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Comparative Study of the Solution of 4th Order Ordinary Differential Equation Using Some Integral Transforms

Aregbesola W. A.[1] , Ekundayo T. D.[2] , Olaniyan A. S.[3] , Vigbe B. S.[4]
[1,2,3]Departments of Mathematics,
Lagos State University,
Ojo, Nigeria.

[4]Departments of Mathematics,
Alvan Ikoku Federal University of Education,
Owerri,
Nigeria.

Email: Adegoke.olaniyan@lasu.edu.ng

Abstract
In mathematics, physics, and engineering, choosing the most effective and sufficient method for the integral transform is crucial since solving any given problem requires finding a precise and distinct solution. Since differential equations are particularly useful for building models in the scientific domain and ordinary differential equations can be solved in a variety of ways. This paper will use Laplace and some updated integral transforms such as Elzaki, Shehu, Aboodh, and Shehu transforms to solve fourth-order ordinary differential equations in order to demonstrate its high accuracy, simplicity, and efficiency.

Keywords: Differential Equations, Elzaki transform , Shehu transform , Aboodh transform , Shehu transform

INTRODUCTION

Differential Equations have become essential studies that play a central role in applied mathematics, physics and engineering. These differential equations are generally difficult to solve and their exact solution are not easy to get. The exact solution and numerical solutions of these differential equations play important role in various fields hence, researchers have developed new methods to obtain analytical solutions that will moderately approximate the exact solutions. In interest of solving these differential equations, integral transforms have been largely used. Integral transforms are constructed from the classical Fourier integral defined by

$$T[f(v)] = \int_{v_1}^{v_2} k\,(s,v)f(v)\,dv\text{s}$$

where the function k(s,v) represents the kernel of a transform, and provided the integral exists. Researchers have introduced new integral transforms such as Laplace, Fourier, Sumudu, Aboodh, Elzaki and Mellin to mention a few as may be distinguished by carefully selecting the kernel and the range of v.

Many integral transforms have been used in the past few years to solve differential and integral equations. The transforms that are most frequently employed in the literature are the Fourier integral and Laplace transforms. The French mathematician Joseph Fourier was

Author for Correspondence

named after Fourier integral transform.The definition of the Fourier integral transforms mathematically is as follows:

$$T[f(t)] = f(w) = [1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt$$

The Laplace integral transform is defined as

$$L[f(t)] = f(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

and is comparable to the Fourier transform. For the purpose of solving a certain class of ordinary and partial differential equations, the Laplace transform is incredibly effective. The Fourier equation can be used to convert the well-known Fourier transform into a Laplace transform and vice versa by substituting the variables for the variable *iw*. In 1993, Watugala introduced a Laplace like integral transform called the Sumudu integral transform. In recent years, Sumudu transform has been applied to many real life problems because of its scale and it preserving properties. The Sumudu transform can be defined mathematically as follows

$$S[f(t)] = G(u) = \frac{1}{u}\int_{0}^{\infty} e^{\frac{-t}{u}} f(t)\, dt$$

The Elzaki transform was first introduced in 2011 and is based on the fundamental concepts of the Laplace and Sumudu integral transforms. The Laplace, Sumudu and natural transforms are intimately associated with the Elzaki transform and is given as

$$E[f(t)] = T(u) = u\int_{0}^{\infty} e\, xp^{\frac{-t}{u}} f(t)\, dt$$

The Fourier integral serves as the basics for the Aboodh transform. Based on the basic characteristics of the Aboodh transform and its simplicity in mathematics, Khalid Aboodh devised the Aboodh transform to speed up the process of solving partial and ordinary differential equations in the time domain defined by the integral equation

$$A[f(t)] = k(v) = \frac{1}{v}\int_{0}^{\infty} f\,(t) e^{-vt} f(t)\, dt$$

In order to solve ordinary differential equations, partial differential equations, integral equations, integro differential equations, partial integro differential equations and delay differential equations, numerous scholars employed these transforms. As a result of the rapid development in science and engineering, many other integral transforms have been developed to deal with this development. However, most of these existing integral transforms have shortcomings and they cannot be directly used to solve non-linear problems or many complex mathematical models of high order. Consequently, researchers became interested to come up with alternative approach to deal with real life problem. In Shehu and Weidong (2019), Laplace type transform called the Shehu transform which is a generalization of the Laplace and Sumudu integral transform for solving differential equations was introduced . In the research work of Katre and Katre (2021), two integral transforms namely Kamal transform and the very famous Laplace transform were studied comparatively. The application of these transforms to solve linear difference equations were demonstrated. Elzaki (2011) developed a new integral transform called Elzaki transform and used it to solve the linear ordinary differential equations. The results of Elzaki transform were compared with the well known integral transform the Laplace transform. Gurpreet and Inderdeep (2020) presented a numerical method based on Laplace transform for solving fourth order ordinary differential equations. Numerical results showed the accuracy of the proposed method by comparing the results with exact solutions. Aboodh (2014) developed a new integral transform namely Aboodh transform which was applied to solve linear ordinary differential equations. Abdebagy *et al* (2016) discussed some relationship between the same integral transform and Elzaki transform. In this study, a comparative study involving Laplace, Sumudu, Aboodh, Elzaki and Shehu to solve 4th order ordinary differential equation will be discussed and analyzed.

MATERIAL AND METHODS

Laplace Transforms (L.T.)

