https://dx.doi.org/10.4314/dujopas.v10i4b.3
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490

Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 10 No. 4b December 2024*

A Hybrid Method of Improved Block Pulse with Bernstein Polynomials for Numerical Solution of Linear Ill-Posed First Kind Fredholm Integral Equation

Hajara Shu'aibu Musa1* and Sirajo Lawan Bichi[1]

1Department of Mathematics
Bayero University
Kano.

Email: slawanbh@gmail.com

Abstract

This paper investigates the approximate solution of linear first-kind Fredholm integral equation (LFFIE). The LFFIE is an ill-posed problem (ILPP) and often requires solving linear system of equations with high condition number, which makes it difficult or impossible to solve. A novel approach is introduced, employing a mix of Benstein polynomials (BEPLs) and improved block-pulse functions (IBLPFs) within the domain $[0,1)$ *as hybrid functions. Various properties of these functions are used to convert the LFFIE in to algebraic equations. Analysis of the method's convergence together with numerical examples are provided to demonstrate how relevant the method is. The numerical results proves that the hybrid function of IBLPFs together with BEPLs solve the LFFIE even with large condition number of the matrix.*

Keywords: Linear First kind Fredholm integral equation (LFFIE), Hybrid functions, Direct method, Condition number, Convergence analysis,

INTRODUCTION

The LFFIEs characterized by the presence of an unknown function under the integral sign:

$$\int_0^1 w(q,p)z(p)dp = e(q) \qquad 0 \le q \le 1, \tag{1}$$

which complicates the solution process due to their inherent ill-posedness. These equations often appear in applications such as signal processing, potential theory, and image restoration, where data recovery is complicated by sensitivity to input perturbations (Baker *et al*., 1964). In particular, the LFFIE exhibit sensitivity where minor changes in input data can result in significant variations in the solution, making direct analytical solutions difficult and necessitating robust numerical approaches Hansen (1994).

Tikhonov regularisation, one of the most widely used approaches, introduces a stabilizing term to control sensitivity to input perturbations; however, it may compromise accuracy, especially for highly oscillatory kernels (Tikhonov & Arsenin, 1977). Polynomial approximations, such as those involving Legendre or Chebyshev polynomials Maleknejad & Sohrabi, (2007)., Chebyshev wavelets Adibi & Assari (2010) are commonly used in the numerical solution of integral equations due to their ease of computation and ability to handle various functional forms. However, these approaches alone often struggle with the balance between stability and accuracy, particularly in ill-posed equations with complex kernel

Author for Correspondence

structures. Methods based on block pulse functions simplify the computation further by breaking down the equation into piecewise constants, though they may lack smoothness, which can limit their applicability in higher-accuracy settings Anderssen & Latham, (1995). These limitations highlight the need for more adaptive and flexible polynomial methods.

To address the challenges posed by traditional polynomial approaches, hybrid methods combining different polynomial bases have gained traction in recent years. These hybrid methods seek to capitalise on the individual strengths of various polynomial functions to enhance stability and accuracy. For example, combining block pulse functions with legendre Maleknejad, & Saeedipoor (2017), and combining block pulse with polynomials wavelets has proven effective in achieving computational simplicity while preserving detailed solution characteristics Temirbekov & Temirbekova, (2022). However, there is still room for innovation, particularly with techniques that combine BEPLs with IBLPFs to enhance smoothness without sacrificing computational simplicity. Such a hybrid approach promises to provide a robust framework for solving LFFIE with improved performance and reduced error.

Despite the progress in hybrid numerical methods, existing techniques still face challenges related to computational efficiency and accuracy when addressing the complexities of LFFIE (Anderssen & Latham, 1995; Maleknejad & Sohrabi, 2007). This study aims to bridge this gap by proposing a novel hybrid method that integrates BEPLs with IBLPFs effectively combining computational efficiency with a high degree of smoothness. This approach represents a unique advancement in terms of efficiency and makes it suitable for large-scale applications. The proposed method thus contributes to the field by offering a practical and effective solution to a longstanding challenge in numerical analysis.

We give some specific definitions and attributes of BEPLs and IBLPFs.

Lemma 1
The v^{th} degree BEPLs are defined over the interval $[a,b]$ as

$$B_{l,v}(q)=\binom{v}{l}\frac{(q-a)^l(b-q)^{v-l}}{(b-a)^v} \quad a\le q\le b,\ \ l=0,1,2,\cdots v, \tag{2}$$

where

$$\binom{v}{l}=\frac{v!}{l!(v-l)!}$$

The BEPLs serve as basis on $L^2[a,b]$. For each v^{th} degree there are $v+1$ polynomials. Therefore, any v^{th} degree polynomial can be expressed using the linear combination of $B_{l,v}(q), l=0,1,\cdots,v$ Maleknejad *et. al* (2012).

