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Age-Structured Modeling of COVID-19 Transmission and Vaccine Response

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Abstract

This research presents an age-structured Susceptible-Vaccinated-Exposed-Asymptomatic-Symptomatic-Hospitalized (SVEASH) model for COVID-19 transmission, which incorporates age-dependent recruitment rates, transmission dynamics, contact patterns, and various infection forces. It assumes perfect vaccination, ensuring permanent immunity. The study investigates the mathematical properties of the model, including the existence of equilibrium states and the local stability of the disease-free equilibrium (E^0). The basic reproduction number (R0), a critical threshold for stability, is derived as a weighted average across both asymptomatic and symptomatic infection classes. The model is solved analytically using the Laplace transform, and simulations demonstrate that vaccinating up to 85% of the global population between the ages of 0 and 80 can significantly reduce the peak of the disease and shorten the duration of the epidemic. Additionally, the research emphasizes the importance of vaccinating not only susceptible and hospitalized individuals but also those who have recovered, further aiding in the control of the outbreak.

Keywords: COVID-19 transmission, vaccine response, age-structured population, disease dynamics, mathematical modeling.

INTRODUCTION

The COVID-19 pandemic, caused by the SARS-CoV-2 virus, has resulted in widespread health, social, and economic disruptions. Since its emergence in 2020, the virus has spread globally, prompting governments to implement a range of public health measures to curb transmission, such as lockdowns, travel restrictions, and vaccination campaigns. Mathematical and epidemiological models have been pivotal in understanding the transmission dynamics of COVID-19 and evaluating the effectiveness of these interventions (Ferguson et al., 2020). One significant advancement in epidemiological modeling has been the inclusion of age structure, where the population is divided into age groups with distinct risks for infection, disease severity, and vaccine response (Keeling & Rohani, 2008; Hethcote, 2000). Age-structured models are particularly important for COVID-19, as evidence shows that the severity and transmission of the disease vary significantly across different age groups (Liu et al., 2020). Older adults face a higher risk of severe disease and mortality from COVID-19, while children and young adults typically experience milder symptoms or remain asymptomatic (Liu et al., 2020). This variation in disease progression underscores the need for models that account for these age-related differences in order to accurately predict the trajectory of the pandemic and inform public health interventions. Additionally, age plays a critical role in immune response to both natural infection and vaccination, with older individuals generally exhibiting a weaker immune response to vaccines compared to younger

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populations (Polack et al., 2020; Hall et al., 2021). This highlights the importance of considering age in both the modeling of transmission dynamics and the development of vaccination strategies.

Traditional epidemiological models, such as the Susceptible-Infected-Recovered (SIR) model, have been instrumental in understanding the spread of infectious diseases (Kermack & McKendrick, 1927). However, these models often assume a homogeneous population, which fails to capture the important variations in disease transmission and vaccine efficacy across age groups. To address this, age-structured models explicitly partition the population into age classes, each with distinct rates of infection, recovery, and vaccination response. These models are particularly effective in simulating how COVID-19 spreads through different cohorts and in assessing the impact of vaccination campaigns that prioritize specific age groups (Anderson & May, 1992; Brauer et al., 2019). The importance of age-structured modeling in the COVID-19 context has been demonstrated in several studies. For example, Ferguson et al. (2020) developed a model that incorporated age-specific transmission rates and assessed the impact of various non-pharmaceutical interventions, including social distancing and quarantine measures. Their work emphasized that age structure must be considered when predicting the outcomes of public health interventions. Similarly, age-structured models have been used to evaluate vaccination strategies, with a focus on prioritizing older adults who are at greater risk of severe disease (Hodges et al., 2021). These models have shown that vaccinating high-risk age groups first can significantly reduce hospitalizations and deaths, even in the absence of widespread vaccination. Another critical factor in the modeling of COVID-19 transmission is the role of vaccine efficacy across age groups. Vaccines have been shown to be highly effective in preventing severe disease, but their efficacy can vary by age. For instance, older adults may experience a lower immune response to COVID-19 vaccines, which influences vaccine effectiveness and the timing of booster doses (Hall et al., 2021). Models that incorporate these differences are essential for simulating the impact of vaccination campaigns. Studies by Hodges et al. (2021) and others have indicated that prioritizing vaccines for older populations can help achieve the greatest reductions in mortality and hospitalizations, even if younger populations are vaccinated later. Additionally, waning immunity over time presents another challenge for modeling the long-term effectiveness of vaccination campaigns. As immunity from both natural infection and vaccination decreases over time, booster doses may be necessary to maintain protection, particularly in older age groups (Ferguson et al., 2020). Models incorporating age-structured immunity dynamics can estimate the long-term impact of vaccination strategies and help inform decisions regarding booster shots.

Methodology

Model Variables and Parameters: Table 2.1 shows the description of model variables and parameters used.

Table 2.1: Description of model variables and parameters

Variable	Description
$P(a,t)$	Total population density of age a at time t
$S(a,t)$	density of susceptible individuals of age a at time t
$V(a,t)$	density of Vaccinated individuals of age a at time t
$E(a,t)$	density of exposed individuals of age a at time t
$M(a,t)$	density of asymptomatic infectious individuals of age a at time t
$\mathbf{I}(a,t)$	density of symptomatic infectious individuals of age a at time t
$H(a,t)$	density of hospitalized individuals of age a at time t
Parameter	**Description**

A	Maximum age attained by individuals in the population, $0 < A < \infty$
$y(a,t)$	The per capita force of infection
$\beta(a)$	Transmission or infection rate
$g(a)$	Contact ratio
$b(a)$	Recruitment rate for all ages a
B	Total number of birth rate (newborns)
K, j	Fraction of susceptible who become vaccinated, exposed who become symptomatic.
$\mu(a), \alpha(a)$	age-specific natural and disease induced death rate respectively
$\emptyset(a)$	age-specific vaccination rate
$\theta(a)$	age- specific exit rate from the exposed class
$z(a), q(a)$	age-specific hospitalized rate for infectious asymptomatic and symptomatic class
$n(a)$	age-specific boost of immunity

Model Description

The proposed model is an age-structured susceptible-vaccinated-exposed-asymptomatic infected-symptomatic infected and hospitalized (SVEMIH) model that considered a total population density of $P(a,t)$,where a denotes the age of individuals at time t. A is the highest age attained by the individuals in the population, where $A < \infty$ with $a \in [0, A)$ or with $a \in [0, \infty)$. The whole population under consideration is divided into six compartments of susceptible, vaccinated, exposed, asymptomatic infectious, symptomatic infectious, and hospitalized age densities denoted by $S(a,t)$, $V(a,t)$, $E(a,t)$, $M(a,t)$, $I(a,t)$ and $H(a,t)$ respectively.

Let $b(a)$, $\mu(a)$ and $\alpha(a)$ be age specific flow or recruitment for all ages a entering only the susceptible compartment, natural mortality and force of mortality rate of the population respectively with a fraction k of susceptible individuals vaccinated at the rate $\emptyset(a)$ and the remaining (1-k) become exposed after contact with the infection at the transition rate $g(a)\lambda(t)$. $\theta(a)$ is the exit rate from the latent class, j is the proportion of exposed individuals who show symptoms while the remaining (1-j) are asymptomatic infectious without symptoms.

Most individual's immune response is capable of controlling and clearing the infection over time when hospitalized and treated (WHO 2021). $n(a)$ is the rate at which individuals who recovered as a result of medical intervention are vaccinated. q(a), $z(a)$ is the rate at which symptomatic infectious and asymptomatic individuals who are noticed as a result of diagnosis are hospitalized respectively.

Here, the force of infection $y(a,t)$ after contact with the symptomatic and asymptomatic infectious in the infective compartment is assumed to be

$$y(a,t) = g(a) \int_0^A \beta(a)[I(a,t) + M(a,t)]\, da \qquad (1)$$

Equation (1) is the force of infection of the inter-cohort separable form, where $\beta(a)$ and $g(a)$ is defined as the transmission coefficient and the contact ratio respectively.

Let $\lambda(t) = \int_0^A \beta(a)[I(a,t) + M(a,t)]\, da$

equation (1) becomes

$$y(a,t) = g(a)\lambda(t) \qquad (2)$$

The integral $\int_0^A I(a,t)\, da$ and $\int_0^A M(a,t)\, da$ stands for the total number of symptomatic infected and asymptomatic infected individuals respectively.

The following diagram describes the transmission dynamics and vaccination of COVID-19 infection.

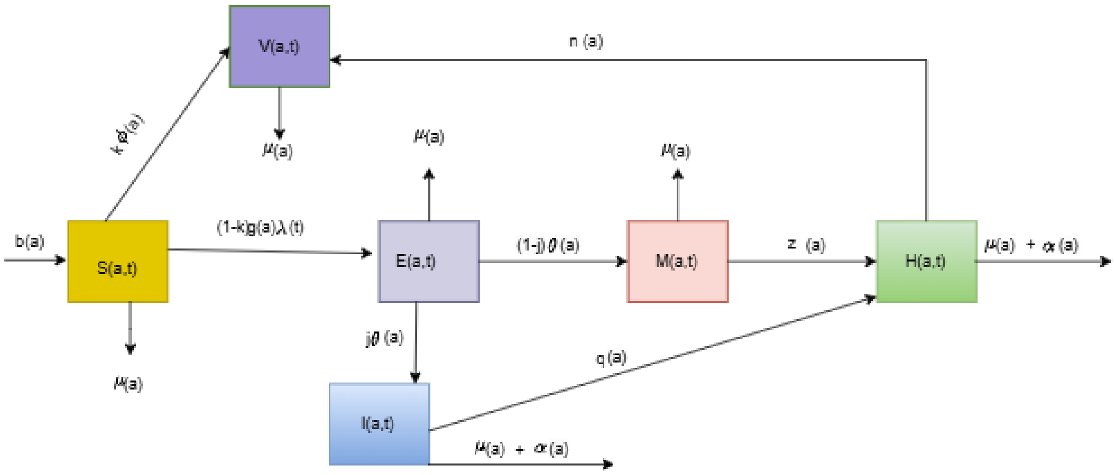

Figure 1 Flow diagram of COVID-19 transmission and vaccine response

The Model Equations

From the assumptions, descriptions and the compartment diagram in figure 1, we obtained the following system of partial differential equations for the transmission dynamics of the diseases.

$$\frac{\partial S(a,t)}{\partial a} + \frac{\partial S(a,t)}{\partial t} = b(a) - [\mu(a) + k\phi(a) + (1-k)g(a)\lambda(t)]S(a,t) \qquad (2)$$

$$\frac{\partial V(a,t)}{\partial a} + \frac{\partial V(a,t)}{\partial t} = k\phi(a)S(a,t) + n(a)H(a,t) - \mu(a)\,V(a,t) \qquad (3)$$

$$\frac{\partial E(a,t)}{\partial a} + \frac{\partial E(a,t)}{\partial t} = (1-k)g(a)\lambda(t)S(a,t) - [\mu(a) + \theta(a)]E(a,t) \qquad (4)$$

$$\frac{\partial M(a,t)}{\partial a} + \frac{\partial M(a,t)}{\partial t} = (1-j)\theta(a)E(a,t) - [z(a) + \mu(a)]M(a,t) \qquad (5)$$

$$\frac{\partial I(a,t)}{\partial a} + \frac{\partial I(a,t)}{\partial t} = j\theta(a)E(a,t) - [q(a) + \mu(a) + \alpha(a)]I(a,t) \qquad (6)$$

$$\frac{\partial H(a,t)}{\partial a} + \frac{\partial H(a,t)}{\partial t} = q(a)I(a,t) + z(a)M(a,t) - [n(a) + \mu(a) + \alpha(a)]H(a,t) \qquad (7)$$

$$\lambda(t) = \int_0^A \beta(a)[I(a,t) + M(a,t)]\,da \qquad (8)$$

with limiting conditions

$$S(0,t) = B, V(0,t) = E(0,t) = I(0,t) = H(0,t) = M(0,t) = 0 \qquad (9)$$

and initial conditions

$$S(a,0) = S_0(a),\ V(a,0) = V_0(a),\ E(a,0) = E_0(a),\ M(a,0) = M_0(a) \qquad (10)$$
$$I(a,0) = I_0(a), H(a,0) = H_0(a)$$

Laplace Transform

The Laplace transform, named after its discoverer Pierre-Simon Laplace, is an integral transform that converts a function of a real variable (usually t, in the time domain) to a function of a complex variables s (in the complex-valued frequency domain, also known as s-domain, or s-plane).

$$L\{u_t(a,t)\} = \int_0^\infty e^{-st} u_t(a,t)\,dt = s\,u^*(a,s) - u(a,0) \qquad (11)$$

$$L\{u_a(a,t)\} = \int_0^\infty e^{-st} u_a(a,t)\,dt = u_a(a,s) \qquad (12)$$

Basic properties of the model

To be sure that the model formulated is well-posed and epidemiological meaningful, there is need to prove the positivity and invariant region of the solutions of equation (2) – (10). These are best done when the model equations are ODE and not PDE and since solution to the transformed ODE along the characteristics curves also provide solution to the PDE. Hence, there is need to transform our model equations to ODE only at these points using the method of characteristics to be able to carry out these proves.