Suppose that f is real or complex valued function of the true variable and s is a real or complex parameter, we define the Laplace transform of f as

$$F(s)] = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral exists. If $F(t) = L^{-1}F(s)$, then the function $F(t)$ is called the inverse Laplace transform of F(s)

Derivative of Laplace Transform (L.T.)

$$L[f'] = SL[f(t)] - f(0)$$
$$L[f''] = S^2L[f(t)] - Sf(0) - f^I(0)$$
$$L[F'''] = S^3L[F(t)] - S^2f(0) - Sf'(0) - f''(0)$$
$$L[F^{iv}] = S^3L[F(t)] - S^3f(0) - S^2f'(0) - Sf''(0) - f'''(0)$$

Elzaki Transform (E.T.)

A new integral transform called the Elazaki transform defined for the functions of exponential order, we consider functions in the set A

$$A = \left[f(t){:}\,\exists M, K_1, K_2 \geq 0, |f(t)| \leq Me^{\frac{|t|}{k_j}}, if t \in (-1)^j x[0,\infty)\right]$$

for a given function in the set A. The constant M must be finite number k_1, k_2 maybe finite or infinite. The Elzaki transform of function $f(t) \in A$ denoted by $E[f(t)]$ or $T(V)$ is defined by the integral equation

$$E[f(t)] = T(v) = v\int_0^\infty e\,xp^{\frac{-t}{u}} f(t)\, dt\ t \geq 0$$

Derivative of Elzaki Transforms (E.T.)

$$E[f'(t)] = \frac{T(v)}{v} - vf(0)$$
$$E[f''] = \frac{T(v)}{v^2} - f(0) - vf'(0)$$
$$E[f'''] = \frac{T(v)}{v^3} - \frac{1}{v}f(0) - f'(0) - vf''(0)$$
$$E[f^{iv}] = \frac{T(v)}{v^3} - \frac{1}{v^2}f(0) - \frac{1}{v}f'(0) - f''(0) - vf'''(0)$$

Aboodh Transform(A.T)

Aboodh transform defined for function of exponential order is considered a function in the set A defined by

$$A = [f(t){:}\,\exists M, K_1, K_2 \geq 0, |f(t)| \leq Me^{-\mathrm{vt}}]$$

for a given function in the set A, the constant must be finite number while k_1,k_2 may be finite or infinite.

The Aboodh transform denoted by the operator A[f(t)] is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty e^{-vt} f(t)\, dt\ t \geq 0$$

Derivative of Aboodh Transforms

$$A(y) = a(y)$$
$$A(ay') = a\left[vA(y) - \frac{y(0)}{v}\right]$$
$$A(ay'') = a\left[v^2A(y) - \frac{y'(0)}{v} - y(0)\right]$$
$$A(ay''') = a\left[v^3A(y) - \frac{y''(0)}{v} - y'(0) - vy(0)\right]$$
$$A(ay^{iv}) = a\left[v^4A(y) - \frac{y'''(0)}{v} - \frac{y''}{v} - y^I(0) - vy(0)\right]$$

Sumudu Transform (S.T.)

The Sumudu transform over the set of functions

$$A = \left[f(t): \exists M, K_1, K_2 \geq 0, |f(t)| \leq Me^{\frac{|t|}{k}}, if\ t \in (-1)^j x[0,\infty)\right]$$

is defined by the integral

$$S[f(t)] = f(W) = \int_0^\infty e^{-t} f(wt)\, dt$$

Derivatives of Sumudu Transform

$$S[f'(t)] = \frac{G(v) - f(0)}{v}$$
$$S[f''(t)] = \frac{G(v)}{v^2} - \frac{f(0)}{v^2} - \frac{f'(0)}{v}$$
$$S[f'''(t)] = \frac{G(v)}{v^3} - \frac{f(0)}{v^3} - \frac{f''(0)}{v}$$
$$S[f^{iv}(t)] = \frac{G(v)}{v^4} - \frac{f(0)}{v^4} - \frac{f'''(0)}{v}$$

Shehu Transform (Sh.T)

The Shehu transform of the function v(t) of exponential order is defined over the set of function

$$A = \left[f(t): \exists M, K_1, K_2 \geq 0, |f(t)| \leq Me^{\frac{|t|}{k_j}}, if\ t \in (-1)^j x[0,\infty)\right]$$

by the following integral

$$S[v(t)] = v(s,u) = \int_0^\infty e^{\frac{-st}{u}} v(t) dt$$

Derivatives of Shehu Transform

$$Sh[v(t)] = v(s,u)$$
$$Sh[v'(t)] = \frac{s}{u} v(s,u) - v(0)$$
$$Sh[v''(t)] = \frac{s^2}{u^2} v(s,u) - \frac{s}{u} v(0) - v'(0)$$
$$Sh[v'''(t)] = \frac{s^3}{u^3} v(s,u) - \frac{s^2}{u^2} v(0) - \frac{s}{u} v'(0) - v''(0)$$
$$Sh[v^{iv}(t)] = \frac{s^4}{u^4} v(s,u) - \frac{s^3}{u^3} v(0) - \frac{s^2}{u^2} v'(0) - sv''(0) - v'''(0)$$

TABLE 1 : Some Standard Functions OF L.T., E.T., A.T., S.T. and Sh.T.