Lemma 2
An $(U+1)$ set of improved block-pulse function (IBLPF's) $Ib_u(q)\ \ u=1,2,\cdots U+1$ is defined over the interval $[0,1)$ with a little modification to the block pulse function,

$$Ib_1(q)=\begin{cases}1 & q\in[0,\frac{h}{2}) \\ 0 & otherwise\end{cases}, \tag{3}$$

$$Ib_u(q)=\begin{cases}1 & q\in[(u-2)h+\frac{h}{2},(u-1)h+\frac{h}{2}) \quad u=2,3,\cdots,U\\ 0 & otherwise\end{cases}, \tag{4}$$

$$Ib_{u+1}(q)=\begin{cases}1 & q\in[(1-\frac{h}{2},1)\\ 0 & otherwise\end{cases}, \tag{5}$$

where $U+1$ is a positive integer and $h=\frac{1}{U+1}$.

The characteristic of these functions adhere to the following: disjointness, orthogonality, and completeness Jiang (1992).
Proof

The property of disjointness can be easily derived from the definition of IBLPFs as follows:

$$Ib_u(q)Ib_v(q)=\begin{cases}Ib_u(q) & u=v\\ 0 & otherwise\end{cases}.$$

Similarly, the IBLPFs are orthogonal with each other in the interval $q\in[0,1)$:

$$\int_0^1 Ib_u(q)Ib_v(q)dq=\begin{cases}\frac{h}{2} & u=v\in 1,U+1\\ h & u=v\in 2,3,\cdots U\\ 0 & otherwise\end{cases}$$

where $u,v=1,2,\cdots,U+1$.

This characteristic can be directly inferred from the disjointness of IBLPFs.

The third property indicates that the IBLPFs set is complete when U approaches infinity. For each $z\in L^2([0,1))$ Parseval's identity holds:

$$\int_0^1 z^2(q)dq=\sum_{u=0}^{\infty} z_u^2 \parallel Ib_u(q)\parallel^2$$

where;

$$z_u=\frac{1}{h}\int_0^1 z(q)Ib_u(q)dq$$

Definition (***Hybrid of IBLPFs and BEPLs***). *Hybrid of IBLPFs and BEPLs have two arguments* u *and* l *where* $u=1,2,\cdots U+1$ *and* $l=0,1,.\cdots V$ *are the order of IBLPFs and BEPLs respectively. Hybrid of IBLPFs and BEPLs defined over the interval* $[0,1)$ *below* Ramadan *et.al* (2020):

$$Ih_{u,l}(q)=\begin{cases} B_{l,V}\left(\frac{2q}{h}\right) & q\in\left[0,\frac{h}{2}\right) \quad u=1, l=0,1,\cdots V \\ 0 & otherwise \end{cases}, \tag{6}$$

$$Ih_{u,l}(q)=\begin{cases} B_{l,V}\left(\frac{q}{h}+\frac{3}{2}-u\right) & q\in\left[(u-2)h+\frac{h}{2},(u-1)h+\frac{h}{2}\right) \quad u=2,3,\cdots U, l=0,1,\cdots V \\ 0 & otherwise \end{cases}, \tag{7}$$

$$Ih_{u,l}(q)=\begin{cases} B_{l,V}\left(\frac{2q}{h}-\frac{2}{h}+1\right) & q\in\left[1-\frac{h}{2},1\right) \quad u=U+1, l=0,1,\cdots V \\ 0 & otherwise \end{cases}. \tag{8}$$

Definition (***Condition number of a matrix(CN)***).*Let D be an invertible m.m matrix. The CN of D is defined as follows* $CN(D)=\|D\|\|D^{-1}\|$. *If D is uninvertible, set* $CN(D) = +\infty$.

Function Approximation

A function $z(q)\in L^2[0,1)$ may be expanded in terms of hybrid functions as follows

$$z(q)\approx\sum_{u=1}^{U+1}\sum_{l=0}^{V} z_{u,l} Ih_{u,l}(q) = Z^T IH(q), \tag{9}$$

where;

$$IH=\left[Ih_{10} Ih_{11} \cdots Ih_{1V} \ \ldots \ Ih_{U+1,0} Ih_{U+1,1}\cdots Ih_{U+1,V-1}\cdots Ih_{U+1,V}\right]^T,$$

and

$$Z=\left[z_{10} z_{11}\cdots z_{1V} \ \ldots \ z_{U+1,0} z_{U+1,1} \ \ldots \ z_{U+1,V-1}\cdots z_{U+1,V}\right]^T$$

