ORIGINAL ARTICLE

Sixth Order Stable Central Difference Method for Self-Adjoint Singularly Perturbed Two-Point Boundary Value Problems

Yitbarek Zeslassie, Gemechis File and Tesfaye Aga

Abstract

A numerical method based on finite difference scheme with uniform mesh is presented for solving self-adjoint singularly perturbed two-point boundary value problems. First, the original problem is reduced to its conventional form and then, the reduced problem is solved by using sixth order stable central difference method. To demonstrate the applicability of the method, numerical illustrations have been given. Graphs are also depicted in support of the numerical results. The stability and convergence properties of the method have been examined. The method approximates the exact solution very well and gives more accurate solutions than some methods found in the literature.

***Key Words*:** Singular perturbation, self-adjoint and stable central difference method

INTRODUCTION

Self-adjoint singularly perturbed differential equation is a differential equation that has the same solution as its adjoint equation in which the highest order derivative is multiplied by a small positive parameter (Delkhon and Delkhosh, 2012; and Miller et al., 1996). Problems related to self-adjoint singularly perturbed differential equations arise frequently in diversified fields of applied mathematics and engineering, for instance, fluid mechanics, elasticity, hydrodynamics, quantum mechanics, plasticity, chemical-reaction theory, aerodynamics, plasma dynamics, rarefied-gas dynamics,

Department of Mathematics, Jimma University, Jimma

oceanography, meteorology, modeling of semiconductor devices, diffraction theory, reaction-diffusion processes and many other allied area (Kadalbajoo and Kumar, 2008). As a result, studying self-adjoint singular perturbation problems became the part of contemporary mathematical circles. In connection to this, there are some numerical methods suggested by various authors for solving self-adjoint singular perturbation problems. Namely, initial value technique (Mishra et al., 2009), quintic spline method (Rashidinia et al., 2010), non-polynomial spline functions method (Tirmizi et al., 2008), difference scheme using cubic spline (Rashidinia et al., 2005), finite difference method with variable mesh (Kadalbajoo and Kumar, 2010), fitted mesh B-spline collocation method (Kadalbajoo and Aggrwal, 2005) etc. Further, Niijima, (1980a, 1980b) gave uniformly convergent second order accurate difference schemes, whereas Miller, (1979) gave sufficient conditions for the uniform first order convergence of a general three point difference scheme. However, the order of convergence of these methods does not exceed quadratic convergence. Moreover, classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory solutions and this is very costly, and time consuming method (Farrel et al., 2000).

In this paper, we have presented a numerical method which is more accurate, which gives good result for reasonable number of mesh points, stable and easily adaptable for computer use.

Description of the Method

Consider the following self-adjoint singularly perturbed equation of the form:

$$-\varepsilon(p(x)y')'+q(x)y(x)=f(x); \quad 0<x<1 \tag{1}$$

with the Dirichlet boundary conditions,

$$y(0)=\alpha,\ y(1)=\beta \tag{2}$$

where $\varepsilon\ (0<\varepsilon<<1)$ is a small parameter; $\alpha,\ \beta$ are given constants and $p(x)$, $q(x)$ and $f(x)$ are assumed to be sufficiently continuous differentiable functions with $p(x)\neq 0$. To describe the method, we divide the interval $[0,1]$ into N equal subintervals of mesh length h. Let $0=x_0,\ x_1,\ x_2,\ ...,\ x_N=1$ be the mesh points. Then, we have $x_i=x_0+ih$, $i=0,1,2,....,N$

Using integration by part, dividing both side of Eq. (1) by $\varepsilon p(x)$ and evaluating at the point x_i, we obtain:

$$-y''(x_i)+a(x_i)y'(x_i)+b(x_i)y(x_i)=c(x_i) \quad (3)$$

where $a(x_i)=\dfrac{-p'(x_i)}{\varepsilon p(x_i)}$, $b(x_i)=\dfrac{q(x_i)}{\varepsilon p(x_i)}$ and $c(x_i)=\dfrac{f(x_i)}{\varepsilon p(x_i)}$

For the sake of simplicity, let us denote the values of the following terms at nodal points as: $a(x_i)=a_i$, $b(x_i)=b_i$, $c(x_i)=c_i$, $y'(x_i)=y_i'$, $y''(x_i)=y_i''$ an $y^{(n)}(x_i)=y_i^{(n)}$

Thus, Eq. (3) can be re-written as:

$$-y_i''+a_iy_i'+b_iy_i=c_i \quad (4)$$

Assuming that $y(x)$ is continuously differentiable in the interval [0 1] and applying Taylor's series expansion for y_i'' and y_i', we obtain:

$$\begin{aligned} y_{i+1}&=y_i+hy_i'+\frac{h^2}{2!}y_i''+\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}+\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}+ \\ &\quad \frac{h^7}{7!}y_i^{(7)}+\frac{h^8}{8!}y_i^{(8)}+O(h^9) \end{aligned} \quad (5)$$

$$\begin{aligned} y_{i-1}&=y_i-hy_i'+\frac{h^2}{2!}y_i''-\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}-\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}- \\ &\quad \frac{h^7}{7!}y_i^{(7)}+\frac{h^8}{8!}y_i^{(8)}+O(h^9) \end{aligned} \quad (6)$$

Then, from Eqs. (5) and (6), we obtain:

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h}-\frac{h^2}{6}y_i'''-\frac{h^4y_i^{(5)}}{120}+T_1 \quad (7)$$

$$y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}-\frac{h^2}{12}y_i^{(4)}-\frac{h^4}{360}y_i^{(6)}+T_2 \quad (8)$$

where $T_1=-\dfrac{h^6y_i^{(7)}}{5040}$ and $T_2=-\dfrac{h^6}{20160}y_i^{(8)}$

Substituting Eqs. (7) and (8) into Eq. (4) for the derivatives, we obtain:

$$\begin{aligned} &\left(-\frac{1}{h^2}-\frac{a_i}{2h}\right)y_{i-1}+\left(\frac{2}{h^2}+b_i\right)y_i+\left(\frac{a_i}{2h}-\frac{1}{h^2}\right)y_{i+1}-\frac{a_ih^2}{6}y_i'''+ \\ &\quad \frac{h^2}{12}y_i^{(4)}-\frac{a_ih^4}{120}y_i^{(5)}+\frac{h^4}{360}y_i^{(6)}=c_i+T_3 \end{aligned} \quad (9)$$

where: $T_3=T_2-a_iT_1$

Differentiating both sides of Eq. (4) successively and rearranging, we obtain:

$$y_i'''=a_iy_i''+\left(a_i'+b_i\right)y_i'+b_i'y_i-c_i' \quad (10)$$

$$
\begin{aligned}
y_i^{(4)} = &\left(a_i^2 + 3a_i' + b_i\right) y_i'' + \left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) y_i' + \\
&\left(a_i b_i' + b_i''\right) y_i - a_i c_i' - c''
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
y_i^{(5)} = &\left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b_i' + 3a_i'^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b''\right) y_i' + \\
&(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b') y'' + \\
&\left(a_i^2 b_i' + a_i b_i'' + 3a_i' b_i' + b_i b_i' + b_i'''\right) y_i - \left(a_i^2 + 3a_i' + b_i\right) c_i' - a_i c_i'' - c_i'''
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
y_i^{(6)} = &(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + \\
&b_i^2 + 4a_i''' + 6b'') y_i'' + \left(a_i^3 a_i' + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' + 7a_i a_i'^2 + \right. \\
&9a_i b_i a_i' + 2a_i b_i^2 + a_i a_i''' + 3a_i b_i'' + 10a_i' a_i'' + 12a_i' b_i' + 7b_i a'' + \\
&\left. 6b_i b_i' + a_i^{(4)} + 4b_i''' \right) y_i' + (a_i^3 b_i' + a_i^2 b_i'' + 7a_i a_i' b_i' + 2a_i b_i b_i' + \\
&a_i b_i''' + 4a_i' b_i'' + b_i b_i'' + 6b_i' a_i'' + 4b_i'^2 + b_i^{(4)}) y_i - (a_i^3 + 7a_i a_i' + \\
&2a_i b_i + 6a_i'' + 4b_i') c_i' - (a_i^2 + 4a_i' + b_i) c_i'' - a_i c_i''' - c_i^{(4)}
\end{aligned} \tag{13}
$$

Substituting Eqs. (10) - (13) into Eq. (9) and rearranging, we obtain:

$$(-\frac{1}{h^2}-\frac{a_i}{2h})y_{i-1}+(\frac{2}{h^2}+b_i-\frac{a_ib_i'h^2}{6}+\frac{h^2}{12}\left(a_ib_i'+b_i''\right)-\frac{a_ih^4}{120}(a_i^2b_i'+$$
$$a_ib_i''+3a_i'b_i'+b_ib_i'+b_i''')+\frac{h^4}{360}(a_i^3b_i'+a_i^2b_i''+7a_ia_i'b_i'+2a_ib_ib_i'+$$
$$a_ib_i'''+4a_i'b_i''+b_ib_i''+6b_i'a_i''+4b_i'^2+b_i^{(4)})y_i+(\frac{a_i}{2h}-\frac{1}{h^2})y_{i+1}+$$
$$((-\frac{a_ih^2}{6}\left(a_i'+b_i\right)+\frac{h^2}{12}\left(a_ia_i'+a_ib_i+a_i''+2b_i'\right)-\frac{a_ih^4}{120}(a_i^2a_i'+a_i^2b_i+$$
$$a_ia_i''+2a_ib_i'+3a_i'^2+4a_i'b_i+b_i^2+a_i'''+3b_i'')+\frac{h^4}{360}(a_i^3a_i'+a_i^3b_i+$$
$$a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+a_ia_i'''+3a_ib''+10a_i'a_i''+$$
$$12a_i'b_i'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i'''))y_i'+(-\frac{a_i^2h^2}{6}+\frac{h^2}{12}(a_i^2+2a_i'+$$
$$b_i)-\frac{a_ih^4}{120}(a_i^3+5a_ia_i'+2a_ib_i+3a_i''+3b_i'\ )+\frac{h^4}{360}(a_i^4+9a_i^2a_i'+$$
$$3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''))y_i''\ =\ c_i+$$
$$(-\frac{a_ih^2}{12}-\frac{a_ih^4}{120}\left(a_i^2+3a_i'+b_i\right)+\frac{h^4}{360}(a_i^3+7a_ia_i'+2a_ib_i+6a_i''+ \tag{14}$$
$$4b_i'))c_i'+(\frac{h^2}{12}-\frac{a_i^2h^4}{120}+\frac{h^4}{360}\left(a_i^2+4a_i'+b_i\right))c_i''+(-\frac{a_ih^4}{120}+$$
$$\frac{a_ih^4}{360})c_i'''+\frac{h^4}{360}c_i^{(4)}+T_3$$

Further, the Taylor series second order approximations of first and second order derivatives of y_i are given as:

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h} \tag{15}$$
$$y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (14), we obtain:

$$
\begin{aligned}
&(-\frac{1}{h^2}-\frac{a_i}{2h}+\frac{a_ih}{12}\left(a_i'+b_i\right)-\frac{h}{24}\left(a_ia_i'+a_ib_i+a_i''+2b_i'\right)+\frac{a_ih^3}{240}(a_i^2a_i'+\\
&a_i^2b_i+a_ia_i''+2a_ib_i'+3a_i^2+4a_i'b_i+b_i^2+a_i'''+3b_i'')-\frac{h^3}{720}(a_i^3a_i'+a_i^3b_i+\\
&a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+a_ia_i'''+3a_ib_i''+10a_i'a_i''+\\
&12a_i'b'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i''')-\frac{a_i^2}{6}+\frac{1}{12}\left(a_i^2+2a_i'+b_i\right)-\\
&\frac{a_ih^2}{120}(a_i^3+5a_ia_i'+2a_ib+3a_i''+3b')\,\frac{h^2}{360}(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+\\
&7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''))y_{i-1}+(b_i+\frac{2}{h^2}-\frac{a_ib_i'h^2}{12}+\\
&\frac{b_i''h^2}{12}-\frac{a_ih^4}{120}(a_i^2b_i'+3a_i'b_i'+a_ib_i''+b_ib_i'+b_i''')+\frac{h^4}{360}(a_i^3b_i'+a_i^2b_i''+\\
&7a_ia_i'b_i'+2a_ib_ib_i'+a_ib_i'''+4a_i'b_i''+b_ib_i''+6b_i'a_i''+4b_i'^2+b_i^{(4)})+\\
&\frac{a_i^2}{3}-\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{60}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''+3b'\right)-\\
&\frac{h^2}{180}(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+\\
&4a_i'''+6b_i''))y_i+(-\frac{1}{h^2}+\frac{a_i}{2h}-\frac{a_ih}{12}\left(a_i'+b_i\right)+\frac{h}{24}(a_ia_i'+a_ib_i+a_i''+\\
&2b_i')-\frac{a_ih^3}{240}(a_i^2a_i'+a_i^2b_i+a_ia_i''+2a_ib_i'+3a_i^2+4a_i'b_i+b_i^2+a_i'''+\\
&3b_i'')+\frac{h^3}{720}(a_i^3a_i'+a_i^3b_i+a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+\\
&a_ia_i'''+3a_ib_i''+10a_i'a_i''+12a_i'b_i'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i''')-\frac{a_i^2}{6}+\\
&\frac{1}{12}\left(a_i^2+2a_i'+b_i\right)-\frac{a_ih^2}{120}(a_i^3+5a_ia_i'+2a_ib+3a_i''+3b_i')+\\
&\frac{h^2}{360}(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+\\
&6b_i''))y_{i+1}=c_i+(-\frac{a_ih^2}{12}-\frac{a_ih^4}{120}\left(a_i^2+3a_i'+b_i\right)+\frac{h^4}{360}(a_i^3+7a_ia_i'+
\end{aligned}
\tag{17}
$$

$$+2a_ib_i+6a_i''+4b_i'))c_i'+(\frac{h^2}{12}-\frac{a_i^2h^4}{120}+\frac{h^4}{360}(a_i^2+4a_i'+$$

$$b_i))c_i''+(-\frac{a_ih^4}{120}+\frac{a_ih^4}{360})c_i'''+\frac{h^4}{360}c_i^{(4)}+T_3 \quad .$$

Eq. (17) can be written as the three- term recurrence relation of the form:

$$-E_iy_{i-1}+F_iy_i-G_iy_{i+1}=H_i, \qquad i=1,\,2,\,3,\,\ldots,\,N-1 \tag{18}$$

where:

$$E_i=\frac{1}{h^2}+\frac{a_i}{2h}-\frac{a_ih}{12}\left(a_i'+b_i\right)+\frac{h}{24}\left(a_ia_i'+a_ib_i+a_i''+2b_i'\right)-$$

$$\frac{a_ih^3}{240}\left(a_i^2a_i'+a_i^2b_i+a_ia_i''+2a_ib_i'+3a_i^2+4a_i'b_i+b_i^2+a_i'''+3b_i''\right)+$$

$$\frac{h^3}{720}\left(a_i^3a_i'+a_i^3b_i+a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+\right.$$

$$\left.a_ia_i'''+3a_ib''+10a_i'a_i''+12a_i'b_i'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i'''\right)+$$

$$\frac{a_i^2}{6}-\frac{1}{12}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{120}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''+3b_i'\right)-$$

$$\frac{h^2}{360}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\right)$$

$$F_i=b_i+\frac{2}{h^2}-\frac{a_ib_i'h^2}{12}+\frac{b_i''h^2}{12}-\frac{a_ih^4}{120}\left(a_i^2b_i'+3a_i'b_i'+a_ib_i''+b_ib_i'+b_i'''\right)+\frac{a_i^2}{3}-$$

$$\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{h^4}{360}(a_i^3b_i'+a_i^2b_i''+7a_ia_i'b_i'+2a_ib_ib_i'+a_i\,b_i'''+4a_i'b_i''+$$

$$b_ib_i''+6b_i'a_i''+4b_i'^2+b_i^{(4)})+\frac{a_ih^2}{60}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''+3b_i'\right)-$$

$$\frac{h^2}{180}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\right)$$

$$G_i = \frac{1}{h^2} - \frac{a_i}{2h} + \frac{a_i h}{12}\left(a_i' + b_i\right) - \frac{h}{24}\left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) +$$

$$\frac{a_i h^3}{240}\left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b' + 3a_i^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b''\right) -$$

$$\frac{h^3}{720}\left(a_i^3 a_i + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' + 7a_i a_i'^2 + 9a_i b_i a_i' + 2a_i b_i^2 + \right.$$

