ORIGINAL ARTICLE

Fitted-Stable Finite Difference Method for Singularly Perturbed Two Point Boundary Value Problems

Gemechis File[1], Awoke Andargie[2] & Y. N. Reddy[3]

Abstract

A fitted-stable central difference method is presented for solving singularly perturbed two point boundary value problems with the boundary layer at one end (left or right) of the interval. A fitting factor is introduced in second order stable central difference scheme (SCD Method) and its value is obtained using the theory of singular perturbations. Thomas Algorithm (also known as Discrete Invariant Imbedding Algorithm) is used to solve the resulting tri-diagonal system. To validate the applicability of the method, some linear and non-linear examples have been solved for different values of the perturbation parameter. The numerical results are tabulated and compared with exact solutions. The error bound and convergence of the proposed method has also been established. From the results, it is observed that the present method approximates the exact solution very well.

Key words: *Singular perturbation problems, stable, central differences, fitted methods*

INTRODUCTION

Singularly perturbed second-order two-point boundary value problems occur very frequently in fluid mechanics, fluid dynamics, chemical reactor theory, elasticity etc and have received a significant amount of attention in past and recent years. The solution of these types of problems exhibits a multi scale characters. That is, there are a narrow region called boundary layer in which their solution changes rapidly and the outer region where solution changes smoothly.

[1]Department of Mathematics, Jimma University, Jimma, P.O .Box 378, Ethiopia, *E-mail*:gammeef@yahoo.com
[2]Department of Mathematics, Bahir Dar University, Bahir Dar, P.O .Box 79, Ethiopia
[3]Department of Mathematics, National Institute of Technology, WARANGAL-506 004, INDIA

Thus, numerical treatment of such problems is not trivial because of the boundary layer behavior of their solutions. There are a wide variety of asymptotic expansion methods available for solving singular perturbation problems. However, it may be difficult to apply these asymptotic expansion methods as finding of the appropriate asymptotic expansions in the inner and outer regions is not routine exercises rather requires skill, insight, and experimentations. Moreover, the matching of the coefficients of the inner and outer solution expansions can also be a demanding process. For detail discussion of solving singular perturbation problems by asymptotic expansion methods, one can refer to the books and high level monographs: (O'Malley, 1974, 1991; Nayfeh, 173, 1981; Cole and Kevorkian, 1979; Bender and Orszag, 1978; Eckhaus, 1973; Van Dyke, 1975; and Bellman, 1964).

Moreover, in the recent times many researchers have been trying to develop and present numerical methods for solving these problems. For instance, based on the asymptotic behavior of singular perturbation problems, the researchers (Kadalbajoo and Patidar, 2002; and Kadalbajoo and Reddy, 1987a, 1988) have discussed numerical schemes for the solution of linear singularly perturbed two-point boundary value problems. Reproducing kernel method (RKM) has been presented for solving a class of singularly perturbed boundary value problems by transforming the original problem in to a new boundary value problem whose solution does not change rapidly (Geng, 2011). RKM has the advantage that it can produce smooth approximate solutions, but it is difficult to apply the method for singularly perturbed boundary value problems without transforming using appropriate transformation. The authors (Padmaja and *et al.*, 2012) have presented a nonstandard explicit method involving the reduction of order for solving singularly perturbed two point boundary value problems. The original problem is approximated by a pair of initial value problems. In order to know the behavior of the solution of the problems in the boundary layer region, these researchers solved the first initial value problem as outer region problem whose solution can be required in the second initial value problem which they considered it as an inner region problem and is modified using the stretching transformation. The Differential Quadrature Method (DQM) has been applied for finding the numerical solution of singularly perturbed two point boundary value problems with mixed condition (Prasad and Reddy, 2011). DQM is based on the approximation of the derivatives of the unknown functions involved in the differential equations at the mesh point of the solution domain and is an efficient discretization technique in solving boundary value problems using a considerably small number of grid points. Moreover, the scholars (Choo and Schultz, 1993) also developed stable central difference methods (they call it SCD methods) for solving singularly perturbed two point boundary value problems.