Then we have

$$Z^T=\frac{\langle z(q), IH(q)\rangle}{\langle IH(q), IH(q)\rangle}, \tag{10}$$

and

$$Z^T=R^{-1}\langle z(q), IH(q)\rangle.$$

Similarly, the function $w(q,p)\in L^2([0,1)\times[0,1)$ may be estimated as:

$$w(q,p)\approx IH^T(q)\times W\times IH(p), \tag{11}$$

where W is an $(U+1)(V+1)\times(U+1)(V+1))$ matrix called the kernel matrix and entries of the matrix is given by

$$w_{r,s}=\frac{\langle Ih_r(q),\langle w(q,p) Ih_s(p)\rangle\rangle}{\langle Ih_r(q), Ih_r(q)\rangle,\langle Ih_s(p), Ih_s(p)\rangle} \quad r,s=1,2,\cdots U+1)(V+1), \tag{12}$$

$$W=R^{-1}\langle IH(q),\langle w(q,p), IH(p)\rangle\rangle R^{-1}.$$

Integration of the Cross Product

The cross product integration of two hybrid function vectors $IH(q)$ is defined by

$$R = \int_0^1 IH(q)IH(q)dq, \tag{13}$$

where;

$$R = \begin{bmatrix} J_1 & \vec{0} & \dots & \vec{0} \\ \vec{0} & J_2 & \dots & \vec{0} \\ \vdots & & & \\ \vec{0} & \vec{0} & \dots & J_{U+1} \end{bmatrix}. \tag{14}$$

$\vec{0}$ is an $(V+1)\times(V+1)$ matrix, and $J_u(u=1,2,\cdots U+1)$ is an $(V+1)\times(V+1)$ matrix. $J_u(u=1,2,\cdots U+1)$ is described as:

$$J_1 = \int_0^{\frac{h}{2}} B(\frac{2q}{h})B(\frac{2q}{h})dq, \tag{15}$$

$$J_u = \int_{(u-2)h+\frac{h}{2}}^{(u-1)h+\frac{h}{2}} B(\frac{q}{h}+\frac{3}{2}-u)B(\frac{q}{h}+\frac{3}{2}-u)^T dq\,, \tag{16}$$

$u = 2,3,\cdots U$

Lastly,

$$J_{U+1} = \int_{1-\frac{h}{2}}^{1} B(\frac{2q}{h}-\frac{2}{h}+1)\times B(\frac{2q}{h}-\frac{2}{h}+1)dq, \tag{17}$$

Material and Methods

We provide a numerical direct method to address equation (1) in this part using hybrid function comprising of IBLPFs and BEPLs. To achieve this, we approximate the functions in equation (1) using the hybrid series expansion of functions Ramadan *et.al* (2020) taking the form:

$$z(q) \approx Z^T IH(q), \tag{18}$$

$$w(q,p) \approx IH^T(q)\times W\times IH(p), \tag{19}$$

and

$$e(q) \approx E^T IH(q), \tag{20}$$

where W is an $(U+1)(V+1)\times(U+1)(V+1)$ dimensional matrix and E is a known $(U+1)(V+1)\times 1$ vector. In equation (18), Z is an unknown $(U+1)(V+1)\times 1$ vector. By substituting equation (18)-(20) in to equation (1), we get

$$\int_0^1 IH^T(q)WIH(p)IH^T(p)Zdp = IH^T(q)E, \tag{21}$$

or

$$IH^T(q)W\int_0^1 IH(p)IH^T(p)Zdp = IH^T(q)E \tag{22}$$

using equation (13), equation (22) can be replaced by

$$IH^T(q)WRZ = IH^T(q)E. \tag{23}$$

Therefore, we arrive at the resulting system of linear equations:

$$(WR)Z = E \tag{24}$$

Solving the system of linear equations in (24) we find the unknown vector E where R represents the dual operational matrix of IH which is a matrix with dimension $(U+1)(V+1)\times(U+1)(V+1)$ derived from the preceding section.

3.0 Results and Discussion

Error analysis

In this section we analyse the convergence of the suggested numerical approach. We will demonstrate that given suitable conditions, the approximate solution derived from our method converges to the precise solution of equation (1). We present the following lemma and theorem in order to perform the convergence analysis.

Lemma 3 . Let $z \in C^{V+1}[0,1)$ is an $V+1$ times continuously differentiable function such that $z \approx \sum_{u=1}^{U+1} z_{u,}$ and let $O_u = span\{Ih_{u0}(q), h_{u1}(q)\cdots h_{uV+1}(q)\}$ $u=1,2,\cdots U+1$. If $Z_u^T IH_u(q)$ is the best approximation to z_u from O_u where $Z_u = [z_{u0}, z_{u1}\cdots z_{uV+1}]^T$ and $IH_u(q) = [Ih_{u0}(q), Ih_{u1}(q)\cdots Ih_{uV+1}(q)]^T$ then $y_{(U+1)(V+1)}z(q) = Y^T IH(q)$ approximates $z(q)$ with the following error bound

$$\left\| z(q) - y_{(U+1)(V+1)} z(q) \right\|_2 \le \frac{\delta}{((V+1)!)(U+1)^{V+1}} \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)}. \quad where\ \delta = \max_{q\in[0,1)} \left| f^{V+1}(q) \right| \tag{25}$$