$$\left. a_i a_i''' + 3a_i b_i'' + 10a_i' a_i'' + 12a_i' b_i' + 7b_i a_i'' + 6b_i b_i' + a_i^{(4)} + 4b''' \right) + \frac{a_i^2}{6} -$$

$$\frac{1}{12}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{120}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i'\right) -$$

$$\frac{h^2}{360}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right)$$

$$H_i = c_i + \Big(-\frac{a_i h^2}{12} - \frac{a_i h^4}{120}\left(a_i^2 + 3a_i' + b_i\right) + \frac{h^4}{360}\big(a_i^3 + 7a_i a_i' +$$

$$2a_i b_i + 6a_i'' + 4b_i'\big)\Big)c_i' + \Big(\frac{h^2}{12} - \frac{a_i^2 h^4}{120} + \frac{h^4}{360}\left(a_i^2 + 4a_i' + b_i\right)c_i'' +$$

$$\Big(-\frac{a_i h^4}{120} + \frac{a_i h^4}{360}\Big)c_i''' + \frac{h^4}{360}c_i^{(4)} + T_3$$

Eq. (18) gives us the tri-diagonal system and we solved the system by using the well-known algorithm known as Thomas Algorithm (Fasika et al., 2016).

Stability and Convergence Analysis

Definition (Keller, 1968): The linear difference operator L_h is stable if, for sufficiently small h, there exists a constant k, independent of h, such that $\left|v_j\right| \le k\left\{ \max\left(|v_0|,|v_N| \right) + \max_{1\le i\le N-1}\left|L_h v_i\right| \right\}, \quad j = 0,1,2,\dots,N$ for any mesh function $\left\{ v_j \right\}_{j=0}^{N}$

Theorem: under the assumption $b(x_i) \equiv \theta > 0$ for positive constant θ,

$$(-a_i^2+2a_i'+b_i+4a_i^2a_i'+a_i^2b_i+6a_i'a_i''+4a_ib_i'+8a_i'^2+ 6a_i'b_i+b_i^2+4a_i'''+6b_i'')>0$$ and

$$h<\min\left\{\frac{2\begin{pmatrix}-a_i^2+2a_i'+b_i+4a_i^2a_i'+a_i^2b_i+6a_i'a_i''+4a_ib_i'+\\ 8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\end{pmatrix}}{\begin{vmatrix}-a_ia_i'-a_ib_i+a_i''+2b_i'+4a_ia_i'^2+5a_ia_i'b_i+a_ib_i^2+\\ 10a_ia_i''+12a_i'b_i'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b'''\end{vmatrix}}\right\}$$

The linear difference operator of Eq. (18) is stable for $k=\max\left\{1,\ \frac{1}{\theta}\right\}$

Proof:

Let $L_h(.)$ denoted the difference operator on left side of Eq. (18) and w_i be any mesh function satisfying:

$$L_h(w_i)=H_i \tag{19}$$

If the $\max|w_i|$ occurs for $i=0$ or $i=N$ then definition holds trivially. Since $k\geq 1$ so assume that $\max|w_i|$ occurs for $i=1,2,\ldots N-1$ under the given assumptions

$$E_i>0\ ,G_i>0\ ,F_i>E_i+G_i \quad \text{and} \quad |E_i|\leq|G_i|$$

This implies the tri-diagonal system in Eq. (18) is diagonally dominant and its solution exists and is unique. Then by rearranging the difference Eq. (18) and using the non-negativity of the coefficients, we have:

$$F_i|w_i|\leq E_i|w_{i-1}|+G_i|w_{i+1}|+|H_i|$$

$$\Rightarrow \quad F_i|w_i|\leq E_i|w_{i-1}|+G_i|w_{i+1}|+|L_h\,w_i| \tag{20}$$

Since $b(x_i)\equiv\theta$ is a constant and by assumption $b'(x_i)=0$.

Thus, from Eq. (18) we have:

$$F_i=\frac{2}{h^2}+\frac{a_i^2}{3}-\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{60}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''\right)$$
$$-\frac{h^2}{180}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''\right)+\theta$$

Now, using the fact that,

$$E_i+G_i=\frac{2}{h^2}+\frac{a_i^2}{3}-\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{60}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''\right)$$
$$-\frac{h^2}{180}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''\ \right)$$

and Eq. (20), we get:

$$\begin{aligned}&[\frac{2}{h^2}+\frac{a_i^2}{3}-\frac{1}{6}(a_i^2+2a_i'+b_i)+\frac{a_ih^2}{60}(a_i^3+5a_ia_i'+2a_ib_i+3a_i'')-\\&\frac{h^2}{180}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''\right)+\\&\theta]\left|w_i\right|\le E_i\left|w_{i-1}\right|+G_i\left|w_{i+1}\right|+\left|L_hw_i\right|\\&\qquad\le\left(E_i+G_i\right)\max_{1\le k\le N-1}\left|w_k\right|+\max_{1\le k\le N-1}\left|L_{hK}\right|\end{aligned}\tag{21}$$

Since the inequality in Eq. (21) holds for every i, it follows that:

$$\begin{aligned}&[\frac{2}{h^2}+\frac{a_i^2}{3}-\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{60}\left(a_i^3+5a_ia_i'+2a_ib_i+3a_i''\right)-\frac{h^2}{180}(a_i^4+\\&9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+8a_i'^2+6a_i'b_i+b_i^2+4a_i''')+\theta]\max_{1\le k\le N-1}\left|w_i\right|\le[\frac{2}{h^2}+\frac{a_i^2}{3}-\\&\frac{1}{6}(a_i^2+2a_i'+b_i)+\frac{a_ih^2}{60}(a_i^3+5a_ia_i'+2a_ib_i+3a_i'')-\frac{h^2}{180}(a_i^4+9a_i^2a_i'+\\&3a_i^2b_i+9a_ia_i''+8a_i'^2+6a_i'b_i+b_i^2+4a_i''')]\max_{1\le k\le N-1}\left|w_k\right|+\max_{1\le k\le N-1}\left|L_hw_k\right|\end{aligned}$$