S|N	F(t)	L[f(t)]	E[f(t)]	A[f(t)]	S[f(t)]	S[v(t)]
1	1	$\frac{1}{s}$	v^2	$\frac{1}{v^2}$	1	$\frac{u}{v}$
2	e^{at}	$\frac{1}{s-a}$	$\frac{v^2}{1-av}$	$\frac{1}{v^2-av}$	$\frac{1}{1-au}$	$\frac{u}{v-au}$
3	$Sinat$	$\frac{a}{s^2+a^2}$	$\frac{av^3}{1+a^2v^2}$	$\frac{a}{v(v^2+a^2)}$	$\frac{ua}{1+a^2u^2}$	$\frac{au^2}{v^2+a^2u^2}$
4	$Cosat$	$\frac{s}{s^2+a^2}$	$\frac{v^2}{1+a^2v^2}$	$\frac{1}{v^2+a^2}$	$\frac{1}{1+a^2u^2}$	$\frac{uv}{v^2+a^2u^2}$
5	t	$\frac{1}{s^2}$	v^3	$\frac{1}{v^3}$	u	$\left[\frac{u}{v}\right]^2$
6	$cosh(at)$	$\frac{s}{s^2-a^2}$	$\frac{v^2}{1-a^2v^2}$	$\frac{1}{v^2-a^2}$	$\frac{1}{1-a^2u^2}$	$\frac{au^2}{v^2-a^2u^2}$
7	$Sinh(at)$	$\frac{a}{s^2-a^2}$	$\frac{av^3}{1-a^2v^2}$	$\frac{a}{v(v^2-a^2)}$	$\frac{ua}{1-a^2u^2}$	$\frac{uv}{v^2-a^2v^2}$

TABLE 2 : Some Standard Inverse Functions OF L.T., E.T., A.T., S.T. and Sh.T.

S/N	$F(t)$	$L^{-I}[f(t)]$	$E^{-I}[f(t)]$	$A^{-I}[f(t)]$	$S^{-I}[f(t)]$	$Sh^{-I}[v(t)]$
1	1	$\frac{1}{s}$	v^2	$\frac{1}{v^2}$	1	$\frac{u}{v}$
2	e^{at}	$\frac{1}{s-a}$	$\frac{v^2}{1-av}$	$\frac{1}{v^2-av}$	$\frac{1}{1-au}$	$\frac{u}{v-au}$
3	$Sinat$	$\frac{a}{s^2+a^2}$	$\frac{av^3}{1+a^2v^2}$	$\frac{a}{v(v^2+a^2)}$	$\frac{ua}{1+a^2u^2}$	$\frac{au^2}{v^2+a^2u^2}$
4	$Cosat$	$\frac{s}{s^2+a^2}$	$\frac{v^2}{1+a^2v^2}$	$\frac{1}{v^2+a^2}$	$\frac{1}{1+a^2u^2}$	$\frac{uv}{v^2+a^2u^2}$
5	t	$\frac{1}{s^2}$	v^3	$\frac{1}{v^3}$	u	$\left[\frac{u}{v}\right]^2$
6	cosh(at)	$\frac{s}{s^2-a^2}$	$\frac{v^2}{1-a^2v^2}$	$\frac{1}{v^2-a^2}$	$\frac{1}{1-a^2u^2}$	$\frac{au^2}{v^2-a^2u^2}$
7	$Sinh(at)$	$\frac{a}{s^2-a^2}$	$\frac{av^3}{1-a^2v^2}$	$\frac{a}{v(v^2-a^2)}$	$\frac{ua}{1-a^2u^2}$	$\frac{uv}{v^2-a^2v^2}$

NUMERICAL EXPERIMENT

Example 1

Consider the fourth order ordinary differential equation:

$$\mathrm{y^{iv}} - 3y'' - 4y = 0;\ y(0) = 1, y'(0) = \frac{1}{3},\ \ y''(0) = 0, \qquad y'''(0) = 0$$

Solution by Laplace Transforms (L.T.)

$[S^4(y_s) - S^3(y_s) - S^2(y_0') - S(y_0'') - y_0'''] - 3[S^2(y_s) - S(y_o) - y_0'] - 4y_s = 0$

$S^4(y_s) - S^3(1) - S^2\left(\frac{1}{3}\right) - S(0) - 0 - 3\left[S^2 y_s - S(1) - \frac{1}{3}\right] - 4y_s = 0$

$S^4(y_s) - S^3 - \frac{S^2}{3} - 3S^2(y_s) + 3S + 1 - 4y_s = 0$, then $y_s = \frac{S^3 + \left(\frac{S^2}{3}\right) - 3S - 1}{S^4 - 3S^2 - 4}$.

Using partial fraction, we get

$y_s = \left(\frac{A}{S-2}\right) + \left(\frac{B}{S+2}\right) + \left(\frac{CS+D}{S^2+1}\right)$. Which will yield

$A(S+2)(S^2+1) + B(S-2)(S^2+1) + (CS+D)(S^2-4) = S^3 + \left(\frac{S^2}{3}\right) - 3S - 1$. Evaluating, we get $D = \frac{4}{15}$, $C = \frac{4}{5}$, $A = \frac{7}{60}$, $B = \frac{1}{12}$. So that

$y_s = \left(\frac{\frac{7}{60}}{S-2}\right) + \left(\frac{\frac{1}{12}}{s+2}\right) + \left(\frac{\frac{4S}{5} + \frac{4}{15}}{S^2+1}\right)$

Taking inverse Laplace transform, we get

$$L^{-1}(y_s) = L^{-1}\left[\frac{\frac{7}{60}}{S-2} + \frac{\frac{1}{12}}{S+2} + \frac{\frac{4s}{5}+\frac{4}{15}}{S^2+1}\right]$$

Then, $y_t = \frac{7}{60}e^{2t} + \frac{1}{12}e^{-2t} + \frac{4}{5}cost + \frac{4}{15}sint$

Solution by Elzaki Transform E.T.

