegraph equation for the solution of Example 1 and also, Figure 2 shows as the values of mesh sizes decrease, the pointwise absolute error also decreases. Moreover, results obtained by presented method is compared with the results of Odejide and Binuyo (2014), and it shows, the obtained result is more accurate.

Therefore, the present scheme that obtained from the finite difference methods is more accurate and convergent method for solving the second order one-dimensional linear hyperbolic telegraph equation.

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