Thus, the objective of this paper is to formulate an easily applicable and efficient computational technique which helps to understand the behavior of the solutions of the problems in the inner region, where the solution of the problem changes rapidly, of the boundary layer where other classical numerical methods fail to give good results for $h \geq \varepsilon$ and to find the computational results at uniform mesh length. Owing to this, a fitted-stable central difference (FSCD) method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) of the interval. A fitting factor is introduced in second order stable central difference scheme (SCD Method) (Choo and Schultz, 1993) and its value is obtained using the theory of singular perturbations. Thomas Algorithm is used to solve the resulting tri-diagonal system. To validate the applicability of the method, some linear and non-linear problems have been solved. The results are computed for both SCD and FSCD methods and compared with exact solutions for different values of perturbation parameter.

DESCRIPTION OF THE METHOD

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$Ly \equiv \varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in [0,1] \tag{1}$$

subjected to the boundary conditions

$$y(0) = \alpha \tag{2}$$

$$y(1) = \beta \tag{3}$$

where $0 < \varepsilon << 1$ is small positive parameter and α, β are known constants. We assume that $p(x)$, $q(x)$ and $f(x)$ are bounded and continuously differentiable functions on $(0,1)$.

Left-End Boundary Layer Problems

In general, the solution of problem (1)-(3) exhibits boundary layer behaviour at one end of the interval [0, 1] depending on the sign of $p(x)$. We assume that $p(x) \geq M > 0$ and $q(x) \leq 0$ throughout the interval $[0,1]$, where M is some constant. Under these assumptions, Eq. (1) has a unique solution $y(x)$ which in general exhibits a boundary layer of width $O(\varepsilon)$ on the left side of the underlying interval (Kadalbajoo and Reddy, 1987b).

Now, we divide the interval $[0,1]$ into N equal parts with uniform mesh length h. Let $0 = x_0, x_1, x_2, \cdots, x_N = 1$ be the mesh points. Then we have $x_i = ih, i = 0, 1, 2, \cdots, N$.

For simplicity let

$p(x_i) = p_i, q(x_i) = q_i, f(x_i) = f_i, y(x_i) = y_i$.

Assuming that $y(x)$ has continuous fourth derivatives on $[0,1]$ and by making use of Taylors series expansion for y_{i-1} and y_{i+1}, the following central difference formulae for y' and y'' can be obtained at x_i.

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + R_1 \tag{4}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + R_2 \tag{5}$$

where $R_1 = {-2h^4 y^5(\eta)}/{5!}$ and $R_2 = {-2h^4 y^6(\xi)}/{6!}$ for $\xi, \eta \in [x_i - h, x_i + h]$.

Substituting Eqs. (4) and (5) into Eq. (1), we obtain the central difference in a form that includes the $O(h^2)$ error term for y'. That is,

$$\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i - \frac{p_i h^2}{6} y''' + R = f_i \qquad (6)$$

where $R = {-\varepsilon h^2 y_i^{(4)}}/{12} + p_i R_1 + \varepsilon R_2$. Further, from Eq. (1) we have

$$\varepsilon y'' = f_i - p_i y_i' - q_i y_i \qquad (7)$$

Differentiating both sides of Eq. (7) and substituting into Eq. (6), we obtain

$$\begin{aligned} &\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i + \frac{p_i h^2}{6\varepsilon}\left(p_i y_i'' + (p_i' + q_i) y_i' + q_i' y_i\right) \\ &+ R = f_i + \frac{h^2 p_i}{6\varepsilon} f_i' \end{aligned} \qquad (8)$$

Approximating the converted error term, which have the stabilizing effect (Choo and Schultz, 1993), in Eq. (8) by using the central difference formula for y_i' and y_i'' from Eqs. (4) and (5), we obtain the SCD scheme:

$$\begin{aligned} &\left(\varepsilon + \frac{h^2 p^2{}_i}{6\varepsilon}\right)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(p_i + \frac{h^2 p_i}{6\varepsilon}(p_i' + q_i)\right)\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + \left(q_i + \frac{h^2 p_i q_i'}{6\varepsilon}\right) y_i \\ &= f_i + \frac{h^2 p_i}{6\varepsilon} f_i' + T_i \end{aligned} \qquad (9)$$

where $T_i = -\dfrac{h^4}{72\varepsilon} y^{(4)}{}_i - \dfrac{h^4}{72\varepsilon} y_i''' + R$ is the local truncation error.