Proof The Taylor expansion for the function $z_u(q)$ for $u=1$ is given as:

$$z_u(q) = z_u(0) + z_u'(0)(q-0) + \frac{z_u''(0)}{2!}(q-0)^2 + \cdots + \frac{z_u^V}{V!}(0)(q-0)^V + \frac{z_u^{V+1}}{V+1!}(0)(q-0)^{V+1} + \cdots \qquad q \in \left[0, \frac{h}{2}\right), \tag{26}$$

for which it is known that

$$\left| z_u(q) - \tilde{z}_u(q) \right| \le \left| z^{V+1}(\theta) \right| \frac{q^{V+1}}{(V+1)!}, \quad \theta \in \left[0, \frac{h}{2}\right), \quad u=1. \tag{27}$$

Similarly, the Taylor expansion for the function $z_u(q)$ for $u = 2,3,\cdots U$ is given as:

$$\begin{aligned} z_u(q) = {} & z_u\left((u-2)h + \frac{h}{2}\right) + z_u'\left((u-2)h + \frac{h}{2}\right)\left(q - \left((u-2)h + \frac{h}{2}\right)\right) + \\ & + \cdots + \frac{z_u^V}{V!}\left((u-2)h + \frac{h}{2}\right)\left(q - \left((u-2)h + \frac{h}{2}\right)\right)^V \\ & + \frac{z_u^{V+1}}{V+1!}\left((u-2)h + \frac{h}{2}\right)\left(q - (u-2)h + \frac{h}{2}\right)^{V+1} + \cdots \\ & q \in \left[(u-2)h + \frac{h}{2}, (u-1)h + \frac{h}{2}\right), \end{aligned} \tag{28}$$

for which it is known that

$$\left| z_u(q) - \tilde{z}_u(q) \right| \le \left| z^{V+1}(\theta) \right| \frac{(q - ((u-2)h + \frac{h}{2}))^{V+1}}{(V+1)!}, \tag{29}$$

$$\theta \in \left[(u-2)h+\frac{h}{2},(u-1)h+\frac{h}{2}\right) \quad u=2,3,\cdots U.$$

Lastly

$$z_u(q)= z_u\left(1-\frac{h}{2}\right)+z_u^{'}\left(1-\frac{h}{2}\right)\left(q-\left(1-\frac{h}{2}\right)\right)+\cdots+\frac{z_u^V}{V!}\left(1-\frac{h}{2}\right)\left(q-\left(1-\frac{h}{2}\right)\right)^V$$
$$+\frac{z_n^{V+1}}{V+1!}\left(1-\frac{h}{2}\right)\left(q-\left(1-\frac{h}{2}\right)\right)^{V+1}+\cdots \tag{30}$$
$$q\in\left[1-\frac{h}{2},1\right),$$

for which it is known that

$$|z_u(q)-\tilde{z}_u(q)|\le |z^{V+1}(\theta)|\frac{(q-(1-\frac{h}{2}))^{V+1}}{(V+1)!},$$
$$\text{where } \theta\in\left[1-\frac{h}{2},1\right),\ u=U+1 \tag{31}$$

Since $Z_u IH_u(q)$ is the best approximation to $\tilde{z}_u$ from O_u then

$$\|z_u(q)-Z_u^T IH_u(q)\|_2^2=\|z_u(q)-\tilde{z}_u(q)\|_2^2$$
$$=\int_0^{\frac{h}{2}}|z_u(q)-\tilde{z}_u(q)|^2dq+\int_{(u-2)h+\frac{h}{2}}^{(u-1)h+\frac{h}{2}}|z_u(q)-\tilde{z}_u(q)|^2dq \tag{32}$$
$$+\int_{1-\frac{h}{2}}^{1}|z_u(q)-\tilde{z}_u(q)|^2dq$$

Now let $\delta=\max_{\theta\in[0,1)}|z^{V+1}(\theta)|$ this implies,

$$\|z_u(q)-Z_u^T IH_u(q)\|_2^2\le\left(\frac{\delta}{(V+1)!}\right)^2\left(\begin{array}{l}\int_0^{\frac{h}{2}}q^{2V+2}dt+\int_{(u-2)h+\frac{h}{2}}^{(u-1)h+\frac{h}{2}}(q-((u-2)h+\frac{h}{2}))^{2V+2}dq\\ \qquad+\int_{1-\frac{h}{2}}^{1}(q-(1-\frac{h}{2}))^{2V+2}dq\end{array}\right) \tag{33}$$
$$=\left(\frac{\delta}{(V+1)!}\right)^2\frac{h^{2V+3}}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right),$$

this implies

$$\|z(q)-Z^T IH(q)\|_2^2\le\sum_{u=1}^{U+1}\|(z_u(q)-Z_u^T IH_u(q))\|_2^2,$$
$$\le\frac{\delta^2 h^{2M+3}}{((V+1)!)^2(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right). \tag{34}$$