First, compare equation (1) with the general form of a first order PDE in equation (2)

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \qquad (13)$$

2.6.8

Positivity of Solution

Since the model studied human population, we need to show that all the state variables remain non-negative for all times**.**

Theorem 2.1 : Let $\Omega = \{\Omega = \{(S,V,E,M,I,H) \in \mathbb{R}_+^6 : S(0) \geq 0, V(0) \geq 0, E(0) \geq 0, M(0) \geq 0, I(0) \geq 0, H(0) \geq 0\}$ then the solutions $\{S(t), V(t), E(t), M(t), I(t), H(t)\}$ of the system of equations (3) – (8) are positive for all $t \geq 0$

Proof: Let $\{(S(0), V(0), E(0), M(0), I(0), H(0)) \geq 0\} \in \mathbb{R}_+^6$

From (3), we have

$$\frac{dS}{dt} = b(a) - (\sigma_0 + \sigma_8\lambda(t))S \qquad (14)$$

Where $\sigma_0 = \mu(a) + k\phi(a)$, $\sigma_8 = (1-k)g(a)$

Then

$$\frac{dS}{dt} \geq -(\sigma_0 + \sigma_8\lambda(t))S \qquad (15)$$

Solving (15) gives

$$S(t) \geq S(0)e^{-(\sigma_0 + \sigma_8 \int \lambda(t)\,dt)} \geq 0 \qquad (16)$$

Since $\sigma_0 \geq 0$ and $\sigma_8\lambda(t) \geq 0$

$$\frac{dV}{dt} = \sigma_1 S + n(a)H - \mu(a)\,V \qquad (17)$$

Where $\sigma_1 = k\phi(a)$

Then

$$\frac{dV}{dt} \geq -\mu(a)\,V \qquad (18)$$

Solving (18) gives

$$V(t) \geq V(0)e^{-\mu(a)t} \geq 0 \qquad (19)$$

Since $\mu(a) \geq 0$

From (19), we have

$$\frac{dE}{dt} = \sigma_8\lambda(t)S - \sigma_2 E \quad (20)$$

Where $\sigma_2 = \mu(a) + \theta(a)$

$$\frac{dE}{dt} \geq -\sigma_2 E \quad (21)$$

Solving (21) gives

$$E(t) \geq E(0)e^{-\sigma_2 t} \geq 0 \quad (22)$$

Since $\sigma_2 \geq 0$

From (6), we have

$$\frac{dM}{dt} = \sigma_4 E - \sigma_3 M \quad (23)$$

Where $\sigma_3 = \mu(a) + z(a)$ and $\sigma_4 = (1 - j)\theta(a)$

Then

$$\frac{dM}{dt} \geq -\sigma_3 M \quad (24)$$

Solving (24) gives

$$M(t) \geq M(0)e^{-\sigma_3 t} \geq 0 \quad (25)$$

Since $\sigma_3 \geq 0$

From (7)

$$\frac{dI}{dt} = \sigma_6 E - \sigma_5 I \quad (26)$$

Where $\sigma_6 = j\theta(a)$ and $\sigma_5 = \mu(a) + q(a) + \alpha(a)$

Then

$$\frac{dI}{dt} \geq -\sigma_5 I \quad (27)$$

Solving (27) gives

$$I(t) \geq I(0)e^{-\sigma_5 t} \geq 0 \quad (28)$$

Since $\sigma_5 \geq 0$

From (8), we have

$$\frac{dH}{dt} = q(a)I + z(a)M - \sigma_7 H \quad (29)$$

Where $\sigma_7 = \mu(a) + n(a) + \alpha(a)$

Then

$$\frac{dH}{dt} \geq -\sigma_7 H \quad (30)$$

Solving (30) gives

$$H(t) \geq H(0)e^{-\sigma_7 t} \geq 0 \quad (31)$$

Since $\sigma_7 \geq 0$

Hence, this completes the proof.

2.4.3 Invariant Region

Theorem 2.2: the region Ω in theorem 3.1 is positively invariant and all solutions are contained in $\Omega \in \mathbb{R}_+^6$.

Proof: Let $\Omega = (S, V, E, M, I, H) \in \mathbb{R}_+^6$ be any solution of the system with non-negative initial conditions. From (3.6.8), we have that in the absence of infection $I(t)$ and $H(t)$ equals zero. Thus, we have

$$\frac{dP}{dt} = b(a) - \mu(a)P \quad (32)$$

$$P = S + V \quad (33)$$

Solving (32) using the method of integrating factor yields

$$P(t) = \frac{b(a)}{\mu(a)} + ce^{-\mu(a)t} \quad (33)$$

Using the initial condition $P(0) = P_0(a)$ and simplifying gives

$$P(t) \leq \frac{b(a)}{\mu(a)} + \left(P_0(a) - \frac{b(a)}{\mu(a)}\right) e^{-\mu(a)t} \qquad (34)$$

Applying Birkoff and Rota's theorem on differential inequality (Birkoff and Rota 1982), gives $0 \leq P \leq \frac{b(a)}{\mu(a)}$ as $t \to \infty$

The total population approaches $\frac{b(a)}{\mu(a)}$. Therefore, the feasible solution set of the model enters the region Ω . in this region, the model equations (2) – (7) are epidemiologically meaningful and mathematically well posed.

Disease Free Equilibrium (DFE)

The disease-free equilibrium (DFE) point is a state where there is absence of COVID-19 infection in the population. Steady state solutions play an important role in studying the qualitative properties of the solution when the explicit form of the solution is not known. The disease-free equilibrium points $\mathcal{E}^0 = \left(S^0(a), V^0(a), E^0(a), M^0(a), I^0(a), H^0(a)\right)$ of model system (2) – (7) is obtained by setting

$$\frac{\partial S}{\partial t} = \frac{\partial V}{\partial t} = \frac{\partial E}{\partial t} = \frac{\partial M}{\partial t} = \frac{\partial I}{\partial t} = \frac{\partial H}{\partial t} = 0 \qquad (35)$$

And in the absence of disease,

$$\lambda = E = M = I = H = 0 \qquad (35)$$

$$\frac{dS^0}{da} + \left(\mu(a) + k\phi(a)\right)S^0 = b(a) \qquad (36)$$

Solving (36) gives

$$S^0(a) = Be^{-\int_0^a \sigma_0(x)\,dx} + \int_0^a b(\tau)e^{-\int_\tau^a \sigma_0(x)\,dx}\,d\tau \qquad (37)$$

Where $\sigma_0(a) = \mu(a) + k\phi(a)$

Also, from equation (3), we have

$$\frac{dV^0}{da} + \mu(a)V^0 = k\phi(a)S^0 \qquad (38)$$

Solving (5) gives

$$V^0(a) = \int_0^a \sigma_1(\tau)e^{-\int_\tau^a \mu(x)\,dx}\left(Be^{-\int_0^a \sigma_0(x)\,dx} + \int_0^\tau b(\sigma)e^{-\int_\sigma^\tau \sigma_0(x)\,dx}\,d\sigma\right)d\tau \qquad (39)$$

Where $\sigma_1(a) = k\phi(a)$

Hence the DFE states $\mathcal{E}^0 = (S^0, V^0, E^0, M^0, I^0, H^0)$

2.6 Endemic Equilibrium (EE).

Let $(S^*, V^*, E^*, M^*, I^*, H^*)$ represents any arbitrary endemic equilibrium point of the model equations (2) – (7). this equilibrium satisfies the following equations:

$$\frac{dS^*(a)}{da} + (\sigma_0 + \sigma_8\lambda^*)S^*(a) = b(a) \qquad (40)$$

$$\frac{dV^*(a)}{da} + \mu(a)V^*(a) = \sigma_1 S^*(a) + n(a)H^*(a) \qquad (41)$$

$$\frac{dE^*(a)}{da} + \sigma_2 E^*(a) = \sigma_8\lambda^* S^*(a) \qquad (42)$$

$$\frac{dM^*(a)}{da} + \sigma_3 M^*(a) = \sigma_4 E^*(a) \qquad (43)$$

$$\frac{dI^*(a)}{da} + \sigma_5 I^*(a) = \sigma_6 E^*(a) \qquad (44)$$

$$\frac{dH^*(a)}{da} + \sigma_7 H^*(a) = q(a)I^*(a) + z(a)M^*(a) \qquad (45)$$

Solving (40) using integrating factor method, gives

$$S^*(a) = e^{-\int_0^a (\sigma_0+\sigma_8\lambda^*)\,dx}\left(\int_0^a b(\tau)e^{-\int_0^\tau (\sigma_0+\sigma_8\lambda^*)\,dy}\,d\tau + B\right) \quad (47)$$

Also, solving (41) – (45) gives

$$E^*(a) = e^{-\int_0^a \sigma_2\,dx}\left(\int_0^a \sigma_8\lambda^* S^*(\tau)e^{\int_0^\tau \sigma_2\,dy}\,d\tau\right) \quad (48)$$

$$M^*(a) = e^{-\int_0^a \sigma_3\,dx}\left(\int_0^a \sigma_4 E^*(\tau)e^{\int_0^\tau \sigma_3\,dy}\,d\tau\right) \quad (49)$$

$$I^*(a) = e^{-\int_0^a \sigma_5\,dx}\left(\int_0^a \sigma_6 E^*(\tau)e^{\int_0^\tau \sigma_5\,dy}\,d\tau\right) \quad (50)$$

$$H^*(a) = e^{-\int_0^a \sigma_7\,dx}\left(\int_0^a \big(q(\tau)I^*(\tau) + z(\tau)M^*(\tau)\big)e^{\int_0^\tau \sigma_7\,dy}\,d\tau\right) \quad (51)$$

$$V^*(a) = e^{-\int_0^a \mu(x)\,dx}\left(\int_0^a \big(\sigma_1 S^*(\tau) + n(\tau)H^*(\tau)\big)e^{\int_0^\tau \mu(x)\,dy}\,d\tau\right) \quad (52)$$

So, the endemic equilibrium state $(S^*, V^*, E^*, M^*, I^*, H^*)$ is given by equations (46) - (51)

Basic Reproduction Number ($\mathfrak{R}_0$)

Epidemiologically, $\mathfrak{R}_0$ is the number of secondary cases produced by one infectious individual in an entirely susceptible population during the lifespan as infectious. Mathematically $\mathfrak{R}_0$ is a reproduction number if it serves as threshold for the stability of the disease-free equilibrium. (Li *et al*, 2020)

One of the fundamental questions of mathematical epidemiology is to find the reproduction number, which determines whether an infectious disease spreads in a susceptible population when the disease is introduced into the population. For an age-structured model, a possible formula for $\mathfrak{R}_0$ can be derived by determining the condition for stability of the disease-free equilibrium (Li & Brauer, 2008). Thus, whether a disease becomes persistent or dies out in a population depends on the value of $\mathfrak{R}_0$

Following the approach by Wang & Zhang (2016) and Ashezua (2015).

Let

$$\mathfrak{R}_0 = \sigma_8(a)S^0(a)\int_0^A \beta(a)\big(I^*(a) + M^*(a)\big)\,da \quad (53)$$

According to Dickmann *et al* (1990), Basic reproduction number $\mathfrak{R}_0$ of our COVID-19 model is in the form (3.9.1) and this is explained as follows, Since the total or overall infectivity at time t is the sum of the infectivity of each infected compartment, we define

$$\mathfrak{R}_0 = \mathfrak{R}_I + \mathfrak{R}_M \quad (54)$$

The basic reproduction number $\mathfrak{R}_0$ can be seen as a weighted value of the basic reproduction number due to asymptomatic infectious class and the basic reproduction number due to symptomatic infectious class. $\mathfrak{R}_0$ is a mix of how much the disease spread from people without symptoms and from people with symptoms. It gives us a big picture of how the disease spreads in the entire population since it reflects the combined impact of both groups.

Where

$$\mathfrak{R}_I = \sigma_8(a)S^0(a)\int_0^A \beta(a)I^*(a)\,da \quad (55)$$

is the number of secondary cases generated by individuals in the symptomatic infected class;

$$\mathfrak{R}_M = \sigma_8(a)S^0(a)\int_0^A \beta(a)M^*(a)\,da \quad (56)$$

is the number of secondary cases generated by individuals in the asymptomatic infected class and

$$S^0(a) = Be^{-\int_0^a \sigma_0(x)\,dx} + \int_0^a b(\tau)e^{-\int_\tau^a \sigma_0(x)\,dx}\,d\tau \tag{57}$$

is the number of susceptible individuals in the absence of COVID-19.
When $\Re_0 < 1$, the number of infections decreases toward zero. The basic reproductive number $\Re_0$ must exceed one for the disease to persist in the population.

Local Stability Analysis of the Disease-Free Equilibrium Point
Here, we investigate the local stability of the DFE state

$$\mathcal{E}^0 = \left(S^0(a), V^0(a), E^0(a), M^0(a), I^0(a), H^0(a)\right) = (S^0(a), V^0(a), 0,0,0,0) \tag{58}$$

Let $x(a,t), y(a,t), u(a,t), h(a,t), w(a,t), r(a,t)$ be the perturbation in $\mathcal{E}^0$ respectively, defined as follows;

$$S(a,t) = S^0(a) + x(a,t)$$
$$V(a,t) = V^0(a) + y(a,t)$$
$$E(a,t) = u(a,t)$$
$$M(a,t) = h(a,t)$$
$$I(a,t) = w(a,t)$$
$$H(a,t) = r(a,t)$$

Linearizing equations (2.5.1) – (2.5.8) about $\mathcal{E}^0$, give the following equations

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial a} = b(a) - \sigma_0(a)x(a,t) - \sigma_8(a)S^0(a)\lambda(t) \tag{59}$$
$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} = \sigma_1(a)x(a,t) + \mathrm{n}(a)r(a,t) - \mu(a)y(a,t) \tag{60}$$
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = \sigma_8(a)S^0(a)\lambda(t) - \sigma_2(a)u(a,t) \tag{61}$$
$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial a} = \sigma_4(a)u(a,t) - \sigma_3(a)h(a,t) \tag{62}$$
$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} = \sigma_6(a)u(a,t) - \sigma_5(a)w(a,t) \tag{63}$$
$$\frac{\partial r}{\partial t} + \frac{\partial r}{\partial a} = \mathrm{q}(a)w(a,t) + z(a)h(a,t) - \sigma_7(a)r(a,t) \tag{64}$$
$$\lambda(\mathrm{t}) = \int_0^{\mathrm{A}} \beta(a)[\mathrm{w(a,t)} + \mathrm{h(a,t)}]\,\mathrm{da} \tag{65}$$
$$x(0,t) = y(0,\mathrm{t}) = \mathrm{u(0,t)} = \mathrm{h(0,t)} = \mathrm{w(0,t)} = \mathrm{r(0,t)} = 0 \tag{66}$$

To analyze the asymptotic behavior of $\mathcal{E}^0$, we look for exponential separable solutions of the form:

$$\left.\begin{aligned} x(a,t) = x(a)e^{\varphi t}, y(a,t) = y(a)e^{\varphi t}, u(a,t) = u(a)e^{\varphi t} \\ h(a,t) = h(a)e^{\varphi t}, w(a,t) = w(a)e^{\varphi t}, r(a,t) = r(a)e^{\varphi t} \end{aligned}\right\} \tag{67}$$