Applying Elzaki derivatives on the differential equation, we get

$$\frac{T(v)}{v^4} - \frac{1}{v^2}f(0) - \frac{1}{v}f'(0) - f''(0) - vf'''(0) - 3\left[\frac{T(v)}{v^2} - f(0) - vf'(0)\right] - 4T(v) = 0$$

Evaluating, we have

T(v)=$\frac{(3+v-9v^2-3v^3)}{(1-2v)(1+2v)(3+3v^2)}$

Then using partial fraction method, we obtain

T(v) = $v^2\left(\frac{A}{1-2v} + \frac{B}{1+2v} + \frac{Cx+D}{3+3v}\right)$

$A(1+2v)(3+3v^2) + B(1-2v)(3+3v^2) + Cv + D = 3 + v - 9v^2 - 3v^3$

Then solving, we have

A = $\frac{7}{60}$, B = $\frac{1}{12}$, C = $\frac{4}{15}$, D = $\frac{12}{5}$.

Substituting these values in T(v) and applying inverse Elzaki Transform, we get

$$E(v) = E^{-1}\left(\left(\frac{\frac{7v^2}{60}}{1-2v}\right) + \left(\frac{\frac{1v^2}{12}}{1+2v}\right) + \frac{1}{3}\left(\frac{\frac{4v^3}{5}+\frac{12v^2}{5}}{3+3v^2}\right)\right)$$

Making use of the inverse functions in table 2, we obtain

$$E(v) = \frac{7}{60}e^{2t} + \frac{1}{12}e^{-2t} + \frac{4}{15}sint + \frac{4}{5}cost$$

Solution by Sumudu Transform (S.T.)

Applying Sumudu derivatives on the differential equation, we have

$$\frac{G(u)}{u^4} - \frac{f(0)}{u^4} - \frac{f'(0)}{u^3} - \frac{f'''(0)}{u} - 3\left(\frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u}\right) - 4\big(G(u)\big) = 0$$

$$\frac{G(u)}{u^4} - \frac{1}{u^4} - \frac{1}{3u^3} - 3\frac{G(u)}{u^2} + \frac{1}{u^2} + \frac{1}{3u} - 4G(u) = 0$$

Then, $G(u) = \left(\frac{3+u-9u^2-3u^3}{(1-2u)(1+2u)(3+3u^2)}\right)$

Using partial fraction,we have

$$G(u) = \frac{A}{1-2u} + \frac{B}{1+2u} + \frac{Cu+D}{3+3u^2}$$

where $A(1+2v)(3+3v^2) + B(1-2v)(3+3v^2) + Cu + D(1-4v^2) = 3 + u - 9u^2 - 3u^3$.

Upon evaluating we get $A = \frac{7}{60},\ B = \frac{1}{12},\ C = \frac{4}{5},\ D = \frac{12}{5}$

Apply inverse Sumudu Transform we get,

$$G^{-1}(u) = G^{-1}\left(\frac{\frac{7}{60}}{1-2u} + \frac{\frac{1}{12}}{1+2u} + \frac{\frac{4v}{5}+\frac{12}{5}}{3+3v^2}\right)$$

So that, $G(t) = \frac{7}{60}e^{2t} + \frac{1}{12}e^{-2t} + \frac{4}{15}sint + \frac{4}{5}cost$

Solution by Shehu Transform (Sh.T.)

Applying Shehu derivatives on the differential equation, we have

$$\frac{S^4}{U^4}v(s,u)-\frac{S^3}{U^3}v(0)-\frac{S^2}{U^2}v'(0)-\frac{S}{U}v''(0)-v'''-3\left(\frac{S^2}{U^2}v(u,s)-\frac{S}{U}v(0)-v'(0)\right)-4v(s,u)=0$$

$$\frac{S^4}{U^4}v(s,u)-\frac{S^3}{U^3}-\frac{1}{3}\frac{S^2}{U^2}-3\frac{S^2}{U^2}v(s,u)-3\frac{S}{U}-1-4v(s,u)=0$$

$$\frac{S^4}{U^4}v(s,u)-3\frac{S^2}{U^2}v(s,u)-4v(s,u)=\frac{S^3}{U^3}-\frac{1}{3}\frac{S^2}{U^2}-3\frac{S}{U}-1$$

$$let\ \ \frac{S}{U}=t$$

Then we have $v(s,u)=\frac{t^3+\frac{1}{3}t^2+3t+1}{(t-2)(t+2)(t^2+1)}$

Using partial fraction,we get

$$v(s,u)=\frac{A}{t-2}+\frac{B}{t+2}+\frac{Ct+D}{t^2+1}$$

and $A=\frac{7}{60},\ B=\frac{1}{12},\ C=\frac{4}{5},\ D=\frac{4}{15}$

$$v(t)=\frac{7}{60}\left(\frac{1}{t-2}\right)+\frac{1}{12}\left(\frac{1}{t+2}\right)+\left(\frac{\frac{4}{5}t+\frac{4}{15}}{t^2+1}\right)$$

Recall $t=\frac{S}{U}$

$$v(t)=\frac{7}{60}\left(\frac{U}{S-U}\right)+\frac{1}{12}\left(\frac{U}{S+2U}\right)+\left(\frac{\frac{4}{5}SU+\frac{4}{15}U^2}{S^2+U^2}\right)$$

Apply Inverse Shehu Transform we have,

$$v(t)=\frac{7}{60}\left(\frac{U}{S-2U}\right)+\frac{1}{12}\left(\frac{U}{S+2U}\right)+\frac{4}{5}\left(\frac{SU}{S^2+U^2}\right)+\frac{4}{15}\left(\frac{U^2}{S^2+U^2}\right)$$

Therefore, $v(t)=\frac{7}{60}e^{2t}+\frac{1}{12}e^{-2t}+\frac{4}{5}cost+\frac{4}{15}$

Solution by Aboodh Transform (A.T.)