By taking the square root of equation (34) gives the bound

$$\left\|z(q)-Z^T IH(q)\right\|_2 \leq \frac{\delta}{((V+1)!)(U+1)^{V+1}}\sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)}. \tag{35}$$

Theorem *In equation (1) suppose that* $w(q,p)$ *is continuous on the square* $[0,1)^2$ *and* $z(q)$ *belong to* $C^{V+1}[0,1)$ *with* $V \geq 1$. *Furthermore assume the approximate solution*

$$z(q) \approx \sum_{u=1}^{U+1}\sum_{l=0}^{V} z_{ul} Ih_{ul}(q) = Z^T IH(q) \tag{36}$$

is given by hybrid IBLPFs and BEPLs method, Now if A *is non-singular then*

$$\left\|z(q)-\sum_{u=1}^{U+1}\sum_{l=0}^{V} z_{ul} Ih_{ul}(q)\right\|_2 \leq \frac{1}{((V+1)!)(U+1)^{V+1}}\sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)} \left(\delta+\left\|A^{-1}\right\| c_2 (U+1)(V+1) \left(\frac{\sqrt{\frac{V(2V+1)(2V-1)}{(V+1)}}}{V\sqrt{2}}\right)\right) \tag{37}$$

Proof The estimation in (37) is derived as follows:
Let

$$z(q) \approx z_{(U+1)(V+1)}(q) = \sum_{u=1}^{U+1}\sum_{l=0}^{V} z_{ul} Ih_{ul}(q), \tag{38}$$

where z_{ul} are unknown coefficients determined by solving the linear system in equation (24). Also suppose

$$y_{(U+1)(V+1)}(q) = \sum_{u=1}^{U+1}\sum_{l=0}^{V} y_{ul} Ih_{ul}(q) \tag{39}$$

our goal is to find a bound for $\left\|z(q)-z_{(U+1)(V+1)}(q)\right\|_2$, so

$$\left\|z(q)-z_{(U+1)(V+1)}(q)\right\|_2 \leq \left\|z(q)-y_{(U+1)(V+1)}(q)\right\|_2 + \left\|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right\|_2, \tag{40}$$

from lemma (1) we have

$$\left\|z(q)-z_{(U+1)(V+1)}(q)\right\|_2 \leq \frac{\delta}{((V+1)!)(U+1)^{V+1}}\sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)}$$
$$\textit{where}\ \ \delta = \max\left|z^{V+1}(q)\right| \quad q\in[0,1), \tag{41}$$

We now find a bound for $\left\|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right\|_2$,

$$\left\|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right\|_2^2 = \int_0^1 \left|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right|^2 dq$$
$$= \int_0^1 \left|\sum_{u=1}^{U+1}\sum_{l=0}^{V}(y_{ul}-z_{ul}) Ih_{ul}(q)\right|^2 dq$$
$$\leq \int_0^1 \left(\sum_{u=1}^{U+1}\sum_{l=0}^{V}\left|(y_{ul}-z_{ul})\right|\left|Ih_{\gamma}(q)\right|\right)^2 dq \tag{42}$$

where $\left|Ih_{\gamma}(q)\right| = \sup\left|Ih_{ul}(q)\right| \quad u=1,2,\cdots U+1 \quad l=0,1,\cdots V$

$$\left\|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right\|_2^2 \le \int_0^1 \sum_{u=1}^{U+1}\sum_{l=0}^{V}\left|(y_{ul}-z_{ul})\right|^2 \left|Ih_\gamma(q)\right|^2 dq$$
$$=\sum_{u=1}^{U+1}\sum_{l=0}^{V}\left|(y_{ul}-z_{ul})\right|^2 \alpha$$
$$=\|Y-Z\|_2^2\alpha, \qquad (43)$$

where $\alpha=\int_0^1 \left|Ih_\gamma(q)\right|^2 dq,$

this implies,

$$\left\|y_{(U+1)(V+1)}(q)-z_{(U+1)(V+1)}(q)\right\|_2 \le \|Y-Z\|_2\sqrt{\alpha}, \qquad (44)$$

if we substitute $z_{(U+1)(V+1)}$ as an approximate solution in equation (1) then

$$\int_0^1 w(q,p)z_{(U+1)(V+1)}(p)dp=e(q)$$

but if we use $y_{(U+1)(V+1)}$ as an approximate solution in equation (1) we obtain

$$\int_0^1 w(q,p)y_{(U+1)(V+1)}(p)dp=\tilde{e}(q),$$

by using the numerical method of hybrid IBLPFs and BEPLs in equation(24) we have

$$\begin{aligned} Z&=(WR)^{-1}E=A^{-1}E,\\ Y&=(WR)^{-1}\tilde{E}=A^{-1}\tilde{E}, \end{aligned} \qquad (45)$$