This implies $\theta\max_{1\le i\le N-1}\left|w_i\right|\le\max_{1\le k\le N-1}\left|L_hw_K\right|$

Hence, $\max_{1\le k\le N-1}\left|w_i\right|\le\frac{1}{\theta}\max_{1\le k\le N-1}\left|L_hw_k\right|\le\frac{1}{\theta}\left\{\max\left(\left|w_0\right|,\left|w_N\right|\right)+\max_{1\le k\le N-1}\left|L_hw_k\right|\right\}$

Therefore, $\left|w_i\right|\le k\left\{\max\left(\left|w_0\right|,\left|w_N\right|\right)+\max_{1\le k\le N-1}\left|L_hw_k\right|\right\}$ where $k=\frac{1}{\theta}$

Hence, L_h is stable and this implies that the solution of the system of the difference Eq. (18) are uniformly bounded, independent of mesh size h and the parameter ε. Hence, Eq. (18) is stable by definition of (Keller, 1968).

Corollary: Under the conditions for the theorem, the error $e_i=y(x_i)-y_i$ between the solutions of $y(x)$ of the continuous problem and y_i of the discrete problem, with boundary conditions satisfies the estimate

$$\left|e_i\right|=k\max_{1\le i\le N-1}\left|T_i\right|\tag{22}$$

where $T_i=\frac{h^6}{20160}\left|y_i^{(8)}\right|+\frac{a_ih^6}{5040}\left|y_i^{(7)}\right|$ is the truncation error and

$$T_i \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{h^6}{20160}\left|y_i^{(8)}\right| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{a_i h^6}{5040}\left| y_i^{(7)}\right| \right\}$$

Proof: under the given conditions it is clear that the error e_i satisfies

$$L_h(e_i) = L_h(y(x_i) - y_i) = T_i\ ,\ i = 1,2,...N-1$$

Then from above theorem, the stability of L_h implies that

$$\left| y(x_i) - y_i \right| = \left| e_i \right| \le k \max_{1\le i\le N-1} \left| T_i \right| \tag{23}$$

Hence the estimate in Eq. (22) establishes the convergence of the scheme for the fixed value of the perturbation parameter ε .

Numerical Examples and Results

To validate the applicability of the method, three model examples of second order self-adjoint singular perturbation problems have been considered.

Example 1. Consider the following self-adjoint singular perturbation problem (Kadalbajoo and Kumar, 2010):

$$-\varepsilon y'' + \frac{4}{(x+1)^4}\left(1+\sqrt{\varepsilon}(x+1)\right)y = f(x), \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = 2$, $y(1) = -1$ and $f(x)$ is chosen, such that the exact solution is given by $y(x) = -\cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\left(\exp\left(\frac{-2x}{\sqrt{\varepsilon}(x+1)}\right) - \exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\right)}{1-\exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)}$

The numerical solutions in terms of maximum absolute errors are given in Table 1 for different values of the perturbation parameter ε and N, and its graph is also given in Figure 1.

Table 1: Maximum Absolute Errors for Example1

N	$\varepsilon = (1/N)^{0.25}$	$\varepsilon = (1/N)^{0.5}$	$\varepsilon = (1/N)^{0.75}$	$\varepsilon = (1/N)^{1.0}$
Our Method				
16	2.9718E-04	4.9658E-04	8.9268E-04	1.7181E-03
32	2.0905E-05	4.1607E-05	9.0798E-05	2.3653E-04
64	1.4884E-06	3.4999E-06	9.8228E-06	3.9036E-05
128	1.0650E-07	3.0026E-07	1.1659E-06	7.4775E-06
256	7.6403E-09	2.6424E-08	1.5321E-07	1.5612E-06
Kadalbajoo and Kumar, 2010				
16	2.0E-02	1.7E-02	1.5E-02	1.4E-02
32	4.7E-03	4.0E-03	3.4E-03	4.1E-03
64	1.1E-03	9.1E-04	9.3E-04	1.1E-03
128	2.6E-04	2.0E-04	2.4E-04	3.2E-04
256	6.1E-05	5.0E-05	6.4E-05	9.6E-05