Applying Aboodh derivatives on the differential equation, we have

$$V^4A(y)-\frac{y'''}{v}-y''(0)-Vy'(0)-V^2y(0)-3\left(V^2A(y)-\frac{y'(0)}{V}-y(0)\right)-4A(y)=0$$

Making use of the initial condition and solving for A(y), we have

$$A(y)=\frac{\frac{1}{V^2}+V^3-1-3V}{V(V-2)(V+2)(V^2+1)}$$

Solving by partial fraction, we have

$$A(y)=\frac{A}{V-2}+\frac{B}{V+2}+\frac{CV+D}{V^2+1}$$

so that $A=\frac{7}{60},\ B=\frac{1}{12},\ C=\frac{4}{5}\ and\ D=\frac{4}{15}$

$$\mathrm{A(y)}=\frac{1}{V}\left(\frac{7}{60}\left(\frac{1}{V-2}\right)+\frac{1}{12}\left(\frac{1}{V+2}\right)+\left(\frac{\frac{4}{5}V+\frac{4}{15}}{v^2+1}\right)\right)$$

Taking the inverse we get,

$$\mathrm{A(t)}=\frac{7}{60}e^2t+\frac{1}{12}e^{-2t}+\frac{4}{5}Cost+\frac{4}{15}Sint$$

Example 2
Consider the fourth order differential equation:

$$\boldsymbol{y^{iv} - 10y'' + 9y = 0;\ y(0) = 5,\ y'(0) = -1,\ y''(0) = 21,\ y'''(0) = -49}$$

Solution by Laplace Transform(L.T.)
Applying Laplace transform derivatives, we have

$$[S^4 y_s - S^3 y_o - S^2 y_0' - S y_0'' - y_0'''] - 10[S^2 y_s - S y_0 - y_o'] + 9y_s = 0$$

so that

$$y_s = \frac{5S^3 - S^2 - 29S - 39}{(S-3)(S+3)(S-1)(S+1)}$$

Using partial fraction method we have,

$$y_s = \frac{A}{S-3} + \frac{B}{S+3} + \frac{C}{S-1} + \frac{D}{S+1}$$

Upon evaluation we get $A = 0,\ B = 2,\ C = 4\ and\ D = -1$.
Then, $y_s = \frac{0}{S-3} + \frac{2}{S+3} + \frac{4}{S-1} - \frac{1}{S+1}$
Apply inverse Laplace Transform we get,

$$L^{-1}(y_s) = L^{-1}\left(\frac{0}{S-3}\right) + L^{-1}\left(\frac{2}{s+3}\right) + L^{-1}\left(\frac{4}{S-1}\right) - L^{-1}\left(\frac{1}{S+1}\right)$$

Therefore, $y_t = 2e^{-3t} + 4e^{t} - e^{-\mathrm{t}}$

Solution by Elzaki Transform (E.T.)
Applying Elzaki transform derivatives, we obtain

$$\frac{T(v)}{v^4} - \frac{1}{v^2}f(0) - \frac{1}{v}f'(0) - f''(0) - Vf'''(0) - 10\left(\frac{T(v)}{v^2} - f(0) - vf'(0)\right) + 9T(v) = 0$$

So that

$$T(v) = v^2\left(\frac{5 - v - 29v^2 - 39v^3}{(1-3v)(1+3v)(1-v)(1+v)}\right)$$

Using partial fraction method we have

$$T(v) = v^2\left(\frac{A}{1-3v} + \frac{B}{1+3v} + \frac{C}{1-v} + \frac{D}{1+v}\right)$$

then A = 0, B = 2, C = 4 and D = -1.
Substituting these values in T(v), we have

$$T(v) = v^2\left(\frac{0}{1-3v} + \frac{2}{1+3v} + \frac{4}{1-v} - \frac{1}{1+v}\right)$$

Applying inverse Elzaki Transform we get,
$E(y) = 2e^{-3t} + 4e^{t} - e^{-t}$

Solution by Sumudu Transform (S.T.)
Applying Sumudu transform derivatives on the differential equation, we have

$$\frac{G(u)}{U^4} - \frac{f(0)}{U^4} - \frac{f'(0)}{U^3} - \frac{f''(0)}{U^2} - \frac{f'''(0)}{U} - 10\left(\frac{G(U)}{U^2} - \frac{f(0)}{U^2} - \frac{f'(0)}{U}\right) + 9G(U) = 0$$

where

$$G(U) = \left(\frac{5 - U - 29U^2 - 39U^3}{(1-U)(1+U)(1-3U)(1+3U)}\right)$$

Making use of partial fraction method, we have

$$G(U) = \frac{A}{1-U} + \frac{B}{1+U} + \frac{C}{1-3U} + \frac{D}{1+3U}$$

Evaluating we have A = 4, B = -1, C = 0 and D = 2
So that $G(U) = \left(\frac{4}{1-u} - \frac{1}{1+U} + \frac{2}{1+3U}\right)$
Then applying inverse Sumudu Transform, we get
$G(t) = 4e^{t} - e^{-t} + 2e^{-3t}$