where

$$W=\left[w_{10}\dots w_{1V} \quad w_{20}\dots y_{2V} \quad \dots \quad w_{U+10}\dots w_{U+1V}\right]^T,$$
$$Y=\left[y_{10}\dots y_{1V} \quad y_{20}\dots y_{2V} \quad \dots \quad y_{U+10}\dots y_{U+1V}\right]^T,$$
$$E=\left[e_{10}\dots e_{1V} \quad e_{20}\dots e_{2V} \quad \dots \quad e_{U+10}\dots e_{U+1V}\right]^T,$$
$$\tilde{E}=\left[\tilde{e}_{10}\dots \tilde{e}_{1V} \quad \tilde{e}_{20}\dots \tilde{e}_{2V} \quad \dots \quad \tilde{e}_{U+10}\dots \tilde{e}_{U+1V}\right]^T,$$

Also, we observe $Y-Z=A^{-1}\left(\tilde{E}-E\right)$

We have the following bound of $\|Y-Z\|_2$ as

$$\|Y-Z\|_2 \le \left\|A^{-1}\right\|_2\left\|\left(\tilde{E}-E\right)\right\|_2, \qquad (46)$$

then it is enough to find a bound for $\left\|\left(\tilde{E}-E\right)\right\|_2$ but before then we need a bound for $\left\|\left(\tilde{e}(q)-e(q)\right)\right\|_2$, Now from equation (1),

$$e(q)=\int_0^1 w(q,p)z(p)dp-\int_0^1 w(q,p)y_{(U+1)(V+1)}(p)dp+\int_0^1 w(q,p)y_{(U+1)(V+1)}(p)\,dp,$$

$$\tilde{e}(q)=e(q)-\int_0^1 w(q,p)\left(z(p)-y_{(U+1)(V+1)}(p)\right)dp,$$

therefore;

$$\left\|\left(\tilde{e}(q)-e(q)\right)\right\|_2^2=\int_0^1\left|\int_0^1 w(q,p)\left(z(p)-y_{(U+1)(V+1)}(p)\right)dp\right|^2 dq,$$

$$\leq \int_0^1 \left(\int_0^1 \left| w(q,p)\left(z(p) - y_{(U+1)(V+1)}(p) \right) dp \right| \right)^2 dq,$$

$$= \int_0^1 \left(\int_0^1 \left| w(q,p) \right| \left| \left(z(p) - y_{(U+1)(V+1)}(p) \right) \right| dp \right)^2 dq,$$

If we assume $c_0 = \sup |w(q,p)| \quad q, p \in [0,1)$ we have

$$\begin{aligned} \left\| \left(\tilde{e}(q) - e(q) \right) \right\|_2^2 &\leq c_0^2 \left(\int_0^1 \left| \left(z(p) - y_{(U+1)(V+1)}(p) \right) \right| dp \right)^2 \\ &\leq c_0^2 \left(\left\| \left(z(p) - y_{(U+1)(V+1)}(p) \right) \right\|_2 \right)^2 \\ &\leq c_0^2 \left(\frac{\delta}{((V+1)!)(U+1)^{V+1}} \sqrt{\frac{1}{(2V+3)} \left(\frac{1}{2^{2V+2}} + 1 \right)} \right)^2 \end{aligned} \quad (47)$$

taking square roots of both sides of equation (47) we have;

$$\left\| \left(\tilde{e}(q) - e(q) \right) \right\|_2 \leq \frac{\delta c_0}{((V+1)!)(U+1)^{V+1}} \sqrt{\frac{1}{(2V+3)} \left(\frac{1}{2^{2V+2}} + 1 \right)}. \quad (48)$$

Now we find a bound for $\left\| \tilde{E} - E \right\|_2$ since

$$e_{ul} = \frac{(2V+1)\binom{2V}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \int_0^1 e(q) Ih_{ul}(q) dq \quad u = 1, 2, \cdots U+1 \quad l, s = 0, 1, \cdots V. \quad (49)$$

and

$$\tilde{e}_{ul} = \frac{(2V+1)\binom{2M}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \int_0^1 \tilde{e}(q) Ih_{ul}(q) dq \quad u = 1, 2, \cdots U+1 \quad l, s = 0, 1, \cdots V. \quad (50)$$

then we have

$$\left\| \tilde{E} - E \right\|_2^2 = \sum_{u=1}^{U+1} \sum_{l=0}^{V} \left| \left(\tilde{e}(q) - e(q) \right) \right|^2$$

$$\leq (U+1)(V+1) \sup_{1 \leq u \leq U+1;\, 0 \leq l,s \leq V} \left(\left| \frac{(2V+1)\binom{2V}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \int_0^1 Ih_{ul}(q) \tilde{e}(q) - e(q) dq \right|^2 \right),$$

$$\leq (U+1)(V+1) \sup_{1\leq u\leq U+1;\, 0\leq l,\, s\leq V} \left(\frac{(2V+1)\binom{2V}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \int_0^1 \left| Ih_{ul}(q)(\tilde{e}(q)-e(q))dq \right| \right)^2$$