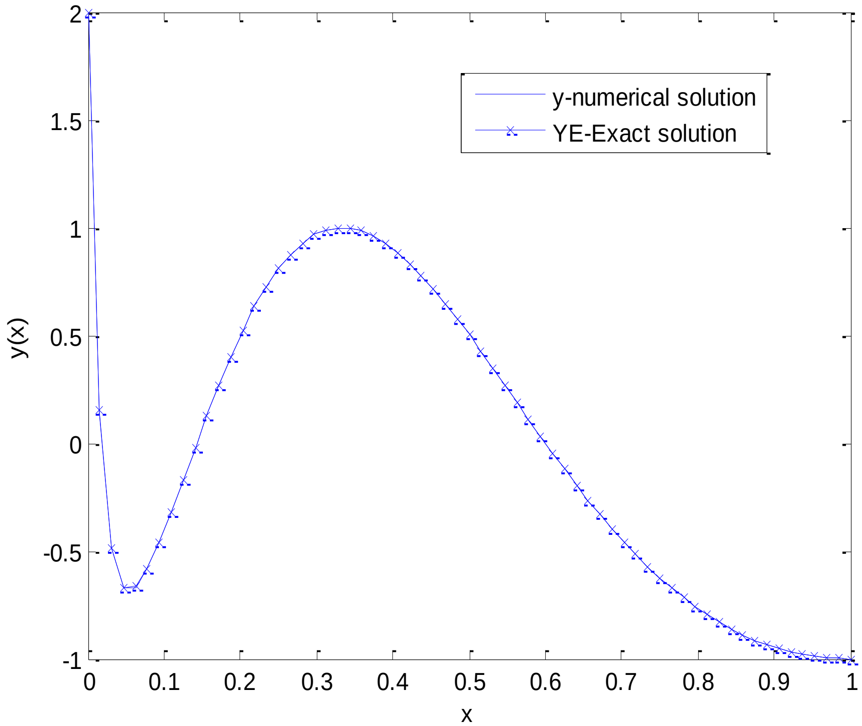

Figure 1: Numerical Solution of Example 1 for $\varepsilon = 10^{-3}$ and $N = 64$

Example 2. Consider the following self-adjoint singular perturbation problem (Kadalbajoo and Kumar, 2008)

$$-\varepsilon((1+x^2)y')' + \left(\frac{\cos x}{(3-x)^3}\right)y = 4(3x^2 - 3x + 1)((x-0.5)^2 + 2), \ \ 0 \le x \le 1$$

with $y(0) = -1, \quad y(1) = 0$

The exact solution for this problem is not available. The numerical results are obtained by using the double mesh principle (Doolan *et al.*, 1980) and tabulated in terms of maximum absolute errors in Table 2 and its graph is given in Figures 2.

Table 2: Maximum Absolute Errors for Example 2

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Our Method						
2^{-2}	5.5954e-05	1.3967e-05	3.4903e-06	8.7250e-07	2.1812e-07	5.4530e-08
2^{-4}	1.9765e-04	4.9382e-05	1.2340e-05	3.0851e-06	7.7125e-07	1.9281e-07
2^{-6}	6.1739e-04	1.5402e-04	3.8510e-05	9.6262e-06	2.4065e-06	6.0163e-07
2^{-8}	1.2821e-03	3.2272e-04	8.0755e-05	2.0193e-05	5.0488e-06	1.2622e-06
2^{-10}	1.4094e-03	3.9358e-04	1.0097e-04	2.5404e-05	6.3616e-06	1.5911e-06
2^{-12}	3.1577e-03	3.2669e-04	9.0865e-05	2.3923e-05	6.1999e-06	1.5637e-06
2^{-14}	8.1928e-02	5.0797e-03	2.6048e-04	2.6985e-05	6.9159e-06	1.7422e-06
Kadalbajoo and Kumar, 2008						
2^{-2}	1.310e-03	3.280e-04	8.210e-05	2.050e-05	5.130e-06	1.280e-06
2^{-4}	4.930e-03	1.230e-03	3.080e-04	7.710e-05	1.930e-05	4.820e-06
2^{-6}	1.600e-02	4.000e-03	1.000e-03	2.500e-04	6.260e-05	1.560e-05
2^{-8}	3.710e-02	9.270e-03	2.320e-03	5.790e-04	1.450e-04	3.620e-05
2^{-10}	6.190e-02	1.540e-02	3.860e-03	9.650e-04	2.410e-04	6.030e-05
2^{-12}	9.390e-02	2.340e-02	5.830e-03	1.460e-03	3.640e-04	9.100e-05
2^{-14}	1.340e-01	3.290e-02	8.150e-03	2.030e-03	5.080e-04	1.270e-04

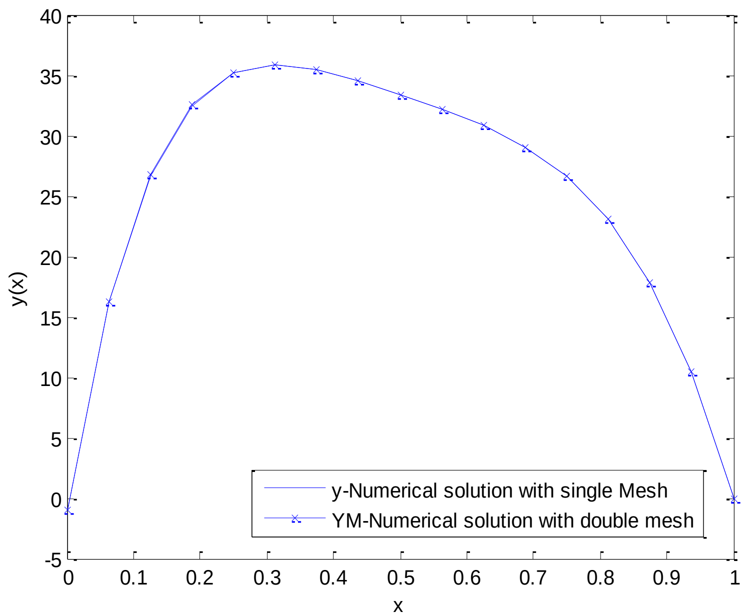

Figure 2: Numerical Solution of Example 2 for $\varepsilon = 2^{-8}$ and $N = 16$

Example 3. Consider the following self-adjoint singular perturbation problem (Beckett and Machenzie, 2001).

$$-\varepsilon y'' + (1+x)^2 y = (12x^2 - 13x + 5)(1+x)^2, \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = 0 = y(1)$

The exact solution of the problem is not known. The numerical results are obtained by using the double mesh principle and tabulated in terms of maximum absolute errors in Table 3 and its graph is given in Figures 3(a) and 3(b).