This is because exponential separable solutions of the form 3.10.10 captures the temporal growth or decay rates of the perturbations. It simplifies the process of finding the eigenvalues and determining the stability of the equilibrium in a system of PDEs. Hence, we consider the following linear eigen-value problems:

$$\frac{dx(a)}{da} + \left(\varphi + \sigma_0(a)\right)x(a) = \left(b(a) - \sigma_8(a)S^0(a)\lambda\right) \tag{68}$$
$$\frac{dy(a)}{da} + \left(\varphi + \mu(a)\right)y(a) = \sigma_1(a)x(a) + n(a)r(a) \tag{69}$$
$$\frac{du(a)}{da} + \left(\varphi + \sigma_2(a)\right)u(a) = \sigma_8(a)S^0(a)\lambda \tag{70}$$

$$\frac{dh(a)}{da} + (\varphi + \sigma_3(a))h(a) = \sigma_4(a)u(a) \quad (71)$$
$$\frac{dw(a)}{da} + (\varphi + \sigma_5(a))w(a) = \sigma_6(a)u(a) \quad (72)$$
$$\frac{dr(a)}{da} + (\varphi + \sigma_7(a))r(a) = q(a)w(a) + z(a)h(a) \quad (73)$$
$$\lambda = \int_0^A \beta(a)[w(a) + h(a)]\, da \quad (74)$$
$$x(0,t) = y(0,t) = u(0,t) = h(0,t) = w(0,t) = r(0,t) = 0 \quad (75)$$
$$u(a) = \lambda \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} e^{-\int_\tau^a \sigma_2(s)\, ds}\, d\tau \quad (76)$$
$$h(a) = \int_0^a \sigma_4(\tau) u(\tau) e^{-\varphi(a-\tau)} e^{-\int_\tau^a \sigma_3(s)\, ds}\, d\tau \quad (77)$$
$$w(a) = \int_0^a \sigma_6(\tau) u(\tau) e^{-\varphi(a-\tau)} e^{-\int_\tau^a \sigma_5(s)\, ds}\, d\tau \quad (78)$$

Substituting (75) into (76) – (78) and changing the order of integration, we get
$$h(a) = \lambda \int_0^a \sigma_4(\tau) \left(\int_0^\tau \sigma_8(\eta) S^0(\eta) e^{-\varphi(a-\eta)} e^{-\int_\eta^\tau \sigma_2(s)\, ds}\, d\eta \right) e^{-\int_\tau^a \sigma_3(s)\, ds}\, d\tau$$
$$= \lambda \int_0^a \sigma_8(\eta) S^0(\eta) e^{-\varphi(a-\eta)} \int_\eta^a \sigma_4(\tau) e^{-\int_\eta^\tau \sigma_2(s)\, ds}\, e^{-\int_\tau^a \sigma_3(s)\, ds}\, d\tau d\eta$$
$$h(a) = \lambda \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} \int_\tau^a \sigma_4(\eta) e^{-\int_\tau^\eta \sigma_2(s)\, ds}\, e^{-\int_\eta^a \sigma_3(s)\, ds}\, d\eta d\tau \quad (79)$$
$$w(a) = \lambda \int_0^a \sigma_6(\tau) \left(\int_0^\tau \sigma_8(\eta) S^0(\eta) e^{-\varphi(a-\eta)} e^{-\int_\eta^\tau \sigma_2(s)\, ds}\, d\eta \right) e^{-\int_\tau^a \sigma_5(s)\, ds}\, d\tau$$
$$= \lambda \int_0^a \sigma_8(\eta) S^0(\eta) e^{-\varphi(a-\eta)} \int_\eta^a \sigma_6(\tau) e^{-\int_\eta^\tau \sigma_2(s)\, ds}\, e^{-\int_\tau^a \sigma_5(s)\, ds}\, d\tau d\eta$$
$$w(a) = \lambda \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} \int_\tau^a \sigma_6(\eta) e^{-\int_\tau^\eta \sigma_2(s)\, ds}\, e^{-\int_\eta^a \sigma_5(s)\, ds}\, d\eta d\tau \quad (80)$$

Substituting (70) and (71) into (77), it follows that
$$\lambda = \lambda \int_0^A \beta(a) \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} \int_\tau^a e^{-\int_\eta^\tau \sigma_2(s)\, ds} \Big(\sigma_4(\eta) e^{-\int_\eta^a \sigma_3(s)\, ds} Z$$
$$+ \sigma_6(\eta) e^{-\int_\eta^a \sigma_5(s)\, ds} \Big)\, d\eta d\tau da \quad (81)$$

By dividing both sides of (81) by λ ($\lambda \neq 0$), we get the following characteristic equation about the eigenvalue φ
$$1 = \int_0^A \beta(a) \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} \int_\tau^a e^{-\int_\eta^\tau \sigma_2(s)\, ds} \Big(\sigma_4(\eta) e^{-\int_\eta^a \sigma_3(s)\, ds}$$
$$+ \sigma_6(\eta) e^{-\int_\eta^a \sigma_5(s)\, ds} \Big)\, d\eta d\tau da \quad (82)$$

denote the expression on the right-hand side of (79) by $F(\varphi)$, i.e.,
$$F(\varphi) = \int_0^A \beta(a) \int_0^a \sigma_8(\tau) S^0(\tau) e^{-\varphi(a-\tau)} \int_\tau^a e^{-\int_\eta^\tau \sigma_2(s)\, ds} \Big(\sigma_4(\eta) e^{-\int_\eta^a \sigma_3(s)\, ds}$$
$$+ \sigma_6(\eta) e^{-\int_\eta^a \sigma_5(s)\, ds} \Big)\, d\eta d\tau da \quad (83)$$

and define the basic reproductive number as $\Re_0 = F(0)$, i.e.,
$$\Re_0 = \int_0^A \beta(a) \int_0^a \sigma_8(\tau) S^0(\tau) \int_\tau^a e^{-\int_\eta^\tau \sigma_2(s)\, ds} \Big(\sigma_4(\eta) e^{-\int_\eta^a \sigma_3(s)\, ds}$$
$$+ \sigma_6(\eta) e^{-\int_\eta^a \sigma_5(s)\, ds} \Big)\, d\eta d\tau da \quad (84)$$

Now, we establish the following results from equation (84)

Theorem 2.3
The disease-free equilibrium of the system (2) – (9) is locally asymptotically stable, if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.
Proof: Suppose we differentiate $F(\varphi)$, we then have

$$F'(\varphi) = -\int_0^A \beta(a) \int_0^a (a-\tau)\sigma_8(\tau)S^0(\tau)e^{-\varphi(a-\tau)} \int_\tau^a e^{-\int_\eta^\tau \sigma_2(s)\,ds} \Big(\sigma_4(\eta)e^{-\int_\eta^a \sigma_3(s)\,ds} + \sigma_6(\eta)e^{-\int_\eta^a \sigma_5(s)\,ds}\Big)\, d\eta\, d\tau\, da$$

it is observed that F is a decreasing function of φ as

$$F'(\varphi) < 0, \quad \lim_{\varphi\to\infty} F(\varphi) = 0, \quad \lim_{\varphi\to-\infty} F(\varphi) = +\infty$$

we know that equation (84) has a unique negative real solution φ^*, if and only if $F(0) < 1$, or $\Re_0 < 1$. and a unique positive (zero) real solution if $F(0) > 1$ ($F(0) = 1$), or $\Re_0 > 1$ ($\Re_0 = 1$). To show that φ^* is the dominant real part of the roots of $F(\varphi)$, we let $\varphi = x + iy$ ($x, y \in \mathbb{R}$, where i is the imaginary unit and $\mathbb{R}$ is the set of real numbers) be an arbitrary complex solution to equation (84). we note that

$$1 = F(\varphi) = |F(x+iy)| \le F(x)$$

which indicates that $\mathrm{Re}\varphi \le \varphi^*$, where Re denotes the real part. It follows that the disease-free equilibrium is locally asymptotically stable if $\Re_0 < 1$, and unstable if $\Re_0 > 1$.

Analytical Solution of the Model Using Laplace Transform
Our model equations are coupled. It cannot be solved analytically without breaking the couplings. To decoupled the equations, we have to perturbed the equations. At this point, S_0, V_0, E_0, M_0, I_0, H_0 are the varying compartments and ${S_0}^*, {V_0}^*, {E_0}^*, {M_0}^*, {I_0}^*, {H_0}^*$ are the Laplace transform of the compartments respectively.
Let $0 < \beta(a) < 1$ and let us define the following

$$S(a,t) = S_0(a,t) + \beta(a)S_1(a,t) + \cdots$$
$$V(a,t) = V_0(a,t) + \beta(a)V_1(a,t) + \cdots$$
$$E(a,t) = E_0(a,t) + \beta(a)E_1(a,t) + \cdots$$
$$M(a,t) = M_0(a,t) + \beta(a)M_1(a,t) + \cdots$$
$$I(a,t) = I_0(a,t) + \beta(a)I_1(a,t) + \cdots$$
$$H(a,t) = H_0(a,t) + \beta(a)H_1(a,t) + \cdots$$

Then, we have for
$\beta(a)^0$ (Order zero):

$$\frac{\partial S_0}{\partial t} + \frac{\partial S_0}{\partial a} + [\mu(a) + k\phi(a)]S_0 = b(a) \qquad (85)$$
$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial a} + \mu(a)V_0 = k\phi(a)S_0 + n(a)H_0 \qquad (86)$$
$$\frac{\partial E_0}{\partial t} + \frac{\partial E_0}{\partial a} + [\mu(a) + \theta(a)]E_0 = 0 \qquad (86)$$
$$\frac{\partial M_0}{\partial t} + \frac{\partial M_0}{\partial a} + [\mu(a) + z(a)]M_0 = (1-j)\theta(a)E_0 \qquad (87)$$
$$\frac{\partial I_0}{\partial t} + \frac{\partial I_0}{\partial a} + [\mu(a) + q(a) + \alpha(a)]I_0 = j\theta(a)E_0 \qquad (88)$$
$$\frac{\partial H_0}{\partial t} + \frac{\partial H_0}{\partial a} + [\mu(a) + n(a) + \alpha(a)]H_0 = q(a)I_0 + z(a)M_0 \qquad (89)$$

$\beta(a)^1$ (Order one):

$$\frac{\partial S_1}{\partial t} + \frac{\partial S_1}{\partial a} + [\mu(a) + k\phi(a)]S_1 = -(1-k)g(a)\left[\int_0^A [I_0 + M_0]\,da\right]S_0 \qquad (90)$$
$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial a} + \mu(a)V_1 = k\phi(a)S_1 + n(a)H_1 \qquad (91)$$

$$\frac{\partial E_1}{\partial t}+\frac{\partial E_1}{\partial a}+[\mu(a)+\theta(a)]E_1=(1-k)g(a)\left[\int_0^A [I_0+M_0]\,da\right]S_0 \quad (92)$$

$$\frac{\partial M_1}{\partial t}+\frac{\partial M_1}{\partial a}+[\mu(a)+z(a)]M_1=(1-j)\theta(a)E_1 \quad (93)$$

$$\frac{\partial I_1}{\partial t}+\frac{\partial I_1}{\partial a}+[\mu(a)+q(a)+\alpha(a)]I_1=j\theta(a)E_1 \quad (94)$$

$$\frac{\partial H_1}{\partial t}+\frac{\partial H_1}{\partial a}+[\mu(a)+n(a)+\alpha(a)]H_1=q(a)I_1+z(a)M_1 \quad (95)$$

$$P(a,t)=S(a,t)+V(a,t)+E(a,t)+M(a,t)+I(a,t)+H(a,t) \quad (96)$$

For conveniency, let

$\sigma_0=\mu(a)+k\phi(a)$, $\sigma_1=k\phi(a)$, $\sigma_2=\mu(a)+\theta(a)$, $\sigma_3=\mu(a)+z(a)$,
$\sigma_5=\mu(a)+q(a)+\alpha(a)$, $\sigma_4=(1-j)\theta(a)$, $\sigma_6=j\theta(a)$,
$\sigma_7=\mu(a)+n(a)+\alpha(a)$, $\sigma_8=(1-k)g(a)$

Now, consider equation (96) in the form

$$\frac{\partial S_0}{\partial t}+\frac{\partial S_0}{\partial a}+\sigma_0 S_0=b(a) \quad (97)$$

Applying the Laplace transform to all terms in (2.11.14), we have

$$L_t\left\{\frac{\partial S_0}{\partial a}\right\}+L_t\left\{\frac{\partial S_0}{\partial t}\right\}+\sigma_0 L_t\{S_0\}=L_t\{b(a)\}$$

That is

$$\frac{dS_0^*(a,s)}{da}+sS_0^*(a,s)-S_0(a,0)+\sigma_0 S_0^*(a,s)=\frac{b(a)}{s}$$

$$\frac{dS_0^*(a,s)}{da}+(s+\sigma_0)S_0^*(a,s)=\frac{b(a)}{s}+S_0(a) \quad (98)$$

Solving equation ((98), we obtain

$$S_0^*(a,s)=e^{-(s+\sigma_0)a}\int_0^a\left(\frac{b(a)}{s}+S_0(a)\right)e^{(s+\sigma_0)x}\,dx+f(s)e^{-(s+\sigma_0)a}$$

$$=e^{-(s+\sigma_0)a}\frac{1}{s+\sigma_0}\left(\frac{b(a)}{s}+S_0(a)\right)e^{(s+\sigma_0)x}\Big|_0^a+f(s)e^{-(s+\sigma_0)a}$$

$$S_0^*(a,s)=\frac{1}{s+\sigma_0}\left(\frac{b(a)}{s}+S_0(a)\right)\left(1-e^{-(s+\sigma_0)a}\right)+f(s)e^{-(s+\sigma_0)a}$$

$$S_0^*(0,s)=\ f(s)=\frac{B}{s}$$

Therefore,

$$S_0^*(a,s)=\frac{b(a)}{\sigma_0}\left(\frac{1}{s}\right)+\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}\right)+\left(B-\frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s}e^{-(s+\sigma_0)a}\right)$$
$$-\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}e^{-(s+\sigma_0)a}\right)$$