$$= (U+1)(V+1) \sup_{1\leq u\leq U+1;\, 0\leq l,\, s\leq V} \left(\frac{(2V+1)\binom{2V}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \int_0^1 \left| Ih_{ul}(q) \right| \left| \tilde{e}(q)-e(q) \right| dq \right)^2$$

$$\leq (U+1)(V+1) \sup_{1\leq u\leq U+1;\, 0\leq l,\, s\leq V} \left(\frac{(2V+1)\binom{2V}{l+s}}{2h\binom{V}{l}\binom{V}{s}} \left\| Ih_{ul}(q) \right\|_2 \left\| (\tilde{e}(q)-e(q)) \right\|_2 \right)^2,$$

$$= (U+1)(V+1) \sup_{1\leq u\leq U+1;\, 0\leq l,\, s\leq V} \left(\frac{1}{\sqrt{\left(\frac{2h\binom{V}{l}\binom{V}{s}}{(2V+1)\binom{2V}{l+s}} \right)}} \left\| (\tilde{e}(q)-e(q)) \right\|_2 \right)^2, \quad (51)$$

substitute equation (48) in equation (51) we obtain,

$$\left\| \tilde{E}-E \right\|_2^2 \leq (U+1)(V+1) \sup_{1\leq u\leq U+1;\, 0\leq l,\, s\leq V} \left(\begin{matrix} \left(\dfrac{\delta c_0}{\sqrt{\left(\frac{2h\binom{V}{l}\binom{V}{s}}{(2V+1)\binom{2V}{l+s}} \right)} ((V+1)!)(U+1)^{V+1}} \right) \\ \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1 \right)} \end{matrix} \right)^2. \quad (52)$$

Consequently, by substituting equation (52) in the inequality (44) we can conclude

$$\left\| Y-E \right\|_2 \leq \left\| A^{-1} \right\| \frac{c_1 (U+1)(V+1)}{((V+1)!)(U+1)^{V+1}} \left(\frac{\sqrt{\frac{V(2V+1)(2V-1)}{(V+1)}}}{V\sqrt{2}} \right)$$

$$\sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1 \right)} \quad (53)$$

where $c_1 = \delta c_0$.

So by substituting equation (53) in equation (44) we obtain

$$\left\| y_{(U+1)(V+1)}(q) - z_{(U+1)(V+1)}(q) \right\|_2 \le \left\| A^{-1} \right\| \frac{c_2 (U+1)(V+1)}{\left((V+1)!\right)(U+1)^{V+1}} \left(\frac{\sqrt{\frac{V(2V+1)(2V-1)}{(V+1)}}}{V\sqrt{2}} \right) \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)} \tag{54}$$

where $c_2 = c_1\sqrt{\alpha}$.

Finally substitute equation (54) and equation (41) in equation (40) we obtain

$$\left\| z(q) - z_{(U+1)(V+1)}(q) \right\|_2 = \left\| z(q) - y_{(U+1)(V+1)}(q) \right\|_2 + \left\| y_{(U+1)(V+1)}(q) - z_{(U+1)(V+1)}(q) \right\|_2$$

$$\le \frac{\delta}{\left((V+1)!\right)(U+1)^{V+1}} \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)} + \left\| A^{-1} \right\| \frac{c_2 (U+1)(V+1)}{\left((V+1)!\right)(U+1)^{V+1}} \left(\frac{\sqrt{\frac{V(2V+1)(2V-1)}{(V+1)}}}{V\sqrt{2}} \right) \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)}$$

$$= \frac{1}{\left((V+1)!\right)(U+1)^{V+1}} \sqrt{\frac{1}{(2V+3)}\left(\frac{1}{2^{2V+2}}+1\right)} \left(\delta + \left\| A^{-1} \right\| c_2 (U+1)(V+1) \left(\frac{\sqrt{\frac{V(2V+1)(2V-1)}{(V+1)}}}{V\sqrt{2}} \right) \right).$$

Numerical examples

In this part, we offer some examples demonstrating the approximation of solutions to LFFIEs using the numerical technique outlined according to the preceding section. These numerical tests are conducted utilising Maple 18 software.

Example 1. Consider the LFFIE

$$\int_0^1 \sin(qp) z(p)\, dp = \frac{\sin(q) - q\cos q}{q^2}$$

with exact solution $z(q) = q$. The absolute errors of improved block pulse method for $q \in [0,1]$
is shown in Table 1.