Solution by Shehu Transform (Sh.T.)
Making use of Shehu transform derivatives, we have

$$\frac{S^4}{U^4}v(s,u)-\frac{S^3}{U^3}v(0)-\frac{S^2}{U^2}v'(0)-\frac{S}{U}v''-v'''(0)-10\left(\frac{S^2}{U^2}-\frac{U}{S}v(0)-v'(0)\right)+9\big(v(s,u)\big)=0$$

If $\frac{S}{U}=g$ then

$$v(s,u)=\frac{5g^3-g^2-29g-39}{(g-3)(g+3)(g^2-1)}$$

Using Partial Fraction Method

$$v(s,u)\ =\frac{A}{g-3}+\frac{B}{g+3}+\frac{C}{g-1}+\frac{D}{g+1}$$

then A = 0, B = 2, C = 4 and D = -1. Substituting these values, we obtain

$$v(s,u)=\frac{2}{g+3}+\frac{4}{g-1}+\frac{1}{g+1}$$

Recall $g=\frac{S}{U'}$ hence

$$\text{v}(s,u)=2\left(\frac{U}{S+3U}\right)+4\left(\frac{U}{S-U}\right)-\left(\frac{U}{S+U}\right)$$

Taking inverse Sumudu we get,
$V(t)=2e^{-3t}+4e^{t}-e^{-t}$

Solution by Aboodh Transform
Making use of Aboodh transform derivatives, we have

$$V^4A(y)-\frac{y'''}{V}-y''-Vy'(0)-V^2y(0)-10\left[V^2A(y)-\frac{y'(0)}{V}-y(0)\right]+9[A(y)]=0$$

Making use of the initial conditions and simplifying, we have

$$A(y)=\frac{5V^3-V^2-29V-39}{V(V^4-10V^2+9)}$$

Then, using partial fraction method we have,

$$A(y)=\frac{1}{V}\left(\frac{5V^3-V^2-29V+39}{(V-3)(V+3)(V-1)(V+1)}\right)$$

$A(y)=\frac{A}{V-3}+\frac{B}{V+3}+\frac{C}{V-1}+\frac{D}{V+1}$. Upon solving we obtain A = 0, B = 2, C = 4, D = -1
Substituting these values, we have $A(y)=\frac{1}{v}\left(\frac{2}{V+3}+\frac{4}{V-1}-\frac{1}{V+1}\right)$
Then taking Inverse we get,
$y=2e^{-3t}+4e^{t}-e^{-t}$

Example 3
Consider this fourth order ordinary differential equation:
$y^{iv}-5y''-36y=0;\ y'''(0)=y''(0)=0,\ y'(0)=y(0)=0$

Solution by Laplace Transform (L.T.)
Applying Laplace transform derivatives, we have
$S^4y_s-S^3y_0-S^2y_0'-Sy_0''-y_0''-5(S^2y_s-Sy_0-y_0')-36y_s=0$
Then making use of the initial conditions and simplifying, we obtain

$$y_s=\frac{S^3+S^2-5S-5}{(S^2-9)(S^2+4)}$$

Using Partial Fraction method

$$y_s=\frac{A}{S+3}+\frac{B}{S-3}+\frac{Cs+D}{S^2+4}$$

Evaluating we have $A = \frac{4}{39},\ B = \frac{8}{39},\ C = \frac{9}{13},\ D = \frac{9}{13}$

so that, $y_s = \frac{\frac{4}{39}}{S+3} + \frac{\frac{8}{39}}{S-3} + \frac{\frac{9}{13}S + \frac{9}{13}}{S^2+4}$

Then taking inverse we have,

$$y_t = L^{-1}\left(\frac{\frac{4}{39}}{s+3} + \frac{\frac{8}{39}}{s-3} + \frac{\frac{9}{13}s + \frac{9}{13}}{S^2+4}\right)$$

Therefore, $y_t = \frac{4}{39}e^{-3t} + \frac{8}{39}e^{3}t + \frac{9}{13}Cos2t + \frac{9}{26}Sin2t$

Solution by Elzaki Transform (E.T)

Applying the derivatives of Elzaki transform on the differential equation and applying the initial conditions, we have

$$T(V) = V^2\left(\frac{1+V-5V^2-5V^3}{1-5V^2-36V^4}\right) = V^2\left(\frac{1+V-5V^2-5V^3}{(1-3V)(1+3V)(1+4V^2)}\right)$$

Using partial fraction method we obtain,

$$A = \frac{8}{39},\ B = \frac{4}{39},\ C = \frac{9}{13},\ D = \frac{9}{13}$$

Taking inverse we get,

$$E(V) = E^{-1}\left(v^2\left(\frac{\frac{8}{39}}{1-3V} + \frac{\frac{4}{39}}{1+3V} + \frac{\frac{9}{13}V + \frac{9}{13}}{1+4V^2}\right)\right)$$

Hence, $E(t) = \frac{8}{39}e^{3}t + \frac{4}{39}e^{-3t} + \frac{9}{26}sin2t + \frac{9}{13}cos2t$

Solution by Sumudu Transform (S.T.)