So,

$$S_0(a,t)=L_t^{-1}\{S_0^*(a,s)\}=\frac{b(a)}{\sigma_0}+\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)e^{-\sigma_0 t}+\left(B-\frac{b(a)}{\sigma_0}\right)e^{-\sigma_0 a}u_a(t)$$
$$-\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)u_a(t)e^{-\sigma_0 a}e^{-\sigma_0(t-a)} \quad (99)$$

Where,

$$u_a(t)=\begin{cases}1, & t\geq a\\ 0, & t<a\end{cases}, \text{ for } a\geq 0$$

Unit step function (delayed)

Consider equation (99) in the form:

$$\frac{\partial E_0}{\partial t}+\frac{\partial E_0}{\partial a}+\sigma_2 E_0=0 \quad (100)$$

Applying the Laplace transform to all terms in (2.11.17), we have

$$\frac{dE_0^*(a,s)}{da} + (s+\sigma_2)E_0^*(a,s) = E_0(a) \tag{101}$$

Solving equation (100), we obtain

$$E_0^*(a,s) = e^{-(s+\sigma_2)a}\int_0^a E_0(a)\, e^{(s+\sigma_2)x}\, dx + f(s)e^{-(s+\sigma_2)a}$$

$$= e^{-(s+\sigma_2)a}\frac{E_0(a)}{s+\sigma_2} e^{(s+\sigma_2)x}\Big|_0^a + f(s)e^{-(s+\sigma_2)a}$$

$$E_0^*(a,s) = \frac{E_0(a)}{s+\sigma_2}\left(1 - e^{-(s+\sigma_2)a}\right) + f(s)e^{-(s+\sigma_2)a}$$

$$E_0^*(0,s) = f(s) = 0$$

Therefore,

$$E_0^*(a,s) = \frac{E_0(a)}{s+\sigma_2}\left(1 - e^{-(s+\sigma_2)a}\right) \tag{102}$$

So,

$$E_0(a,t) = L_t^{-1}\{E_0^*(a,s)\} = E_0(a)\left(e^{-\sigma_2 t} - e^{-\sigma_2 a}u_a(t)e^{-\sigma_2(t-a)}\right)$$

$$E_0(a,t) = E_0(a)e^{-\sigma_2 t}\left(1 - u_a(t)\right) \tag{103}$$

Also, consider equation (103) in the form

$$\frac{\partial M_0}{\partial a} + \frac{\partial M_0}{\partial t} + \sigma_3 M_0 = \sigma_4 E_0(a,t)$$

That is,

$$\frac{\partial M_0}{\partial a} + \frac{\partial M_0}{\partial t} + \sigma_3 M_0 = \sigma_4 E_0(a)e^{-\sigma_2 t}\left(1 - u_a(t)\right) \tag{104}$$

Applying the Laplace transform to all terms in (2.11.2), we have

$$\frac{dM_0^*(a,s)}{da} + (s+\sigma_3)M_0^*(a,s) = M_0(a) + \sigma_4 E_0(a)\left(\frac{1}{s+\sigma_2} - \frac{e^{-as}}{s(s+\sigma_2)}\right) \tag{105}$$

Solving equation (105), we obtain

$$M_0^*(a,s) = e^{-(s+\sigma_3)a}\int_0^a e^{(s+\sigma_3)x}\left(M_0(a) + \sigma_4 E_0(a)\left(\frac{1}{s+\sigma_2} - \frac{e^{-xs}}{s(s+\sigma_2)}\right)\right)dx + f(s)e^{-(s+\sigma_3)a}$$

$$= e^{-(s+\sigma_3)a}\int_0^a \left(\left(M_0(a) + \sigma_4 E_0(a)\left(\frac{1}{s+\sigma_2}\right)\right)e^{(s+\sigma_3)x} - \frac{\sigma_4 E_0(a)}{s(s+\sigma_2)}e^{\sigma_3 x}\right)dx + f(s)e^{-(s+\sigma_3)a}$$

$$= e^{-(s+\sigma_3)a}\left(\frac{1}{s+\sigma_3}\left(M_0(a) + \sigma_4 E_0(a)\left(\frac{1}{s+\sigma_2}\right)\right)e^{(s+\sigma_3)x} - \frac{\sigma_4 E_0(a)}{\sigma_3 s(s+\sigma_2)}e^{\sigma_3 x}\right)\Bigg|_0^a + f(s)e^{-(s+\sigma_3)a}$$

$$= \frac{1}{s+\sigma_3}\left(M_0(a) + \sigma_4 E_0(a)\left(\frac{1}{s+\sigma_2}\right)\right)\left(1 - e^{-(s+\sigma_3)a}\right) - \frac{\sigma_4 E_0(a)}{\sigma_3 s(s+\sigma_2)}\left(e^{-as} - e^{-(s+\sigma_3)a}\right) + f(s)e^{-(s+\sigma_3)a}$$

$$M_0^*(a,s) = \frac{M_0(a)}{s+\sigma_3} + \frac{\sigma_4 E_0(a)}{(s+\sigma_2)(s+\sigma_3)} - \frac{M_0(a)}{s+\sigma_3}e^{-(s+\sigma_3)a} - \frac{\sigma_4 E_0(a)}{(s+\sigma_2)(s+\sigma_3)}e^{-(s+\sigma_3)a} - \frac{\sigma_4 E_0(a)}{\sigma_3 s(s+\sigma_2)}e^{-as} + \frac{\sigma_4 E_0(a)}{\sigma_3 s(s+\sigma_2)}e^{-(s+\sigma_3)a} + f(s)e^{-(s+\sigma_3)a}$$

$$M_0^*(0,s) = f(s) = 0$$

Therefore,

$$M_0^*(a,s) = \frac{M_0(a)}{s+\sigma_3} + \frac{\sigma_4 E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_3}\right) - M_0(a)e^{-\sigma_3 a}\left(\frac{e^{-as}}{s+\sigma_3}\right) - \frac{\sigma_4 E_0(a)}{(\sigma_3-\sigma_2)}e^{-\sigma_3 a}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_3}\right)e^{-as} - \frac{\sigma_4 E_0(a)}{\sigma_3\sigma_2}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)e^{-as} + \frac{\sigma_4 E_0(a)}{\sigma_3\sigma_2}e^{-\sigma_3 a}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)e^{-as}$$

So,

$$\mathrm{M}_0(\mathrm{a},\mathrm{t}) = L_t^{-1}\{\mathrm{M}_0^*(\mathrm{a},\mathrm{s})\}$$

$$= \mathrm{M}_0(\mathrm{a})e^{-\sigma_3 t} + \frac{\sigma_4\mathrm{E}_0(\mathrm{a})}{(\sigma_3-\sigma_2)}(e^{-\sigma_2 t} - e^{-\sigma_3 t}) - \mathrm{M}_0(\mathrm{a})e^{-\sigma_3 a}u_a(t)e^{-\sigma_3(t-a)} - \frac{\sigma_4\mathrm{E}_0(\mathrm{a})}{(\sigma_3-\sigma_2)}e^{-\sigma_3 a}u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_3(t-a)}\right) - \frac{\sigma_4\mathrm{E}_0(\mathrm{a})}{\sigma_3\sigma_2}u_a(t)\left(1-e^{-\sigma_2(t-a)}\right) + \frac{\sigma_4\mathrm{E}_0(\mathrm{a})}{\sigma_3\sigma_2}e^{-\sigma_3 a}u_a(t)\left(1-e^{-\sigma_2(t-a)}\right) \quad (106)$$

Also, consider equation (106) in the form:

$$\frac{\partial \mathrm{I}_0}{\partial \mathrm{a}} + \frac{\partial \mathrm{I}_0}{\partial \mathrm{t}} + \sigma_5\mathrm{I}_0 = \sigma_6\mathrm{E}_0(a,t)$$

That is,

$$\frac{\partial \mathrm{I}_0}{\partial \mathrm{a}} + \frac{\partial \mathrm{I}_0}{\partial \mathrm{t}} + \sigma_5\mathrm{I}_0 = \sigma_6\mathrm{E}_0(\mathrm{a})e^{-\sigma_2 t}\left(1-\mathrm{u_a}(\mathrm{t})\right) \quad (107)$$

Applying the Laplace transform to all terms in (107), we have

$$\frac{d\mathrm{I}_0^*(\mathrm{a},\mathrm{s})}{d\mathrm{a}} + (\mathrm{s}+\sigma_5)\mathrm{I}_0^*(\mathrm{a},\mathrm{s}) = \mathrm{I}_0(\mathrm{a}) + \sigma_6\mathrm{E}_0(\mathrm{a})\left(\frac{1}{\mathrm{s}+\sigma_2} - \frac{e^{-as}}{s(\mathrm{s}+\sigma_2)}\right) \quad (108)$$

Solving equation (108), we obtained

$$\mathrm{I}_0^*(\mathrm{a},\mathrm{s}) = e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}\int_0^{\mathrm{a}} e^{(\mathrm{s}+\sigma_5)\mathrm{x}}\left(\mathrm{I}_0(\mathrm{a}) + \sigma_6\mathrm{E}_0(\mathrm{a})\left(\frac{1}{\mathrm{s}+\sigma_2} - \frac{e^{-xs}}{s(\mathrm{s}+\sigma_2)}\right)\right)d\mathrm{x} + \mathrm{f}(\mathrm{s})e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}$$

$$= e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}\int_0^{\mathrm{a}}\left(\left(\mathrm{I}_0(\mathrm{a}) + \sigma_6\mathrm{E}_0(\mathrm{a})\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\right)e^{(\mathrm{s}+\sigma_5)\mathrm{x}} - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{s(\mathrm{s}+\sigma_2)}e^{\sigma_5\mathrm{x}}\right)d\mathrm{x} + \mathrm{f}(\mathrm{s})e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}$$

$$= \frac{1}{\mathrm{s}+\sigma_5}\left(\mathrm{I}_0(\mathrm{a}) + \sigma_6\mathrm{E}_0(\mathrm{a})\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\right)\left(1-e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}\right) - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5 s(\mathrm{s}+\sigma_2)}\left(e^{-as} - e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}\right) + \mathrm{f}(\mathrm{s})e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}$$

That is,

$$\mathrm{I}_0^*(\mathrm{a},\mathrm{s}) = \frac{\mathrm{I}_0(\mathrm{a})}{\mathrm{s}+\sigma_5} + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\mathrm{s}+\sigma_2)(\mathrm{s}+\sigma_5)} - \frac{\mathrm{I}_0(\mathrm{a})}{\mathrm{s}+\sigma_5}e^{-(\mathrm{s}+\sigma_5)\mathrm{a}} - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\mathrm{s}+\sigma_2)(\mathrm{s}+\sigma_5)}e^{-(\mathrm{s}+\sigma_5)\mathrm{a}} - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5 s(\mathrm{s}+\sigma_2)}e^{-as} + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5 s(\mathrm{s}+\sigma_2)}e^{-(\mathrm{s}+\sigma_5)\mathrm{a}} + \mathrm{f}(\mathrm{s})e^{-(\mathrm{s}+\sigma_5)\mathrm{a}}$$

$$\mathrm{I}_0^*(0,\mathrm{s}) = \mathrm{f}(\mathrm{s}) = 0$$

Therefore,

$$\mathrm{I}_0^*(\mathrm{a},\mathrm{s}) = \frac{\mathrm{I}_0(\mathrm{a})}{\mathrm{s}+\sigma_5} + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\sigma_5-\sigma_2)}\left(\frac{1}{\mathrm{s}+\sigma_2} - \frac{1}{\mathrm{s}+\sigma_5}\right) - \mathrm{I}_0(\mathrm{a})e^{-\sigma_5 a}\left(\frac{e^{-as}}{\mathrm{s}+\sigma_5}\right) - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\sigma_5-\sigma_2)}e^{-\sigma_5 a}\left(\frac{1}{\mathrm{s}+\sigma_2} - \frac{1}{\mathrm{s}+\sigma_5}\right)e^{-as} - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5\sigma_2}\left(\frac{1}{\mathrm{s}} - \frac{1}{\mathrm{s}+\sigma_2}\right)e^{-as} + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5\sigma_2}e^{-\sigma_5 a}\left(\frac{1}{\mathrm{s}} - \frac{1}{\mathrm{s}+\sigma_2}\right)e^{-as}$$

So,

$$\mathrm{I}_0(\mathrm{a},\mathrm{t}) = L_t^{-1}\{\mathrm{I}_0^*(\mathrm{a},\mathrm{s})\}$$

$$= \mathrm{I}_0(\mathrm{a})e^{-\sigma_5 t} + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\sigma_5-\sigma_2)}(e^{-\sigma_2 t} - e^{-\sigma_5 t}) - \mathrm{I}_0(\mathrm{a})e^{-\sigma_5 a}u_a(t)e^{-\sigma_5(t-a)} - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{(\sigma_5-\sigma_2)}e^{-\sigma_5 a}u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_5(t-a)}\right) - \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5\sigma_2}u_a(t)\left(1-e^{-\sigma_2(t-a)}\right) + \frac{\sigma_6\mathrm{E}_0(\mathrm{a})}{\sigma_5\sigma_2}e^{-\sigma_5 a}u_a(t)\left(1-e^{-\sigma_2(t-a)}\right) \quad (109)$$

Also, consider equation (109) in the form:

$$\frac{\partial \mathrm{H}_0}{\partial \mathrm{a}} + \frac{\partial \mathrm{H}_0}{\partial \mathrm{t}} + \sigma_7\mathrm{H}_0 = \mathrm{q}(\mathrm{a})\mathrm{I}_0 + \mathrm{z}(\mathrm{a})\mathrm{M}_0 \quad (110)$$