Table 3: Maximum Absolute Errors for Example3

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	3.1259E-05	1.9577E-06	1.2244E-07	7.6565E-09	4.7870E-10	3.0028E-11
2^{-5}	1.0705E-04	6.7125E-06	4.2143E-07	2.6349E-08	1.6469E-09	1.0297E-10
2^{-6}	3.8490E-04	2.4186E-05	1.5140E-06	9.4658E-08	5.9180E-09	3.6995E-10
2^{-7}	1.3555E-03	8.9366E-05	5.6001E-06	3.5125E-07	2.1957E-08	1.3725E-09
2^{-8}	5.3582E-03	3.3894E-04	2.1282E-05	1.3318E-06	8.3267E-08	5.2046E-09
2^{-9}	1.9166E-02	1.2357E-03	8.1656E-05	5.1156E-06	3.2064E-07	2.0043E-08
2^{-10}	5.9855E-02	5.0279E-03	3.1782E-04	1.9951E-05	1.2484E-06	7.8048E-08
2^{-11}	1.6013E-01	1.8343E-02	1.1787E-03	7.8013E-05	4.8869E-06	3.0617E-07

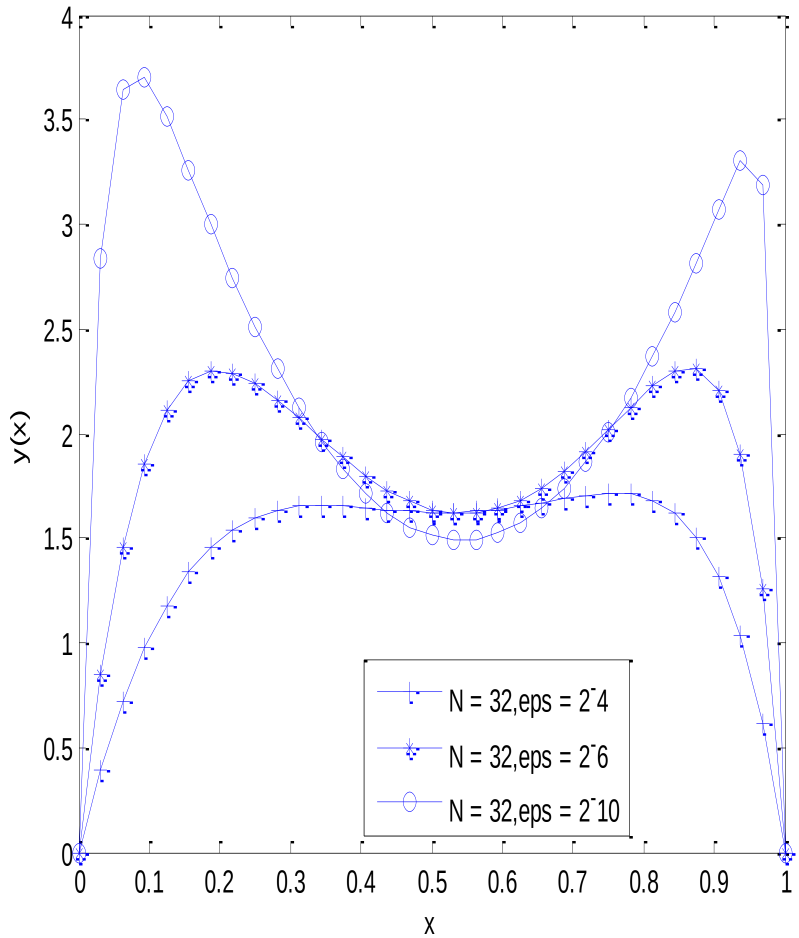

Figure 3(a): Numerical solution of Example 3 for different perturbation parameter $N = 32.$

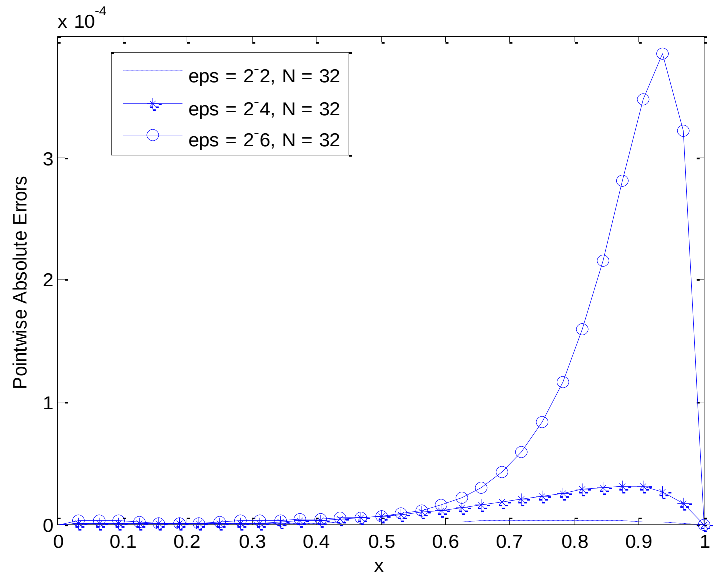

Figure 3(b): The absolute errors of Example 3 for different values of the perturbation parameter.