Introducing the fitting factor σ into Eq. (9), we obtain

$$\frac{\sigma}{h^2}\left(\varepsilon+\frac{h^2 p^2{}_i}{6\varepsilon}\right)\left(y_{i+1}-2y_i+y_{i-1}\right)+\frac{1}{2h}\left(p_i+\frac{h^2 p_i}{6\varepsilon}\left(p_i'+q_i\right)\right)\left(y_{i+1}-y_{i-1}\right)$$
$$+\left(q_i+\frac{h^2 p_i q_i'}{6\varepsilon}\right)y_i=f_i+\frac{h^2 p_i}{6\varepsilon}f_i' \tag{10}$$

$$y_0=\alpha,\quad y_N=\beta \tag{11}$$

where σ is a fitting factor which is to be determined in such a way that the solution of Eqs. (10)- (11) converges uniformly to the solution of Eqs. (1)-(3). Multiplying Eq. (10) by h and taking the limit as $h\to 0$; we obtain

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(1+\frac{\rho^2 p^2(ih)}{6}\right)\left(y(ih+h)-2y(ih)+y(ih-h)\right)$$
$$+\lim_{h\to 0}\frac{1}{2}p(ih)\left(1+\frac{h\rho}{6}\right)\left(p'(ih)+q(ih)\right)\left(y(ih+h)-y(ih-h)\right)=0 \tag{12}$$

where $\rho=h/\varepsilon$ and $f_i+\frac{h^2 p_i}{6\varepsilon}f_i'-\left(q_i+\frac{h^2 p_i q_i'}{6\varepsilon}\right)y_i$ is bounded.

From the theory of singular perturbations it is known that the solution of Eq. (1)-(3) is of the form (O'Malley, 1974):

$$y(x)=y_0(x)+\frac{p(0)}{p(x)}\left(\alpha-y_0(0)\right)e^{-\int_0^x\left(\frac{p(x)}{\varepsilon}-\frac{q(x)}{p(x)}\right)dx}+O(\varepsilon) \tag{13}$$

where $y_0(x)$ is the solution of the reduced problem

$$p(x)y_0'(x)+q(x)y_0(x)=f(x),\quad y_0(1)=\beta \tag{14}$$

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '0' and restricting to their first term, Eq. (13) becomes

$$y(x)=y_0(x)+\left(\alpha-y_0(0)\right)e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)x}+O(\varepsilon) \tag{15}$$

Further, considering Eq. (15) at the point $x = x_i = ih,\ i = 0, 1, 2, \cdots, N$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{p^2(0)-\varepsilon q(0)}{p(0)}\right)i\rho} + O(\varepsilon) \tag{16}$$

By substituting Eq. (16) into Eq. (12) and simplifying we obtain the fitting factor

$$\sigma = \left(\frac{3\rho p(0)}{6+\rho^2 p^2(0)}\right)\coth\left[\left(\frac{p^2(0)-\varepsilon q(0)}{p(0)}\right)\frac{\rho}{2}\right] \tag{17}$$

Finally, by making use of Eq. (10) and σ given by Eq. (17), we can get the three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,\ 2,\ 3, \cdots, N-1 \tag{18}$$

where

$$E_i = \frac{\sigma}{h^2}\left(\varepsilon + \frac{h^2 p^2(x_i)}{6\varepsilon}\right) - \frac{p(x_i)}{2h}\left(1 + \frac{h^2}{6\varepsilon}\left(p'(x_i) + q(x_i)\right)\right)$$

$$F_i = \frac{2\sigma}{h^2}\left(\varepsilon + \frac{h^2 p^2(x_i)}{6\varepsilon}\right) - \left(q(x_i) + \frac{h^2 p(x_i) q'(x_i)}{6\varepsilon}\right)$$

$$G_i = \frac{\sigma}{h^2}\left(\varepsilon + \frac{h^2 p^2(x_i)}{6\varepsilon}\right) + \frac{p(x_i)}{2h}\left(1 + \frac{h^2}{6\varepsilon}\left(p'(x_i) + q(x_i)\right)\right)$$

$$H_i = f(x_i) + \frac{h^2 p^2(x_i)}{6\varepsilon} f'(x_i)$$

This gives us the tri-diagonal system which can be easily solved by Thomas Algorithm.