Applying the Laplace transform to all terms in (110), we have

$$\frac{dH_0^*(a,s)}{da}+(s+\sigma_7)H_0^*(a,s)$$
$$=H_0(a)$$
$$+q(a)\left[\frac{I_0(a)}{s+\sigma_5}+\frac{\sigma_6E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_5}\right)-\frac{I_0(a)}{s+\sigma_5}e^{-(s+\sigma_5)a}\right.$$
$$-\frac{\sigma_6E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_5}\right)e^{-(s+\sigma_5)a}-\frac{\sigma_6E_0(a)}{\sigma_5\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as}$$
$$\left.+\frac{\sigma_6E_0(a)}{\sigma_5\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-(s+\sigma_5)a}\right]$$
$$+z(a)\left[\frac{M_0(a)}{s+\sigma_3}+\frac{\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)-\frac{M_0(a)}{s+\sigma_3}e^{-(s+\sigma_3)a}\right.$$
$$-\frac{\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)e^{-(s+\sigma_3)a}-\frac{\sigma_4E_0(a)}{\sigma_3\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as}$$
$$\left.+\frac{\sigma_4E_0(a)}{\sigma_3\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-(s+\sigma_3)a}\right] \qquad (111)$$

Solving equation (111), we obtained

$$H_0^*(a,s)=e^{-(s+\sigma_7)a}\int_0^a e^{(s+\sigma_7)x}\left[H_0(a)\right.$$
$$+q(a)\left[\frac{I_0(a)}{s+\sigma_5}+\frac{\sigma_6E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_5}\right)-\frac{I_0(a)}{s+\sigma_5}e^{-(s+\sigma_5)x}\right.$$
$$-\frac{\sigma_6E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_5}\right)e^{-(s+\sigma_5)x}-\frac{\sigma_6E_0(a)}{\sigma_5\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-sx}$$
$$\left.+\frac{\sigma_6E_0(a)}{\sigma_5\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-(s+\sigma_5)x}\right]$$
$$+z(a)\left[\frac{M_0(a)}{s+\sigma_3}+\frac{\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)-\frac{M_0(a)}{s+\sigma_3}e^{-(s+\sigma_3)x}\right.$$
$$-\frac{\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)e^{-(s+\sigma_3)x}-\frac{\sigma_4E_0(a)}{\sigma_3\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-sx}$$
$$\left.\left.+\frac{\sigma_4E_0(a)}{\sigma_3\sigma_2}\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-(s+\sigma_3)x}\right]\right]dx+f(s)e^{-(s+\sigma_7)a}$$

$$
\begin{aligned}
H_0^*(a,s) = {} & \frac{H_0(a)}{s+\sigma_7}\left(1-e^{-(s+\sigma_7)a}\right) + \frac{q(a)I_0(a)}{(s+\sigma_5)(s+\sigma_7)}\left(1-e^{-(s+\sigma_7)a}\right) \\
& + \frac{q(a)\sigma_6 E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{(s+\sigma_2)(s+\sigma_7)} - \frac{1}{(s+\sigma_5)(s+\sigma_7)}\right)\left(1-e^{-(s+\sigma_7)a}\right) \\
& - \frac{q(a)I_0(a)}{(\sigma_7-\sigma_5)(s+\sigma_5)}\left(e^{-(s+\sigma_5)a} - e^{-(s+\sigma_7)a}\right) \\
& - \frac{q(a)\sigma_6 E_0(a)}{\sigma_2\sigma_5\sigma_7}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-sa} - e^{-(s+\sigma_7)a}\right) \\
& + \frac{q(a)\sigma_6 E_0(a)}{\sigma_2\sigma_5(\sigma_7-\sigma_5)}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-(s+\sigma_5)a} - e^{-(s+\sigma_7)a}\right) \\
& + \frac{z(a)M_0(a)}{(s+\sigma_3)(s+\sigma_7)}\left(1-e^{-(s+\sigma_7)a}\right) \\
& + \frac{z(a)\sigma_4 E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{(s+\sigma_2)(s+\sigma_7)} - \frac{1}{(s+\sigma_3)(s+\sigma_7)}\right)\left(1-e^{-(s+\sigma_7)a}\right) \\
& - \frac{z(a)M_0(a)}{(\sigma_7-\sigma_3)(s+\sigma_3)}\left(e^{-(s+\sigma_3)a} - e^{-(s+\sigma_7)a}\right) \\
& - \frac{z(a)\sigma_4 E_0(a)}{(\sigma_7-\sigma_3)(\sigma_3-\sigma_2)}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_3}\right)\left(e^{-(s+\sigma_3)a} - e^{-(s+\sigma_7)a}\right) \\
& + \frac{z(a)\sigma_4 E_0(a)}{\sigma_2\sigma_3\sigma_7}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-sa} - e^{-(s+\sigma_7)a}\right) \\
& + \frac{z(a)\sigma_4 E_0(a)}{\sigma_2\sigma_3(\sigma_7-\sigma_3)}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-(s+\sigma_3)a} - e^{-(s+\sigma_7)a}\right) + f(s)e^{-(s+\sigma_7)a}
\end{aligned}
$$

$$H_0^*(0,s) = f(s) = 0$$

$$
\begin{aligned}
H_0^*(a,s) = {} & \frac{H_0(a)}{s+\sigma_7}\left(1-e^{-\sigma_7 a}.e^{-as}\right)+\frac{q(a)I_0(a)}{\sigma_7-\sigma_5}\left(\frac{1}{(s+\sigma_5)}-\frac{1}{(s+\sigma_7)}\right)\left(1-e^{-\sigma_7 a}.e^{-as}\right) \\
& +\frac{q(a)\sigma_6E_0(a)}{(\sigma_5-\sigma_2)}\left[\frac{1}{(\sigma_7-\sigma_2)}\left(\frac{1}{(s+\sigma_2)}-\frac{1}{(s+\sigma_7)}\right)\right. \\
& \left.-\frac{1}{(\sigma_7-\sigma_5)}\left(\frac{1}{(s+\sigma_5)}-\frac{1}{(s+\sigma_7)}\right)\right]\left(1-e^{-\sigma_7 a}.e^{-as}\right) \\
& -\frac{q(a)I_0(a)}{(\sigma_7-\sigma_5)(s+\sigma_5)}\left(e^{-\sigma_5 a}-e^{-\sigma_7 a}\right)e^{-as} \\
& -\frac{q(a)\sigma_6E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)}\left(e^{-\sigma_5 a}-e^{-\sigma_7 a}\right)\left(\frac{1}{(s+\sigma_2)}-\frac{1}{(s+\sigma_5)}\right)e^{-as} \\
& -\frac{q(a)\sigma_6E_0(a)}{\sigma_2\sigma_5\sigma_7}\left(1-e^{-\sigma_7 a}\right)\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as} \\
& +\frac{q(a)\sigma_6E_0(a)}{\sigma_2\sigma_5(\sigma_7-\sigma_5)}\left(e^{-\sigma_5 a}-e^{-\sigma_7 a}\right)\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as} \\
& +\frac{z(a)M_0(a)}{(\sigma_7-\sigma_3)}\left(\frac{1}{s+\sigma_3}-\frac{1}{s+\sigma_7}\right)\left(1-e^{-\sigma_7 a}.e^{-as}\right) \\
& +\frac{z(a)\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left[\frac{1}{(\sigma_7-\sigma_2)}\left(\frac{1}{(s+\sigma_2)}-\frac{1}{(s+\sigma_7)}\right)\right. \\
& \left.-\frac{1}{(\sigma_7-\sigma_3)}\left(\frac{1}{(s+\sigma_3)}-\frac{1}{(s+\sigma_7)}\right)\right]\left(1-e^{-\sigma_7 a}.e^{-as}\right) \\
& -\frac{z(a)M_0(a)}{(\sigma_7-\sigma_3)(s+\sigma_3)}\left(e^{-\sigma_3 a}-e^{-\sigma_7 a}\right)e^{-as} \\
& -\frac{z(a)\sigma_4E_0(a)}{(\sigma_7-\sigma_3)(\sigma_3-\sigma_2)}\left(e^{-\sigma_3 a}-e^{-\sigma_7 a}\right)\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)e^{-as} \\
& -\frac{z(a)\sigma_4E_0(a)}{\sigma_2\sigma_3\sigma_7}\left(1-e^{-\sigma_7 a}\right)\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as} \\
& +\frac{z(a)\sigma_4E_0(a)}{\sigma_2\sigma_3(\sigma_7-\sigma_3)}\left(e^{-\sigma_3 a}-e^{-\sigma_7 a}\right)\left(\frac{1}{s}-\frac{1}{s+\sigma_2}\right)e^{-as}
\end{aligned}
$$

So,

$$H_0(a,t) = L_t^{-1}\{H_0^*(a,s)\}$$

$$
\begin{aligned}
&= \mathrm{H}_0(\mathrm{a})\left(e^{-\sigma_7 \mathrm{t}} - e^{-\sigma_7 \mathrm{a}} u_a(t) e^{-\sigma_7(t-a)}\right) \\
&\quad + \frac{\mathrm{q(a)I_0(a)}}{\sigma_7 - \sigma_5}\left(e^{-\sigma_5 \mathrm{t}} - e^{-\sigma_7 \mathrm{t}} - e^{-\sigma_7 \mathrm{a}} u_a(t)\left(e^{-\sigma_5(t-a)} - e^{-\sigma_7(t-a)}\right)\right) \\
&\quad + \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)}\left[\frac{1}{(\sigma_7 - \sigma_2)}\left(e^{-\sigma_2 t} - e^{-\sigma_7 t}\right) - \frac{1}{(\sigma_7 - \sigma_5)}\left(e^{-\sigma_5 t} - e^{-\sigma_7 t}\right)\right. \\
&\quad - \frac{e^{-\sigma_7 \mathrm{a}}}{(\sigma_7 - \sigma_2)} u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_7(t-a)}\right) \\
&\quad \left. + \frac{e^{-\sigma_7 \mathrm{a}}}{(\sigma_7 - \sigma_5)} u_a(t)\left(e^{-\sigma_5(t-a)} - e^{-\sigma_7(t-a)}\right)\right] \\
&\quad - \frac{\mathrm{q(a)I_0(a)}}{(\sigma_7 - \sigma_5)}\left(e^{-\sigma_5 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t) e^{-\sigma_5(t-a)} \\
&\quad - \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)(\sigma_7 - \sigma_5)}\left(e^{-\sigma_5 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_5(t-a)}\right) \\
&\quad - \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{\sigma_2 \sigma_5 \sigma_7}\left(1 - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(1 - e^{-\sigma_2(t-a)}\right) \\
&\quad + \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{\sigma_2 \sigma_5(\sigma_7 - \sigma_5)}\left(e^{-\sigma_5 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(1 - e^{-\sigma_2(t-a)}\right) \\
&\quad + \frac{\mathrm{z(a)M_0(a)}}{(\sigma_7 - \sigma_3)}\left(e^{-\sigma_3 \mathrm{t}} - e^{-\sigma_7 \mathrm{t}} - e^{-\sigma_7 \mathrm{a}} u_a(t)\left(e^{-\sigma_3(t-a)} - e^{-\sigma_7(t-a)}\right)\right) \\
&\quad + \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_3 - \sigma_2)}\left[\frac{1}{(\sigma_7 - \sigma_2)}\left(e^{-\sigma_2 \mathrm{t}} - e^{-\sigma_7 \mathrm{t}}\right) - \frac{1}{(\sigma_7 - \sigma_3)}\left(e^{-\sigma_3 \mathrm{t}} - e^{-\sigma_7 \mathrm{t}}\right)\right. \\
&\quad - \frac{e^{-\sigma_7 \mathrm{a}}}{(\sigma_7 - \sigma_2)} u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_7(t-a)}\right) \\
&\quad \left. + \frac{e^{-\sigma_7 \mathrm{a}}}{(\sigma_7 - \sigma_3)} u_a(t)\left(e^{-\sigma_3(t-a)} - e^{-\sigma_7(t-a)}\right)\right] \\
&\quad - \frac{\mathrm{z(a)M_0(a)}}{(\sigma_7 - \sigma_3)}\left(e^{-\sigma_3 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t) e^{-\sigma_3(t-a)} \\
&\quad - \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_7 - \sigma_3)(\sigma_3 - \sigma_2)}\left(e^{-\sigma_3 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(e^{-\sigma_2(t-a)} - e^{-\sigma_3(t-a)}\right) \\
&\quad - \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{\sigma_2 \sigma_3 \sigma_7}\left(1 - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(1 - e^{-\sigma_2(t-a)}\right) \\
&\quad + \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{\sigma_2 \sigma_3(\sigma_7 - \sigma_3)}\left(e^{-\sigma_3 \mathrm{a}} - e^{-\sigma_7 \mathrm{a}}\right) u_a(t)\left(1 - e^{-\sigma_2(t-a)}\right) \qquad (112)
\end{aligned}
$$

Also, consider equation (112) in the form:

$$\frac{\partial \mathrm{V}_0}{\partial t} + \frac{\partial \mathrm{V}_0}{\partial a} + \mu(\mathrm{a})\mathrm{V}_0 = \sigma_1 \mathrm{S}_0 + \mathrm{n(a)H}_0 \qquad (113)$$