Applying the derivatives of Sumudu transform on the differential equation, we have

$$\frac{G(u)}{U^4} - \frac{f(0)}{U^4} - \frac{f'}{U^3} - \frac{f''}{U}(0) - 5\left(\frac{G(U)}{U^2} - \frac{f(0)}{U^2} - \frac{f'}{U}\right) - 36\big(G(u)\big) = 0$$

Making use of the initial conditions and simplifying, we get

$$G(U) = \frac{1+U-5U^2-5U^3}{(1-3U)(1+3U)(1+4U^2)}$$

Using partial fractions, we obtain

$$G(U) = \frac{A}{1-3U} + \frac{B}{1+3U} + \frac{CU+D}{1+4U^2}$$

So that $A = \frac{8}{39},\ B = \frac{4}{39},\ C = \frac{9}{13}\ and\ D = \frac{9}{13}$

By Taking the inverse we get,

$$S^{-1}\big(G(U)\big) = S^{-1}\left(\frac{\frac{8}{39}}{1-3U} + \frac{\frac{4}{39}}{1+3V} + \frac{\frac{9}{13}V + \frac{9}{13}}{1+4V^2}\right)$$

Then, $y(t) = \frac{8}{39}e^{3t} + \frac{4}{39}e^{-3t} + \frac{9}{26}sin2t + \frac{9}{13}cos2t$

Solution by Shehu Transform (Sh.T.)

Making use of Sumudu transform derivatives on the differential equation, we have

$\frac{S^4}{U^4}v(s,u) - \frac{S^3}{U^3}v(0) - \frac{S^2}{U^2}v'(0) - \frac{S}{U}V''(0) - V'''(0)] - 5\left[\frac{S^2}{U^2}V(s,u) - \frac{S}{U}V(0) - V'(0)\right] -$
$36V(s,u) = 0$

Then applying the initial conditions, given that $p = \frac{S}{U}$ and simplifying, we obtain

$$V(s,u) = \frac{p^3+p^2-5p-5}{p^4-5p^2-36}$$

Using partial fraction method we have,
$$\frac{p^3+p^2-5p-5}{p^4-5p^2-36}=\frac{A}{P-3}+\frac{B}{p+3}+\frac{Cp+D}{p^2+4}$$
Upon evaluation, we get $A=\frac{4}{39},\ B=\frac{8}{39},\ C=\frac{9}{13},\ D=\frac{9}{13}$

So that, $V(s,u)=\frac{4}{39}\left(\frac{1}{p-3}\right)+\frac{8}{39}\left(\frac{1}{p+3}\right)+\left(\frac{\frac{9}{13}p+\frac{9}{13}}{p^2+4}\right)$

Recall p $=\frac{S}{U}$

Taking inverse we get,
$$V(s,u)=\frac{4}{39}\left(\frac{U}{S-3U}\right)+\frac{8}{39}\left(\frac{U}{S+3U}\right)+\frac{9}{13}\left(\frac{SU+U^2}{S^2+4U^2}\right)$$
Hence, $V(t)=\frac{4}{39}e^{-3t}+\frac{9}{13}Cos2t+\frac{9}{26}Sin2t$

Solution by Aboodh Transform (A.T)

Making use of Aboodh transform derivatives on the differential equation, we have
$$V^4A(y)-\frac{y'''}{V}-y''-Vy'(0)-V^2y(0)-5\left[V^2A(y)-\frac{y'}{V}(0)-y(0)\right]-36A(y)=0$$
Applying the initial conditions and simplifying, we get
$$A(y)=\frac{1}{V}\frac{V^3+V^2-5V-36}{(v-3)(v+3)(v^+4)}$$
Using Partial Fraction we get,
$$\frac{V^3+V^2-5V-36}{(v-3)(v+3)(v^+4)}=\frac{A}{v-3}+\frac{B}{V+3}+\frac{CV+D}{V^2+4}$$
Evaluating, we get: $A=\frac{4}{39},\ B=\frac{8}{39},\ C=\frac{9}{13},\ D=\frac{9}{13}$

So that $A(y)=\frac{1}{V}\left(\frac{1}{V-3}\right)+\frac{8}{39}\left(\frac{1}{V+3}\right)+\frac{9}{13}\left(\frac{V+1}{V^2+4}\right)$

then taking the inverse we get,

$A^{-1}(y)=\frac{4}{39}e^{3t}+\frac{8}{49}e^{-3t}+\frac{9}{13}cos2t+\frac{9}{26}sin2t$

Example 4

Consider the following differential equation:
$$y^{iv}-4y'''+6y''-4y'+y=0;\ y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y'''(0)=1.$$

Solution by Laplace Transform

Applying Laplace transform derivatives, we have

$S^4y_s+S^2-1+4S-4S^3y_s+6S^2y_s-6-4Sy_s+y_s=0$

Then making use of the initial conditions and simplifying, we obtain
$$y_s=\frac{S^2-4S+7}{(S-1)^4}$$
Using partial fraction we have
$$y_s=\frac{A}{S-1}+\frac{B}{(S-1)^2}-\frac{C}{(S-1)^3}+\frac{D}{(S-1)^4}$$
Solving we obtain A = 0, B = 1 , C = -2 and D = 4
$$y_s=\frac{1}{(S-1)^2}-\frac{2}{(S-1)^3}+\frac{4}{(S-1)^4}$$
Taking inverse we have
$$L^{-1}(y_s)=L^{-1}\left[\frac{1}{(s-1)^2}-\frac{2}{(s-1)^3}+\frac{4}{(s-1)^4}\right]$$

Then, $y_t = te^t - t^2e^t + \frac{2}{3}t^3e^4$

Solution by Elzaki Transform(E.T.)