Right End Boundary Layer Problems

Now we assume that $p(x) \le M < 0$ throughout the interval $[0,1]$, where M is some constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$. Thus, from the theory of singular perturbations the solution of Eqs. (1)-(3) is of the form (O'Malley, 1974)

$$y(x) = y_0(x) + \frac{p(1)}{p(x)}(\beta - y_0(1))e^{\int_x^1\left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + O(\varepsilon) \tag{19}$$

where $y_0(x)$ is the solution of the reduced problem

$$p(x)y_0'(x) + q(x)y_0(x) = f(x), \quad y_0(0) = \alpha \tag{20}$$

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '1' and restricting to their first term, Eq. (19) becomes

$$y(x) = y_0(x) + \left(\beta - y_0(1)\right)e^{\left(\frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)}\right)(1-x)} + O(\varepsilon) \tag{21}$$

Similarly, considering Eq. (21) at the point $x = x_i = ih,\ i = 0,\ 1,\ 2, \cdots, N$ and taking the limit as $h \to 0$, we obtain

$$\lim_{h\to 0} y(ih) = y_0(0) + \left(\beta - y_0(1)\right)e^{-\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} + O(\varepsilon) \tag{22}$$

where $\rho = h/\varepsilon$. Applying the same procedures as in section 2.1 (*i.e*. Eqs. (10) - (12)) and making use of Eq. (22), we can get the tri-diagonal system given by Eq. (18) with a fitting factor

$$\sigma = \left(\frac{3\rho p(0)}{6 + \rho^2 p^2(0)}\right)\coth\left[\left(\frac{p^2(1) - \varepsilon q(1)}{p(1)}\right)\frac{\rho}{2}\right] \tag{23}$$

and which can easily be solved by a well known algorithm known as Discrete Invariant Imbedding algorithm or Thomas Algorithm.

As given in (Angel and Bellman, 1972; Kadalbajoo and Reddy, 1986), the stability of the discrete invariant imbedding algorithm is guaranteed by the conditions

$$E_i > 0,\ G_i > 0,\ F_i \geq E_i + G_i \text{ and } |E_i| \leq |G_i|. \tag{24}$$

In our method, under the assumptions $q(x) \leq 0$ and $p(x) > 0$ one can easily show that the conditions in Eq. (24) hold and thus, the Discrete Invariant Imbedding algorithm is stable.

CONVERGENCE ANALYSIS

Writing the tri-diagonal system (18) in matrix-vector form, we obtain

$$AY = C \tag{25}$$

where, $A = (m_{i\,j}),\ 1 \leq i, j \leq N\text{-}1$ is a tri-diagonal matrix of order N-1 , with

$$m_{ii+1} = -\sigma\varepsilon - \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^{\,2} - \frac{h^3}{12\varepsilon} p_i\left(p_i' + q_i\right)$$

$$m_{ii} = 2\sigma\varepsilon + \frac{h^2\sigma}{3\varepsilon} p_i^{\,2} - h^2 q_i - \frac{h^4}{6\varepsilon} p_i q_i'$$

$$m_{ii-1} = -\sigma\varepsilon + \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^{\,2} + \frac{h^3}{12\varepsilon} p_i\left(p_i' + q_i\right)$$

and $C = (d_i)$ is a column vector with $d_i = -h^2\left(f_i + \frac{h^2 p_i^2}{6\varepsilon} f'\right)$, where $i = 1, 2, \ldots, N-1$ with local truncation error

$$T_i(h) = h^4\left(\frac{\varepsilon}{12} - \frac{h^2 p_i^{\,2}}{12\varepsilon}\right) y_i^{(4)} + O(h^6) \quad (26)$$

We also have

$$A\,\overline{Y} - T(h) = C \quad (27)$$

where $\overline{Y} = \left(\overline{y}_0, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_N\right)^t$ and $T(h) = \left(T_1(h), T_2(h), \ldots, T_N(h)\right)^t$ denote the actual solution and the local truncation error respectively.
From Eqs. (25) and (27), we have

$$A\left(\overline{Y} - Y\right) = T(h) \quad (28)$$

This gives the error equation

$$AE = T(h) \quad (29)$$

where $E = \overline{Y} - Y = \left(e_0, e_1, e_2, \ldots, e_N\right)^t$.