Applying the Laplace transform to all the terms in (113), we have

$$
\begin{aligned}
&\frac{dV_0^*(a,s)}{da} + \left(s + \mu(a)\right)V_0^*(a,s) \\
&\qquad = V_0(a) + \sigma_1\left[\frac{b(a)}{\sigma_0}\left(\frac{1}{s}\right) + \left(S_0(a) - \frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}\right) + \left(B - \frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s}e^{-(s+\sigma_0)a}\right)\right. \\
&\qquad \left. - \left(S_0(a) - \frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}e^{-(s+\sigma_0)a}\right)\right] \\
&\qquad + n(a)\left[\frac{H_0(a)}{s+\sigma_7}\left(1 - e^{-(s+\sigma_7)a}\right) + \frac{q(a)I_0(a)}{\sigma_7 - \sigma_5}\left(\frac{1}{(s+\sigma_5)} - \frac{1}{(s+\sigma_7)}\right)\left(1 - e^{-(s+\sigma_7)a}\right)\right. \\
&\qquad + \frac{q(a)\sigma_6 E_0(a)}{(\sigma_5 - \sigma_2)}\left[\frac{1}{(\sigma_7 - \sigma_2)}\left(\frac{1}{(s+\sigma_2)} - \frac{1}{(s+\sigma_7)}\right)\right. \\
&\qquad \left. - \frac{1}{(\sigma_7 - \sigma_5)}\left(\frac{1}{(s+\sigma_5)} - \frac{1}{(s+\sigma_7)}\right)\right]\left(1 - e^{-(s+\sigma_7)a}\right) \\
&\qquad - \frac{q(a)I_0(a)}{(\sigma_7 - \sigma_5)(s+\sigma_5)}\left(e^{-(s+\sigma_5)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad - \frac{q(a)\sigma_6 E_0(a)}{(\sigma_5 - \sigma_2)(\sigma_7 - \sigma_5)}\left(\frac{1}{(s+\sigma_2)} - \frac{1}{(s+\sigma_5)}\right)\left(e^{-(s+\sigma_5)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad - \frac{q(a)\sigma_6 E_0(a)}{\sigma_2\sigma_5\sigma_7}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-as} - e^{-(s+\sigma_7)a}\right) \\
&\qquad + \frac{q(a)\sigma_6 E_0(a)}{\sigma_2\sigma_5(\sigma_7 - \sigma_5)}\left(\frac{1}{s} - \frac{1}{s+\sigma_2}\right)\left(e^{-(s+\sigma_5)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad + \frac{z(a)M_0(a)}{(\sigma_7 - \sigma_3)}\left(\frac{1}{s+\sigma_3} - \frac{1}{s+\sigma_7}\right)\left(1 - e^{-(s+\sigma_7)a}\right) \\
&\qquad + \frac{z(a)\sigma_4 E_0(a)}{(\sigma_3 - \sigma_2)}\left[\frac{1}{(\sigma_7 - \sigma_2)}\left(\frac{1}{(s+\sigma_2)} - \frac{1}{(s+\sigma_7)}\right)\right. \\
&\qquad \left.\left. - \frac{1}{(\sigma_7 - \sigma_3)}\left(\frac{1}{(s+\sigma_3)} - \frac{1}{(s+\sigma_7)}\right)\right]\left(1 - e^{-(s+\sigma_7)a}\right)\right] \\
&\qquad - \frac{z(a)M_0(a)}{(\sigma_7 - \sigma_3)(s+\sigma_3)}\left(e^{-(s+\sigma_2)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad - \frac{z(a)\sigma_4 E_0(a)}{(\sigma_7 - \sigma_3)(\sigma_3 - \sigma_2)}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_3}\right)\left(e^{-(s+\sigma_3)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad - \frac{z(a)\sigma_4 E_0(a)}{(\sigma_7 - \sigma_2)(\sigma_3 - \sigma_2)}\left(\frac{1}{s+\sigma_2}\right)\left(e^{-(s+\sigma_2)a} - e^{-(s+\sigma_7)a}\right) \\
&\qquad + \frac{z(a)\sigma_4 E_0(a)}{(\sigma_7 - \sigma_3)(\sigma_3 - \sigma_2)}\left(\frac{1}{s+\sigma_2}\right)\left(e^{-(s+\sigma_3)a} - e^{-(s+\sigma_7)a}\right) \qquad (114)
\end{aligned}
$$

Solving equation (114) we obtained

$$
\begin{aligned}
V_0^*(\mathrm{a},\mathrm{s}) = e^{-(\mathrm{s}+\mu(a))\mathrm{a}} \int_0^{\mathrm{a}} e^{(\mathrm{s}+\mu(a))\mathrm{x}} \Bigg[& V_0(a) + \sigma_1 \left[\frac{\mathrm{b(a)}}{\sigma_0}\left(\frac{1}{s}\right) + \left(\mathrm{S_0(a)} - \frac{\mathrm{b(a)}}{\sigma_0}\right)\left(\frac{1}{\mathrm{s}+\sigma_0}\right) \right. \\
& \left. + \left(\mathrm{B} - \frac{\mathrm{b(a)}}{\sigma_0}\right)\left(\frac{1}{s} e^{-(\mathrm{s}+\sigma_0)\mathrm{x}}\right) - \left(\mathrm{S_0(a)} - \frac{\mathrm{b(a)}}{\sigma_0}\right)\left(\frac{1}{\mathrm{s}+\sigma_0} e^{-(\mathrm{s}+\sigma_0)\mathrm{x}}\right) \right] \\
& + \mathrm{n(a)} \left[\frac{\mathrm{H_0(a)}}{\mathrm{s}+\sigma_7}\left(\frac{1}{(\mathrm{s}+\sigma_5)} - \frac{1}{(\mathrm{s}+\sigma_7)}\right)\left(1 - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \right. \\
& + \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)} \left[\frac{1}{(\sigma_7 - \sigma_2)}\left(\frac{1}{(\mathrm{s}+\sigma_2)} - \frac{1}{(\mathrm{s}+\sigma_7)}\right) \right. \\
& \left. - \frac{1}{(\sigma_7 - \sigma_5)}\left(\frac{1}{(\mathrm{s}+\sigma_5)} - \frac{1}{(\mathrm{s}+\sigma_7)}\right) \right] \left(1 - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& - \frac{\mathrm{q(a)I_0(a)}}{(\sigma_7 - \sigma_5)(\mathrm{s}+\sigma_5)}\left(e^{-(\mathrm{s}+\sigma_5)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& - \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)(\sigma_7 - \sigma_5)}\left(\frac{1}{(\mathrm{s}+\sigma_2)} - \frac{1}{(\mathrm{s}+\sigma_5)}\right)\left(e^{-(\mathrm{s}+\sigma_5)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& - \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)(\sigma_7 - \sigma_5)}\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\left(e^{-(\mathrm{s}+\sigma_2)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& + \frac{\mathrm{q(a)}\sigma_6 \mathrm{E_0(a)}}{(\sigma_5 - \sigma_2)(\sigma_7 - \sigma_5)}\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\left(e^{-(\mathrm{s}+\sigma_5)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& + \frac{\mathrm{z(a)M_0(a)}}{(\sigma_7 - \sigma_3)}\left(\frac{1}{\mathrm{s}+\sigma_3} - \frac{1}{\mathrm{s}+\sigma_7}\right)\left(1 - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& + \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_3 - \sigma_2)} \left[\frac{1}{(\sigma_7 - \sigma_2)}\left(\frac{1}{(\mathrm{s}+\sigma_2)} - \frac{1}{(\mathrm{s}+\sigma_7)}\right) \right. \\
& \left. \left. - \frac{1}{(\sigma_7 - \sigma_3)}\left(\frac{1}{(\mathrm{s}+\sigma_3)} - \frac{1}{(\mathrm{s}+\sigma_7)}\right) \right] \left(1 - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \right] \\
& - \frac{\mathrm{z(a)M_0(a)}}{(\sigma_7 - \sigma_3)(\mathrm{s}+\sigma_3)}\left(e^{-(\mathrm{s}+\sigma_2)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& - \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_7 - \sigma_3)(\sigma_3 - \sigma_2)}\left(\frac{1}{\mathrm{s}+\sigma_2} - \frac{1}{\mathrm{s}+\sigma_3}\right)\left(e^{-(\mathrm{s}+\sigma_3)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& - \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_7 - \sigma_2)(\sigma_3 - \sigma_2)}\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\left(e^{-(\mathrm{s}+\sigma_2)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \\
& + \frac{\mathrm{z(a)}\sigma_4 \mathrm{E_0(a)}}{(\sigma_7 - \sigma_3)(\sigma_3 - \sigma_2)}\left(\frac{1}{\mathrm{s}+\sigma_2}\right)\left(e^{-(\mathrm{s}+\sigma_3)\mathrm{x}} - e^{-(\mathrm{s}+\sigma_7)\mathrm{x}}\right) \Bigg] dx + f(s) e^{-(\mathrm{s}+\mu(a))\mathrm{a}}
\end{aligned}
$$

$$V_0^*(a,s) = \frac{1}{s+\mu(a)}\Bigg[V_0(a) + \sigma_1\left(\frac{b(a)}{\sigma_0}\left(\frac{1}{s}\right) + \left(S_0(a) - \frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}\right)\right)\left(1-e^{-(s+\mu(a))a}\right)$$
$$+ \frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(B - \frac{b(a)}{\sigma_0}\right)\frac{1}{s}\left(e^{-(s+\sigma_0)a} - e^{-(s+\mu(a))a}\right)$$
$$- \frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(S_0(a) - \frac{b(a)}{\sigma_0}\right)\frac{1}{(s+\sigma_0)}\left(e^{-(s+\sigma_0)a}\right.$$
$$\left. - e^{-(s+\mu(a))a}\right)\frac{n(a)H_0(a)}{(s+\sigma_7)}\left(\frac{1}{(s+\mu(a))}\left(1-e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$
$$+ \frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)}\left(\frac{1}{s+\sigma_5} - \frac{1}{s+\sigma_7}\right)\left(\frac{1}{(s+\mu(a))}\left(1-e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$
$$+ \frac{n(a)q(a)\sigma_6 E_0(a)}{(\sigma_5-\sigma_2)}\left(\frac{1}{(\sigma_7-\sigma_2)}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_7}\right)\right.$$
$$- \frac{1}{(\sigma_7-\sigma_5)}\left(\frac{1}{s+\sigma_5} - \frac{1}{s+\sigma_7}\right)\left(\frac{1}{(s+\mu(a))}\left(1-e^{-(s+\mu(a))a}\right)\right.$$
$$\left.\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)\right)$$
$$- \frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(s+\sigma_5)}\left(\frac{1}{(\mu(a)-\sigma_5)}\left(e^{-(s+\sigma_5)a} - e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$
$$- \frac{n(a)q(a)\sigma_6 E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)}\left(\frac{1}{s+\sigma_2} - \frac{1}{s+\sigma_5}\right)\left(\frac{1}{(\mu(a)-\sigma_5)}\left(e^{-(s+\sigma_5)a} - e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$
$$- \frac{n(a)q(a)\sigma_6 E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)}\left(\frac{1}{s+\sigma_2}\right)\left(\frac{1}{(\mu(a)-\sigma_2)}\left(e^{-(s+\sigma_2)a} - e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$
$$+ \frac{n(a)q(a)\sigma_6 E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)}\left(\frac{1}{s+\sigma_2}\right)\left(\frac{1}{(\mu(a)-\sigma_5)}\left(e^{-(s+\sigma_5)a} - e^{-(s+\mu(a))a}\right)\right.$$
$$\left. - \frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a} - e^{-(s+\mu(a))a}\right)\right)$$

$$+\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)}\left(\frac{1}{s+\sigma_3}-\frac{1}{s+\sigma_7}\right)\left(\frac{1}{\left(s+\mu(a)\right)}\left(1-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)$$
$$+\frac{n(a)z(a)\sigma_4E_0(a)}{(\sigma_3-\sigma_2)}\left(\frac{1}{\sigma_7-\sigma_2}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_7}\right)\right.$$
$$\left.-\frac{1}{\sigma_7-\sigma_3}\left(\frac{1}{s+\sigma_3}-\frac{1}{s+\sigma_7}\right)\right)\left(\frac{1}{\left(s+\mu(a)\right)}\left(1-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)$$
$$-\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(s+\sigma_3)}\left(\frac{1}{(\mu(a)-\sigma_3)}\left(e^{-(s+\sigma_3)a}-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)$$
$$-\frac{n(a)z(a)\sigma_4E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\sigma_3}\right)\left(\frac{1}{(\mu(a)-\sigma_3)}\left(e^{-(s+\sigma_3)a}-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)$$
$$-\frac{n(a)z(a)\sigma_4E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)}\left(\frac{1}{s+\sigma_2}\right)\left(\frac{1}{\left(s+\mu(a)\right)}\left(e^{-(s+\sigma_2)a}-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)$$
$$+\frac{n(a)z(a)\sigma_4E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)}\left(\frac{1}{s+\sigma_2}\right)\left(\frac{1}{(\mu(a)-\sigma_3)}\left(e^{-(s+\sigma_3)a}-e^{-\left(s+\mu(a)\right)a}\right)\right.$$
$$\left.\left.-\frac{1}{(\mu(a)-\sigma_7)}\left(e^{-(s+\sigma_7)a}-e^{-\left(s+\mu(a)\right)a}\right)\right)+f(s)e^{-\left(s+\mu(a)\right)a}\right]$$

$V_0^*(0,s)=f(s)=0$

$$,V_0^*(a,s)=V_0(a)\left(\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{\sigma_1b(a)}{(\mu(a)-\sigma_0)}\left(\frac{1}{s}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)\left(\frac{1}{s+\sigma_0}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_0}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{n(a)H_0(a)}{(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}\left(\frac{1}{s+\sigma_5}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-\frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{n(a)\sigma_6q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-\frac{n(a)\sigma_6q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-\frac{n(a)\sigma_6q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}\left(\frac{1}{s+\sigma_5}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+$$