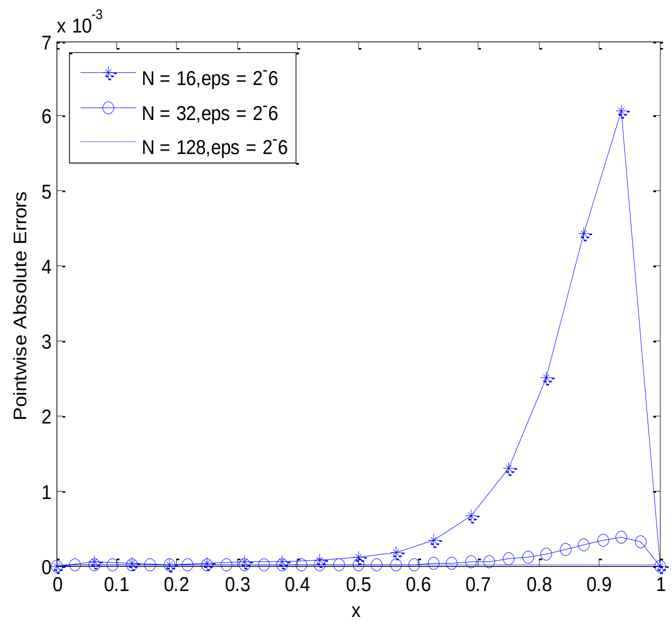

Figure 3(c): The absolute errors of Example 3 for different number of mesh points.

DISCUSSION AND CONCLUSION

A higher order finite difference method, sixth-order stable central difference method, has been presented for solving self-adjoint singularly perturbed two point boundary value problems. Three model examples are given to demonstrate the applicability of the proposed method. The maximum absolute errors are tabulated in the tables (Tables 1-3) for different values of the perturbation parameter ε and the number of mesh points N. The numerical results also presented in graphs (Figures 1 - 3).

The numerical results presented in Tables 1 and 2 clearly indicate that the proposed scheme is more accurate than the methods given in (Kadalbajoo and Kumar, 2010; Kadalbajoo and Kumar, 2008). It can also be observed from (Tables 2-3) and (Figure 3c) that the maximum absolute error decreases rapidly as the mesh size h decreases and this in turn shows that the method is convergent. This further substantiates the theoretical convergence analysis given. The graphs (Figures 1-2) also depicts that the proposed method approximates the exact solutions very well. As $\varepsilon \to 0$ the formation of thin layer, known as boundary layer, is observed at the two end points of the interval (See figure 3a).

In a concise manner, the present method is more accurate, approximates the exact solution very well and easily applicable for solving second order singularly perturbed self-adjoint boundary value problems.

ACKNOWLEDGMENT

The authors would like to thank Jimma University for the financial and materials support as the work is the part of the M. Sc Thesis of Mr. Yitbarek Zeslassie which is supported by the university.

REFERENCES

Beckett G. and Machenzie J. A. (2001). On a uniformly accurate finite difference approximation of a singularly perturbed reaction-diffusion problem using grid equi-distribution, *Journal of Computational and Applied Mathematics* 131, 381–405.

Delkhon M. and Delkhosh M. (2012). Analytic solution of some self-adjoint equations by using variable change method and its application, *Journal of Applied Mathematics*, Volume 2012 (2012), http://dx.doi.org/ 10.1155/2012/180806.

Doolan, E.P. et.al. (1980). Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole 334 Press, Dublin.

Fasika W., Gemechis F. and Tesfaye A. (2016). Sixth order compact finite difference method for solving singularly perturbed 1D Reaction Diffusion Equations with Dirichlet Boundary conditions, *Journal of Taibah University for Science*, http://dx.doi.org/10.1016/j.jtusci. 2015.12.010

Farrell P. A., Hegarty A. F., Miller J. J. H., Riordan E. O and Shishkin G. I. (2000). Robust Computational Techniques for Boundary Layers, *Applied Mathematics* (Boca Raton), vol.16, Chapman and Hall/CRC Press.

Kadalbajoo M. K. and Aggrwal V.K. (2005). Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems, *Applied Mathematics and Computation*, 161, 973-987.

Kadalbajoo, M. K and Kumar, D. (2008). Parameter-uniform fitted operator B-spline collocation method for self-adjoint singularly perturbed two-point boundary value problems. *Electronic Transactions on Numerical Analysis* **30**: 346-358.

Kadalbajoo M. K. and Kumar D. (2010). Variable mesh finite difference method for self- adjoint singularly perturbed two-point boundary value problems, *Journal of Computational Mathematics*, doi:10.4208/ jcm.1003-m2809.

KellerH.B. 1968). *Numerical Methods for two point boundary value problems*, Blaisdell publishing Company, Walthan.

Miller J. J. H. (1979). On the convergence uniformly in ε of difference schemes for a two-point boundary singular perturbation problem, in: *Numerical Analysis of Singular Perturbation Problems*, Academic Press, New York, 467–474.

Miller J.J.H., Riordan E. O', Shishkin G.I., (1996). *Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore*.

Niijima K. (1980a). On a three-point difference scheme for a singular perturbation problem without a first derivative term. I, Mem. *Numer. Math*., 7, 1–10.

Niijima K. (1980b). On a three-point difference scheme for a singular perturbation problem without a first derivative term. II, *Mem. Numer. Math.*, 7, 11–27.

Mishra H. K., Kumar M. and Singh P. (2009). Initial-value Technique for self-adjoint singular perturbation boundary value problems, *Computational Mathematics and Modeling*, 20(2).

Rashidinia J., Mahmoodi Z., Ghasemi M., (2005).Cubic spline solution of singularly perturbed boundary value problems with significant first derivatives, *Applied Mathematics and Computation*, 190, 1762-1766.

Rashidinia J., Mohammadi R., Moatamedoshariati S.H., (2010). Quitic spline method for solution of singularly perturbed boundary value problems, *International Journal for Computational Methods in Engineering science and Mechanics*, 11, 247-257.

Tirmizi A.I., Haq F., Islam. (2008). Non-polynomial spline solution of singularly perturbed boundary value problems, *Applied Mathematics and Computation*, 196, 6-16.