Let S_i be the sum of elements of the i^{th} row of A, then we have

$$S_1 = \sum_{j=1}^{N-1} m_{1j} = \sigma\varepsilon - \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^{\,2} - h^2 q_i - \frac{h^3}{12\varepsilon} p_i\left(p_i' + q_i\right) + O\left(h^4\right)$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -h^2 q_i + O(h^4), \quad \text{for} \quad i = 2,\ 3,\ \ldots,\ N-2$$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{N-1j} = \sigma\varepsilon + \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^{\,2} - h^2 q_i + \frac{h^3}{12\varepsilon} p_i\left(p_i' + q_i\right) + O\left(h^4\right)$$

Since $0 < \varepsilon << 1$, for sufficiently small h the matrix A is irreducible and monotone (Mohanty and Jha, 2005). Then, it follows that A^{-1} exists and its elements are non negative. Hence, from Eq. (29) we have

$$E = A^{-1}T(h) \tag{30}$$

and

$$\|E\| \le \|A^{-1}\|.\|T(h)\| \tag{31}$$

Let $\overline{m}_{ki}$ be the $(ki)^{th}$ element of A^{-1}. Since $\overline{m}_{ki} \ge 0$, from the operations of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{ki}\ S_{\mathrm{i}} = 1 \ , \quad k = 1\ , 2,\ \ldots,\ N\text{-}1 \tag{32}$$

Therefore, it follows that

$$\sum_{i=1}^{N\text{-}1} \overline{m}_{ki} \le \frac{1}{\min\limits_{1\le\ i\ \le N-1} S_i} = \frac{1}{h^2 |q_i|} \tag{33}$$

We define $\|A^{-1}\| = \max\limits_{1\le k\le N-1} \sum_{i=1}^{N-1} |\overline{m}_{ki}|$ and $\|T(h)\| = \max\limits_{1\le i\le N-1} |T_i(h)|$.

Therefore, from Eqs. (26), (30), (31) and (33), we obtain

$$e_j = \sum_{i=1}^{N-1} \ \overline{m}_{ki}\ T_{\mathrm{i}}(h), \quad j = 1, 2, 3, \ldots,\ N\text{ -}1$$

which implies

$$e_j \le \frac{kh^2}{|q_i|} \ , \quad j = 1, 2, 3, \ldots,\ N\text{ -}1 \tag{34}$$

where $k = \left(\frac{\varepsilon}{12} - \frac{h^2 p_i{}^2}{12\varepsilon} \right) \left| y_i{}^{(4)} \right|$.

Therefore, using the definitions and Eq. (34)

$$\|E\| = o\left(h^2\right)$$

This implies that our method gives a second order convergence.

NUMERICAL EXAMPLES AND THEIR CORRESPONDING NUMERICAL RESULTS

Numerical Examples with Left End Boundary Layer

To demonstrate the applicability of the method, three singular perturbations with left end boundary layer are provided. The corresponding approximate solutions are compared with their exact solution.

Linear Problems

Example 1. Consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity (Reinhardt, 1980).

$$\varepsilon y''(x) + y'(x) = 1 + 2x \text{ ; } x \in [0,1] \text{ with } y(0) = 0 \text{ and } y(1) = 1.$$

The exact solution is given by $y(x) = x(x+1\text{-}2\varepsilon) + (2\varepsilon - 1)\dfrac{\left(1-e^{-x/\varepsilon}\right)}{\left(1-e^{-1/\varepsilon}\right)}$.