$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}\left(\frac{1}{s+\sigma_3}-\right.$$
$$\left.\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+\right.$$
$$\left.e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}\left(\frac{1}{s+\sigma_2}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-$$
$$\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)-$$
$$\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}\left(\frac{1}{s+\sigma_3}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+$$
$$\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(\frac{1}{s+\sigma_7}-\frac{1}{s+\mu(a)}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\mu(a)}\right)+\frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(B-\right.$$
$$\left.\frac{b(a)}{\sigma_0}\right)e^{-\sigma_0 a}\frac{e^{-as}}{s}-\frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(B-\frac{b(a)}{\sigma_0}\right)e^{-\mu(a)a}\frac{e^{-as}}{s}-\frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)e^{-\sigma_0 a}\frac{e^{-as}}{s+\sigma_0}+$$
$$\frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(S_0(a)-\frac{b(a)}{\sigma_0}\right)e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_0}-\frac{n(a)H_0(a)}{(\mu(a)-\sigma_7)}e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+\frac{n(a)H_0(a)}{(\mu(a)-\sigma_7)}e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}-$$
$$\frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_5}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}-e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)+$$
$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)-$$
$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_5}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}-e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)+$$
$$\frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_5}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}\right)+\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-\right.$$
$$\left.e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_5}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}\right)+\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-$$
$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_3}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}-\right.$$
$$\left.e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)-\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+\right.$$
$$\left.e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)+\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_3}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}-e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_7}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_7}\right)+$$
$$\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_3}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}\right)+\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-\right.$$
$$\left.e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_3}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}\right)+\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-$$
$$\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}\left(e^{-\sigma_7 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)+\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}\left(e^{-\sigma_5 a}\frac{e^{-as}}{s+\sigma_2}-\right.$$
$$\left.e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-\frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}\left(e^{-\sigma_5 a}\frac{e^{-as}}{s+\sigma_5}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}\right)-$$
$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}\left(e^{-\sigma_5 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-e^{-\sigma_5 a}\frac{e^{-as}}{s+\sigma_5}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_5}\right)-$$
$$\frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}\left(e^{-\sigma_2 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}\left(e^{-\sigma_2 a}\frac{e^{-as}}{s+\sigma_2}-\right.$$
$$\left.e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)+\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}\left(e^{-\sigma_3 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}\right)-$$
$$\frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}\left(e^{-\sigma_3 a}\frac{e^{-as}}{s+\sigma_2}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_2}-e^{-\sigma_3 a}\frac{e^{-as}}{s+\sigma_3}+e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}\right)-$$
$$\frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}\left(e^{-\sigma_3 a}\frac{e^{-as}}{s+\sigma_3}-e^{-\mu(a)a}\frac{e^{-as}}{s+\sigma_3}\right)$$

So

$$V_0(a,t)=L_t^{-1}\{V_0^*(a,s)\}=V_0(a)\left(e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{11}\left(1-e^{-\mu(a)t}-\right.$$
$$\left.u_a(t)e^{-\mu(a)a}+e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{12}\left(e^{-\sigma_0 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_0(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{13}\left(e^{-\sigma_7 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{14}\left(e^{-\sigma_5 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)-B_{15}\left(e^{-\sigma_7 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{16}\left(e^{-\sigma_2 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)-B_{17}\left(e^{-\sigma_7 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)-B_{18}\left(e^{-\sigma_5 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}+\right.$$
$$\left.e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}\right)+B_{19}\left(e^{-\sigma_7 t}-e^{-\mu(a)t}-e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}+\right.$$

$$
\begin{aligned}
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) + B_{20}(e^{-\sigma_3 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) - B_{21}(e^{-\sigma_7 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) + B_{22}(e^{-\sigma_2 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) - B_{23}(e^{-\sigma_7 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) - B_{24}(e^{-\sigma_3 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) + B_{25}(e^{-\sigma_7 t} - e^{-\mu(a)t} - e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\mu(a)(t-a)}) + B_{26}e^{-\sigma_0 a}u_a(t) - B_{26}e^{-\mu(a)a}u_a(t) - B_{12}e^{-\sigma_0 a}u_a(t)e^{-\sigma_0(t-a)} + \\
& B_{12}e^{-\mu(a)a}u_a(t)e^{-\sigma_0(t-a)} - B_{13}e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + B_{13}e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)} - \\
& B_{15}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_5(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) + B_{17}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) - B_{17}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_5(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) + \\
& B_{15}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_5(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}) + B_{19}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_5(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}) + \\
& B_{17}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - B_{19}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - B_{21}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_3(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)} - \\
& e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) - B_{23}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) + \\
& B_{25}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_3(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_7(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_7(t-a)}) + B_{21}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_3(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)}) + \\
& B_{25}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\sigma_7 a}u_a(t)e^{-\sigma_3(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)}) + B_{23}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - \\
& B_{25}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) + B_{18}(e^{-\sigma_7 a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - B_{14}(e^{-\sigma_5 a}u_a(t)e^{-\sigma_5(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}) - \\
& B_{18}(e^{-\sigma_5 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\sigma_5 a}u_a(t)e^{-\sigma_5(t-a)} + \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_5(t-a)}) - B_{16}(e^{-\sigma_2 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - \\
& B_{22}(e^{-\sigma_2 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) + B_{24}(e^{-\sigma_3 a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)}) - B_{24}(e^{-\sigma_3 a}u_a(t)e^{-\sigma_2(t-a)} - e^{-\mu(a)a}u_a(t)e^{-\sigma_2(t-a)} - \\
& e^{-\sigma_3 a}u_a(t)e^{-\sigma_3(t-a)} + e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)}) - B_{20}(e^{-\sigma_3 a}u_a(t)e^{-\sigma_3(t-a)} - \\
& e^{-\mu(a)a}u_a(t)e^{-\sigma_3(t-a)})
\end{aligned}
$$

$$B_{11} = \frac{\sigma_1 b(a)}{(\mu(a)-\sigma_0)}, B_{12} = \frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(S_0(a) - \frac{b(a)}{\sigma_0}\right), B_{13} = \frac{n(a)H_0(a)}{(\mu(a)-\sigma_7)}, B_{14} = \frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}$$

$$B_{15} = \frac{n(a)q(a)I_0(a)}{(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}, B_{16} = \frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}, B_{17} = \frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}$$

$$B_{18} = \frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_5)}, B_{19} = \frac{n(a)\sigma_6 q(a)E_0(a)}{(\sigma_5-\sigma_2)(\sigma_7-\sigma_5)(\mu(a)-\sigma_7)}, B_{20} = \frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}$$

$$B_{21} = \frac{n(a)z(a)M_0(a)}{(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}, B_{22} = \frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_2)}, B_{23} = \frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_2)(\mu(a)-\sigma_7)}$$

$$B_{24} = \frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_3)}, B_{25} = \frac{n(a)\sigma_4 z(a)E_0(a)}{(\sigma_3-\sigma_2)(\sigma_7-\sigma_3)(\mu(a)-\sigma_7)}, B_{26} = \frac{\sigma_1}{(\mu(a)-\sigma_0)}\left(B - \frac{b(a)}{\sigma_0}\right)$$

Table 2.2: Values for Variables used for the Graphical Presentation

Variables	Values per year	Source
$S(a,0)$	2000	Assumed
$V(a,0)$	1000	Assumed
$E(a,0)$	1800	Assumed
$M(a.0)$	900	Assumed
$I(a,0)$	600	Assumed
$H(a,0)$	100	Assumed
$P(a,0)$	6400	Estimated

Table 2.3: Values for Parameters used for the Graphical Presentation

Parameters	Value	Description	Unit	Source
$\beta(a)$	0.0404	Transmission rate	/Day	Wang et al. (2021)
$b(a)$	0.075	Recruitment rate for all ages	/Day	Assumed
a	[0, 80]	Age of individual at time t	/Year	Assumed
$\alpha(a)$	0.01	Force of mortality rate	/Day	Assumed
$g(a)$	0.033	Contact ratio	/Day	Assumed
$\phi(a)$	0.85	Vaccinated rate for susceptible individuals	/Year	Signorelli and Odone (2020)
$\mu(a)$	0.018	Natural death rate	/Year	Assumed
$\theta(a)$	0.1923	Exit rate from latent class	/Year	Wang et al. (2021)
$n(a)$	0.85	Vaccinated rate for hospitalized individuals	/Year	Signorelli and Odone (2020)
$q(a)$	0.06	Hospitalized rate of symptomatic individuals	/Day	Wang et al. (2021)
$z(a)$	0.04	Hospitalized rate of asymptomatic individuals	/Day	Wang et al. (2021)

RESULTS AND DISCUSSION

Simulation Graphs

Graphical representations showing the variations in human population in relation to age a and time t are provided in Figures below;

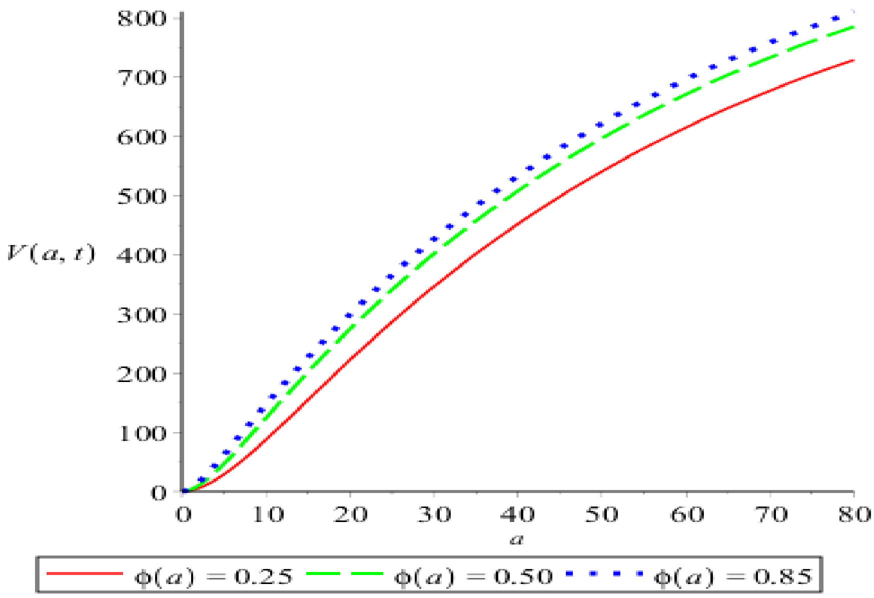

Figure 1: Plot of vaccinated population versus age a

Figure 1 Here, we found that an increase in vaccination rate $\phi(a)$, lead to corresponding increase in vaccinated population across different age group. Figure 1 also indicates that up to 85% of world populace between the age of 0 to 80 accepted vaccination campaign despite the propaganda on the COVID-19 vaccines by both some health workers and the society.

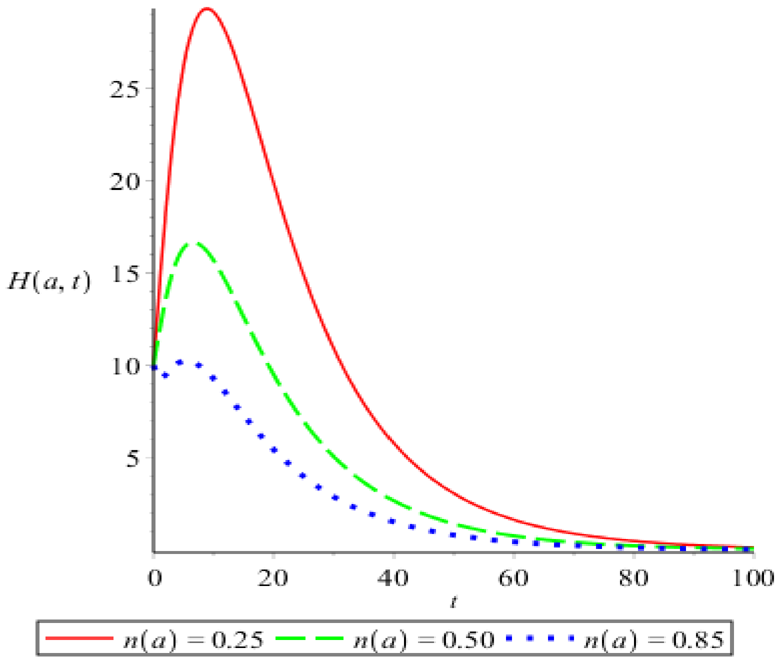

Figure 2: Plot of hospitalized population versus time t

Figure 2 showed that there is a decrease in the hospitalized population as vaccination rate $n(a)$ of recovered individuals increases over time. This occurs because as hospitalized people received vaccines after treatment, they gain immunity. Hence, reduce the population of the susceptible class

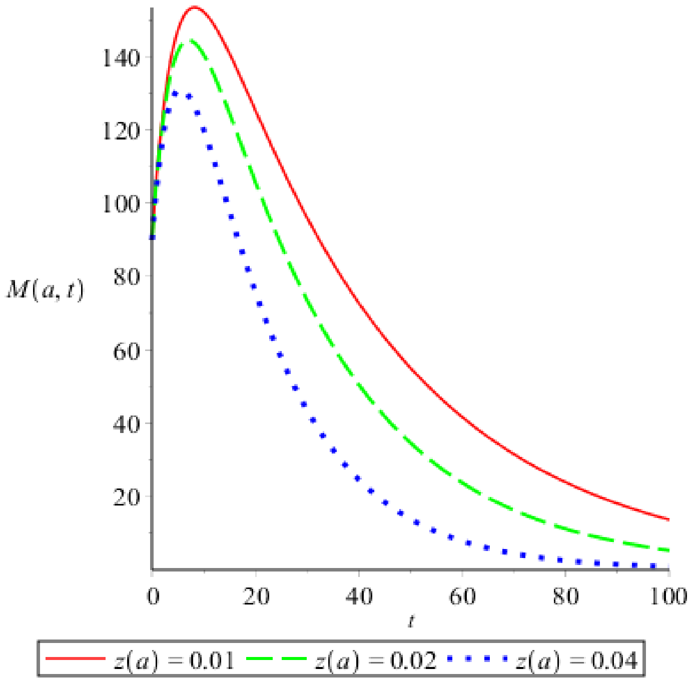

Figure 3: Plot of asymptomatic infectious population versus time t

Figure 3 showed that there is a decrease in the asymptomatic infectious population over time as asymptomatic hospitalized rate $z(a)$ increases. This occurs because asymptomatic infected individuals who are detected are hospitalized, treated and vaccinated after recovery.