Applying the derivatives of Elzaki transform on the differential equation we have

$$\frac{T(v)}{v^4} - \frac{1}{v^2}f(0) - \frac{1}{v}f'(0) - f''(0) - vf'''(0) - 4(\frac{T(v)}{v^3} - \frac{f(0)}{v} - f^{-\prime}(0) - vf'''(0) + 6(\frac{T(v)}{v^2} - f(0) - vf'(0) - 4\left(\frac{T(v)}{v} - vf(0)\right) + T(v) = 0$$

Applying the initial conditions and simplifying, $T(v) = \frac{7v}{(1-v)^2} - \frac{10v}{(1-v^3)} + \frac{4v}{(1-v)^4}$

Taking inverse we get,

$$T^{-1}(v) = T^{-1}\left[\frac{7v}{(1-v)^2} - \frac{10v}{(1-v)^3} + \frac{4v}{(1-v)^4}\right]$$

Hence, $T^{-1}(t) = 7te^t - 5t^2e^t + \frac{2}{3}t^3e^t$

Solution by Sumudu Transform (S.T.)

Making use of the derivatives of Sumudu tranform, we have

$$\frac{G(u)}{U^4} - \frac{f(0)}{U^4} - \frac{f'(0)}{U^3} - \frac{f''(0)}{U^2} - \frac{f'''(0)}{U} - 4(\frac{G(u)}{U^3} - \frac{f(0)}{U^3} - \frac{f'(0)}{U^2} - \frac{f''}{U} + 6\left(\frac{G(u)}{U^2} - \frac{f(0)}{U^2} - \frac{f'(0)}{U}\right) - 4\left(\frac{G(u)}{U} - \frac{f(0)}{U}\right) + G(U) = 0$$

Then applying the initial conditions and simplifying, we get $G(u) = U\left(\frac{1-4U+7U^2}{(1-U)^4}\right)$

Using Partial Fraction we obtain,

$$G(u) = U\left(\frac{1-4U+7U^2}{(1-U)^4}\right) = U\left(\frac{A}{1-U} + \frac{B}{(1-U)^2} + \frac{C}{(1-U)^3} + \frac{D}{(1-U)^4}\right)$$

Solving to get $A = 0, B = 7, C = -10\ and\ D = 4$

So that $G(u) = \frac{7}{(1-U)^2}v - \frac{10}{(1-U)^3}U + \frac{4v}{(1-V)^4}$

Then taking inverse we have,

$$G^{-1}(U) = G^{-1}\left(\frac{7}{(1-U)^2}U - \frac{10}{(1-U)^3}U + \frac{4U}{(1-U)^4}\right)$$

Therefore, $G(t) = 7te^t - 5t^2e^t + \frac{2}{3}t^3e^t$

Solution by Shehu Transform (Sh.T.)

Applying Shehu transform derivatives, we have

$$\frac{S^4}{U^4}v(s,u) - \frac{S^3}{U^3}v(0) - \frac{S^2}{U^2}v'(0) - \frac{S}{U}V''(0) - V'''(0) - 4(\frac{S^3}{U^3}v(s,u) - \frac{S^2}{U^2}v(0) - \frac{S}{U}V'(0) - \frac{S}{U}V'(0) - V''(0) + 6\left(\frac{S^2}{U^2}V(s,u) - \frac{S}{U}v'(0) - V'(0)\right) - 4\left(\frac{S}{U}v(s,u) - V(0)\right) + V(s,u) = 0$$

$Let \frac{S}{U} = t$ and making use of the initial condition, we have

$$V(s,u) = \frac{t^2 - 4t + 7}{(t-1)^4}$$

Using Partial fraction method we obtain,

$$V(s,u) = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{(t-1)^3} + \frac{D}{(t-1)^4}$$

Solving, we obtain $A = 0$, $B = 1$, $C = -2$ and $D = 4$

So that $V(s,u) = \frac{1}{(t-1)^2} - \frac{2}{(t-1)^3} + \frac{4}{(t-1)^4}$

Recall t = $\frac{S}{U}$, then

$$V(s,u) = \frac{U^2}{(S-U)^2} - \frac{2U^3}{(S-U)^3} + \frac{4U^4}{(S-U)^4}$$

Taking inverse Shehu Transform we get,

$$V(t) = te^t - t^2e^t + \frac{2}{3}t^3e^t$$

Solution by Aboodh Transform

Making use of Aboodh transform derivatives, we have

$$\left[V^{iv}(y) - \frac{y'''(0)}{V} - y''(0) - Vy'(0) - V^2y(0)\right] - 4\left[V^3A(y) - \frac{y''(0)}{V} - y'(0) - Vy(0)\right] + 6\left[V^2A(y) - \frac{y'(0)}{V} - y(0)\right] - 4\left[VA(y) - \frac{y(0)}{V}\right] + A(y) = 0$$

Applying the initial conditions,

$$A(y) = \frac{1}{V}\left(\frac{V^2 - 4V + 7}{(V-1)^4}\right)$$

Using Partial Fraction we have,

$$A(y) = \frac{A}{V-1} + \frac{B}{(V-1)^2} + \frac{C}{(V-1)^3} + \frac{D}{(V-1)^4}$$

Solving we get. A = 0 , B = 1 , C = -2 , D = 4. So that

$$A(y) = \frac{1}{V}\left(\frac{1}{(V-1)^2} - \frac{2}{(V-1)^3} + \frac{4}{(V-1)^4}\right)$$

Taking inverse Aboodh Transform we obtain,

$$A(t) = te^t - t^2e^t + \frac{2}{3}e^t$$

CONCLUSION

This study compares five integral transforms that can be used to solve fourth order ordinary differential equations. Any integral transform can be used to get results that are comparable to those achieved with the Laplace transform. These integral transformations are effective and vibrant. The above modifications turn the problem into an algebraic form, which facilitates solving.

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