Table 1. Numerical Results of Example 1 for h=0.001, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$

x	SCD	FSCD [Our Method]	Exact Solution
ε=0.001			
0.000	0.0000000	0.0000000	0.0000000
0.001	-0.5978010	-0.6299651	-0.6298573
0.002	-0.8363199	-0.8610827	-0.8609354
0.003	-0.9311249	-0.9454712	-0.9453095
0.004	-0.9684431	-0.9758798	-0.9757130
0.006	-0.9878881	-0.9896713	-0.9895022
0.008	-0.9892978	-0.9897863	-0.9896172
ε=0.0001			
0.001	-0.4400840	-1.0012850	-0.9987538
0.002	-0.6855697	-1.0003250	-0.9977964
0.003	-0.8223085	-0.9993175	-0.9967916
0.004	-0.8982761	-0.9983081	-0.9957848
0.006	-0.9633093	-0.9962835	-0.9937652
0.008	-0.9822213	-0.9942508	-0.9917376
1.000	1.0000000	1.0000000	1.0000000

Nonlinear Problems

Nonlinear singular perturbation problems were linearized by using the Quasilinearization process (Bellman and Kalaba, 1965). The reduced solution (the solution of the reduced problem by putting $\varepsilon = 0$) is taken to be the initial approximation.

Example 2. Consider the following singular perturbation problem from (Bender and Orszag, 1978, page 463; equation: 9.7.1).

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad 0 \le x \le 1, \text{ with } y(0)=0 \text{ and } y(1)=0.$$

The linear form of this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

We have chosen to use Bender and Orszag's uniformly valid approximation (Bender and Orszag, 1978; page 463; equation: 9.7.6) for comparison.

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)\, e^{-2x/\varepsilon}$$

For this example, we have boundary layer of thickness O(ε) at x=0. (cf. Bender and Orszag, 1978).

Table 2. Numerical Results of Example 2 for h=0.001, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$

x	SCD	FSCD [Our Method]	Exact Solution
ε=0.001			
0.000	0.0000000	0.0000000	0.0000000
0.001	0.5190554	0.5984589	0.5983404
0.002	0.6481187	0.6786520	0.6784537
0.003	0.6796483	0.6886508	0.6884335
0.004	0.6867860	0.6891432	0.6889226
0.006	0.6873376	0.6873811	0.6871608
0.008	0.6855088	0.6853981	0.6851789
ε=0.0001			
0.001	0.1775080	0.6913452	0.6921477
0.002	0.3090479	0.6891994	0.6911492
0.003	0.4064577	0.6882086	0.6901517
0.004	0.4785268	0.6872169	0.6891552
0.006	0.5710662	0.6852362	0.6871651
0.008	0.6211921	0.6832594	0.6851790
1.000	0.0000000	0.0000000	0.0000000

Example 3: Let us consider the following singular perturbation problem from (Kevorkian and Cole, 1981 page 56; equation 2.5.1).

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0 \,;\; 0 \le x \le 1 \text{ with } y(0)= -1 \text{ and } y(1)=3.9995$$

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation (Kevorkian and Cole, 1981; pages 57-58; Eqs. 2.5.5, 2.5.11 and 2.5.14) for comparison.

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right),$$ where $c_1=2.9995$ and $c_2=(1/c_1)\log_e[(c_1-1)/(c_1+1)]$

For this example also we have a boundary layer of width O(ε) at x=0 (cf. Kevorkian and Cole, 1981, pages 56-66).

Table 3. Numerical Results of Example 3 for h=0.001, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$

x	SCD	FSCD [Our Method]	Exact Solution
ε=0.001			
0.000	1.0000000	1.0000000	1.0000000
0.001	2.5007470	2.9010340	2.4569400
0.002	2.8765900	2.9965840	2.9718740
0.003	2.9712890	3.0022920	3.0010170
0.004	2.9957140	3.0035270	3.0034260
0.006	3.0050350	3.0055390	3.0055000
0.008	3.0074930	3.0075390	3.0075000
ε=0.0001			
0.001	1.3623540	3.0018710	3.0005000
0.002	1.6593130	3.0015340	3.0015000
0.003	1.9027300	3.0025350	3.0025000
0.004	2.1023020	3.0035350	3.0035000
0.006	2.4002370	3.0055340	3.0055000
0.008	2.6008820	3.0075340	3.0075000
1.000	3.9995000	3.9995000	3.9995000

Numerical Examples with Right-End boundary Layer

To demonstrate the applicability of the method, two singular perturbation problems with right end boundary layer are provided. The corresponding approximate solution is compared with the exact solution.