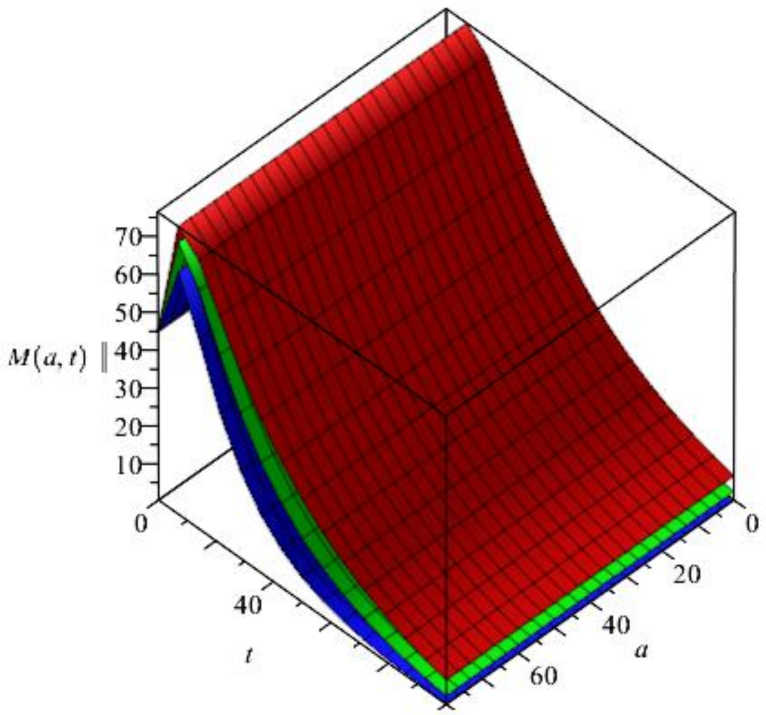

Figure 4: Plot of asymptomatic infectious population versus age a and time t

Figure 2.4 showed that there is a decrease in the asymptomatic infectious class over time and across all ages as the rate of asymptomatic hospitalized increases. This occurs because asymptomatic infected individuals who are detected are hospitalized, treated as a result of medical attention are vaccinated and move to vaccination class, hence increasing the vaccinated population and reducing the susceptible class which in turns lead to the decrease in the number of individuals who become exposed to the disease. Figure 2.4 also revealed that this rate of movement of asymptomatic infectious individuals into the hospitalized class for treatment is not influenced by age. This means that asymptomatic infectious individuals are detected and treated equally across all age groups.

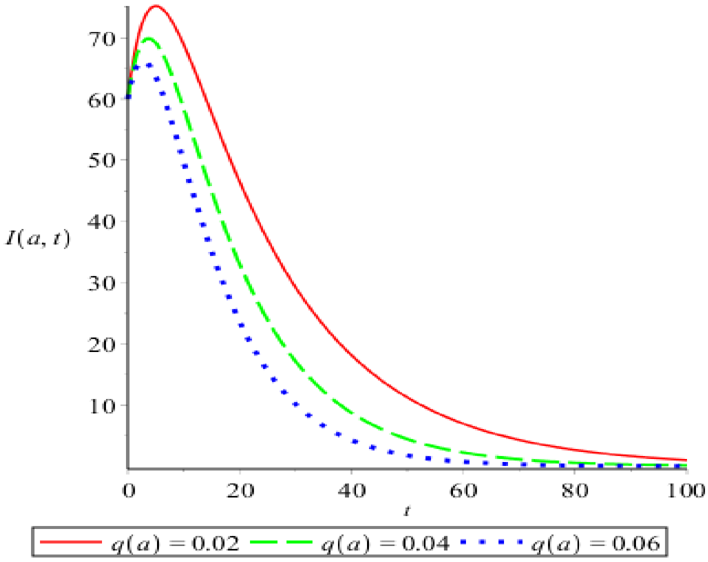

Figure 5: Plot of symptomatic infectious population versus time t

Figure 5 Here, we found that there is a decrease in the symptomatic infectious population as symptomatic hospitalized rate $q(a)$ increases. This occurs because symptomatic infectious individuals are hospitalized, treated as a result of medical attention are vaccinated.

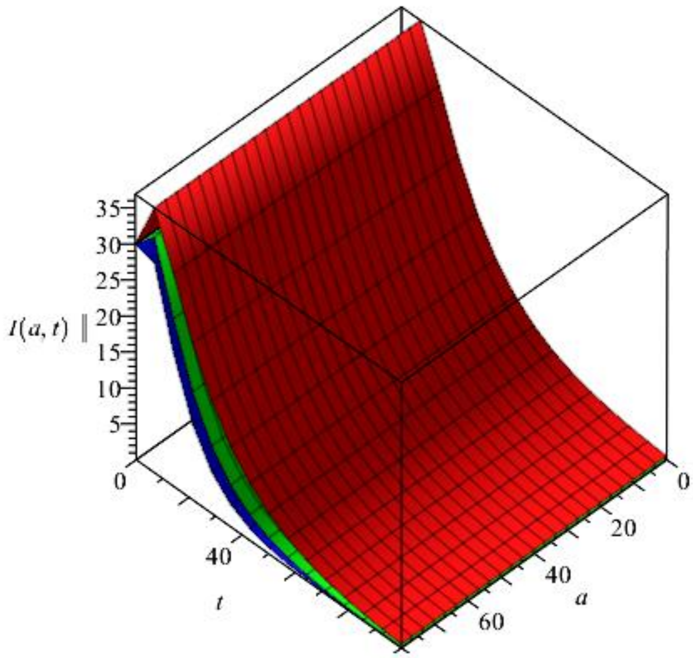

Figure 6: Plot of symptomatic infectious population versus age a and time t

Figure 2.6 showed that there is a decrease in the symptomatic infectious class over time and across all ages as symptomatic hospitalized rate increases. This occurs because symptomatic individuals are hospitalized, treated, get vaccinated and move to vaccination class. Figure.6 also revealed that the rate of hospitalization of symptomatic infectious individuals has nothing to do with age of people.

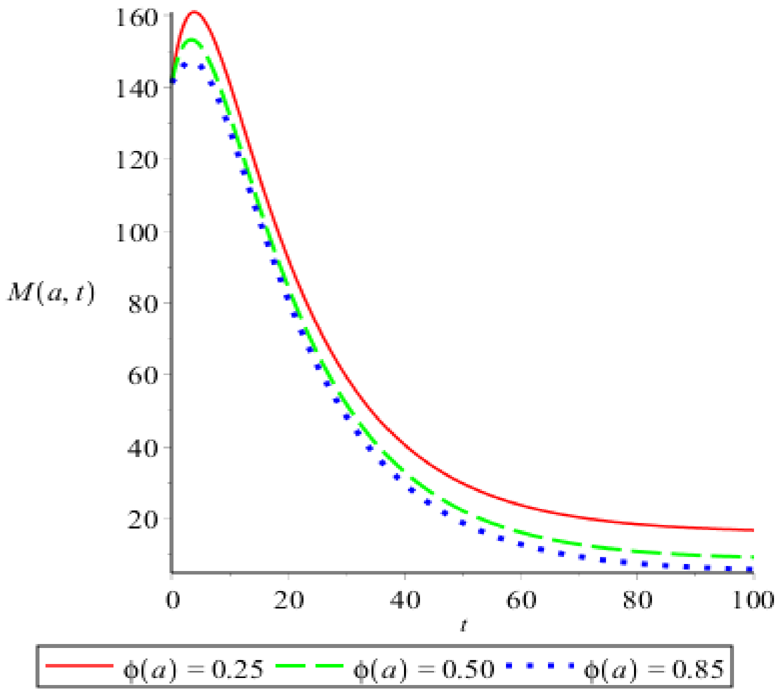

Figure 7: Plot of asymptomatic infectious population versus time t

Figure 7 showed that there is a decrease in the asymptomatic infectious class as vaccination rate $\phi(a)$ of susceptible individuals increases. This suggests that vaccinating more people helps reduce the number of asymptomatic carriers.

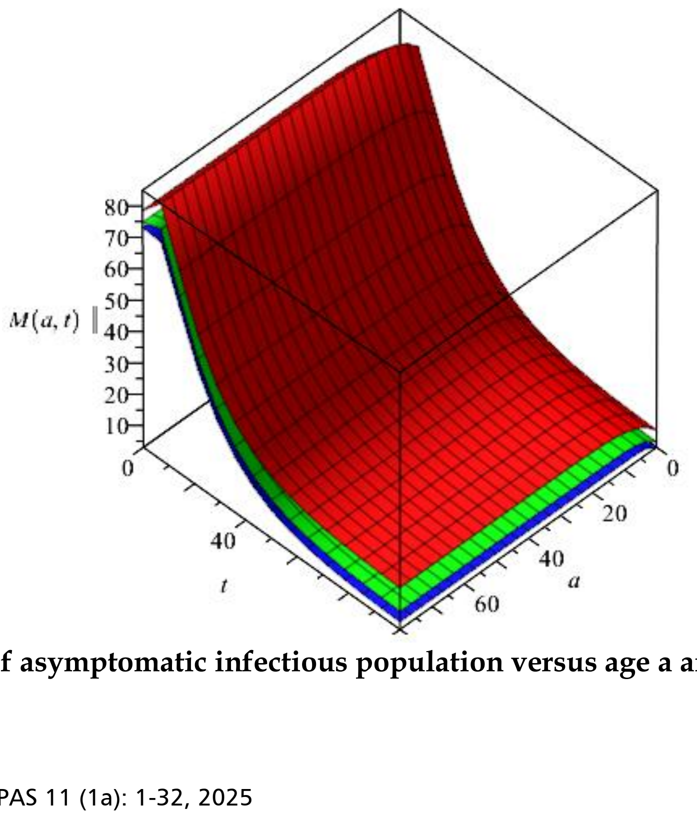

Figure 8: Plot of asymptomatic infectious population versus age a and time t

Figure8 showed that over time, there is a decrease in the asymptomatic infectious population across all ages as the vaccination rate of susceptible individuals increases. This occurs as a result of decrease in the number of both the susceptible and exposed population as vaccination rate of susceptible individuals increases.

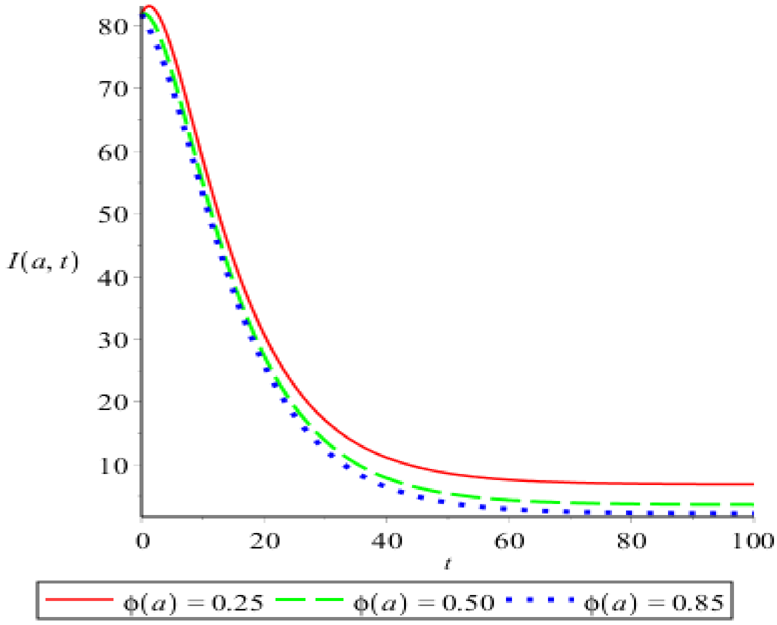

Figure 9: Plot of symptomatic infectious population versus time t

Figure 9 showed that there is a decrease in the symptomatic infectious class as vaccination rate $\phi(a)$ of susceptible individuals increases.

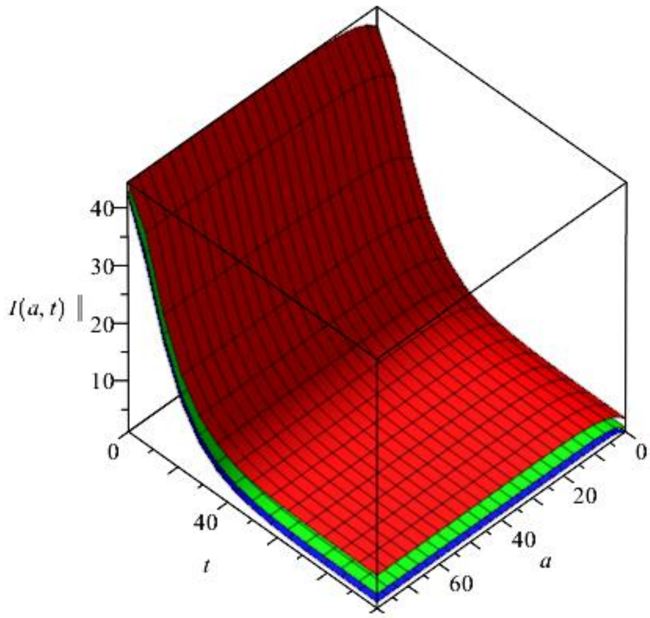

Figure 10: Plot of symptomatic infectious population versus age a and time t

Figure 10 showed that there is a decrease in the symptomatic infectious population over time and across all ages as the vaccination rate of susceptible individuals increases. This occurs as a result of decrease in the number of both the susceptible and exposed population as vaccination rate of susceptible individuals increases.
From figure 7 – 10, the infectious compartments decrease towards 0 as the vaccination rate

increases to 85%. This indicates that vaccination is effective in reducing the number of infectious classes due to fewer people being susceptible and exposed to the disease as more get vaccinated. Furthermore, figure 2-10 suggest that medical interventions, such as hospitalization and vaccination, effectively reduce the number of infectious compartments. Simulations results obtained showed that the nationwide eradication of COVID-19 can be done if at least 85% of the population is vaccinated.

Conclusion

In this research, developed and analyzed an age-structured SVEMIH epidemic model for COVID-19 transmission dynamics, incorporating vaccinated and hospitalized compartments. Determined the steady states of the model and established local stability conditions for the disease-free equilibrium (DFE). Also derived the basic reproduction number and proved that the DFE is locally asymptotically stable under specific conditions.

Numerical simulations, performed using the Laplace transform method, illustrated how different parameters affect disease dynamics. Our results show that COVID-19 can be controlled through vaccination and hospitalization strategies. Both measures significantly reduce the spread of the disease, with perfect vaccination and treatment not only lowering the peak of outbreaks but also shortening the disease duration, as demonstrated in figures 10. Additionally, figure 2 emphasizes the role of boosting immunity post-treatment, showing that increased vaccination among recovered individuals reduces the hospitalized population. Overall, this study concludes that COVID-19 vaccines, along with hospitalization and treatment of infected individuals, are crucial for effectively controlling and eradicating the disease within a population.

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