Example 4. Consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) = 0$; $x \in [0, 1]$ with $y(0) = 1$ and $y(1) = 0$.

The exact solution is given by $y(x) = \dfrac{\left(e^{\frac{x-1}{\varepsilon}} - 1\right)}{\left(e^{\frac{-1}{\varepsilon}} - 1\right)}$

Table 4. Numerical Results of Example 4 for h=0.001, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$

x	SCD	FSCD [Our Method]	Exact Solution
ε=0.001			
0.000	1.0000000	1.0000000	1.0000000
0.992	0.9993449	0.9997575	0.9996645
0.994	0.9959043	0.9976141	0.9975212
0.996	0.9744003	0.9817759	0.9816834
0.997	0.9360003	0.9503017	0.9502110
0.998	0.8400003	0.8647456	0.8646612
0.999	0.6000002	0.6321797	0.6320939
ε=0.0001			
0.992	0.9904894	1.0000000	1.0000000
0.994	0.9695455	1.0000000	1.0000000
0.996	0.9024787	1.0000000	1.0000000
0.997	0.8254883	1.0000000	1.0000000
0.998	0.6877161	1.0000000	1.0000000
0.999	0.4411764	0.9999546	0.9999546
1.000	0.0000000	0.0000000	0.0000000

Example 5. Now we consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) - (1+\varepsilon) y(x) = 0$; $x \in [0, 1]$ with $y(0) = 1 + \exp\left(-\dfrac{1+\varepsilon}{\varepsilon}\right)$ and $y(1) = 1 + 1/e$.

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$

Table 5. Numerical Results of Example 5 for h=0.001, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$

x	SCD	FSCD [Our Method]	Exact Solution
ε=0.001			
0.000	1.0000000	1.0000000	1.0000000
0.992	0.3714640	0.3711784	0.3711671
0.994	0.3741479	0.3725589	0.3725573
0.996	0.3948488	0.3875510	0.3875974
0.997	0.4328066	0.4185158	0.4186246
0.998	0.5283343	0.5034753	0.5036843
0.999	0.7679049	0.7354721	0.7357859
ε=0.0001			
0.992	0.3803346	0.3703522	0.3708343
0.994	0.4005406	0.3696112	0.3700933
0.996	0.4668741	0.3688718	0.3693539
0.997	0.5434996	0.3685026	0.3689847
0.998	0.6809073	0.3681346	0.3686159
0.999	0.9270799	0.3686444	0.3682929
1.000	1.3678790	1.3678790	1.3678790

DISCUSSION AND CONCLUSION

Exponentially fitted-stable central difference method for solving singularly perturbed two-point boundary value problems has been presented. The present method has been implemented on one linear and two non-linear examples with left-end boundary layer; and two examples with right-end boundary layer by taking different values of the perturbation parameter ε. Although the solutions are computed at all points of the mesh size h, only few values specifically in the inner region (boundary layers) have been reported. Numerical results are presented in tables for both SCD and FSCD methods and compared with the exact solutions. It can be observed from the tables (Tables 1-5) that the present method, FSCD, approximates the exact solution very well than SCD for $h \geq \varepsilon$. In fact, the existing classical numerical methods produce good results only for $h < \varepsilon$, but this gives us a very large number of systems of equations that may require high capacity machines/computers or more time to run and to get the results easily. For $h \geq \varepsilon$ the existing methods produce oscillatory solutions (Gemechis and Reddy, 2013). However, the present method, FSCD gives good result in the *inner layer region* where other classical finite difference methods fail to give good results. Moreover, the present method is easy to understand and efficient technique for solving singular perturbation problems. Thus, the present method provides a good alternative technique to the conventional ways of solving singularly perturbed boundary value problems.

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