Table 1 The numerical outcomes derived from the suggested approach for Example 1.

q_i	Exact	HIBB Method With U=2,V=2
0.1	0.1	0.0798652262
0.2	0.2	0.0767057343
0.3	0.3	0.0131597932
0.4	0.4	0.0086792069
0.5	0.5	0.0041986208
0.6	0.6	0.0002819653
0.7	0.7	0.0047625515
0.8	0.8	0.0020062129
0.9	0.9	0.0005623737

Example 2

Consider the first kind Fredholm integral equation

$$\int_0^1 e^p(\sin(q-p+1)+1)z(p)dp = 1+\cos(q)-\cos(q+1)$$

with exact solution $z(q)=e^{-q}$. The absolute errors of improved block pulse method for $q\in[0,1]$ is shown in Table 2.

Table 2 The numerical outcomes derived from the suggested approach for Example 2.

q_i	Exact	HIBB method Wih U=2,V=2
0.1	0.904837418039	0.5389568730358
0.2	0.818730753079	0.8974269336529
0.3	0.740818220687	0.9941483491454
0.4	0.670320046036	0.9645529123312
0.5	0.606530659716	0.9282486871939
0.6	0.548811636090	0.8858740993523
0.7	0.496585303794	0.8380068201946
0.8	0.449328964112	0.7102212282173
0.9	0.406569659745	0.3593715802343

A hybrid of BEPLs and IBLPFs was used as basis in $L^2[0,1)$ space by using the excellent characteristic of the hybrid functions to discretise the ill-posed LFFIE in to a system of linear equation in (24). We now solved the resulting system using Maple 18 software programme. We used the method to solve the second kind Fredholm integral equation as presented by Ramadan *et.al* (2020) and it was proven that the approach appears to be effective for solving integral equations as comparisons with other methods give lesser error. However, due to the ill-poseness of the LFFIE, a large condition number is reflected in the matrix equation $Az=e$ of the descretised linear system of equation in (24) there by making the solution to be unstable which is illustrated using some numerical examples above. The CN of the matrix plays a vital role for obtaining a stable approximate solution as it determines how unstable the linear system $Az=e$ is when the data e gets altered. A CN almost infinity makes the matrix to be uninvertible there by making it impossible to solve to get an approximate solution. Therefore, in practice a small CN close to one is wanted. But despite the matrix's high CN, we were still able to get a numerical approximation for the LFFIE, proving the suggested method's effectiveness which is illustrated using the above numerical examples. We also analyse the convergence of the numerical approach and it was proven to us that the approximate solution obtained by our scheme converges to the exact solution of equation (1).

CONCLUSION

The above-mentioned theoretical and numerical results show that reducing the CN of the matrix can improve accuracy and stability of the approximate solution, therefore incorporating regularisation with the presented method can improve efficiency of the method which is under investigation by the authors. However other methods like preconditioning methods can also be used if the CN of the matrix A is large. The objective is to substitute the linear system with a similar system, such as one that is obtained by multiplying by a preconditioning matrix D, whose matrix has a less CN to get a similar system $DAz = De$, where CN(DA) is lesser than CN(A). Usually a straightforward preconditioning matrix can be found Babolian *et.al* (2005).

REFERENCES

Anderssen, R. S., & Prenter, P. M. (1981). A formal comparison of methods proposed for the numerical solution of first kind integral equations. *The ANZIAM Journal*, *22*(4), 488-500..

Adibi, H., & Assari, P. (2010). Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. *Mathematical problems in Engineering*, *2010*.

Baker, C. T., Fox, L., Mayers, D. F., & Wright, K. (1964). Numerical solution of Fredholm integral equations of first kind. *The Computer Journal*, *7*(2), 141-148. Chen, Z., Micchelli, C. A., & Xu, Y. (2015). *Multiscale methods for Fredholm integral equations* (Vol. 28). Cambridge University Press.

Maleknejad, K., Lotfi, T., & Mahdiani, K. (2007). Numerical solution of first kind Fredholm integral equations with wavelets-Galerkin method (WGM) and wavelets precondition. *Applied Mathematics and computation*, *186*(1), 794-800.

Hansen, P. C. (1994). Regularization tools: A Matlab package for analysis and solution of discrete ill-posed problems. *Numerical algorithms*, *6*(1), 1-35.

Jiang, Z., & Schaufelberger, W. (1992). *Block pulse functions and their applications in control systems* (Vol. 179). Berlin: Springer-Verlag.

Luo, X., Li, F., & Yang, S. (2012). A posteriori parameter choice strategy for fast multiscale methods solving ill-posed integral equations. *Advances in Computational Mathematics*, *36*(2), 299-314.

Maleknejad, K., & Sohrabi, S. (2007). Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets. *Applied Mathematics and Computation*, *186*(1), 836-843.

Maleknejad, K., Basirat, B., & Hashemizadeh, E. (2012). A Bernstein operational matrix approach for solving a system of high order linear Volterra–Fredholm integro-differential equations. *Mathematical and Computer Modelling*, *55*(3-4), 1363-1372.

Maleknejad, K., & Saeedipoor, E. (2017). An efficient method based on hybrid functions for Fredholm integral equation of the first kind with convergence analysis. *Applied Mathematics and Computation*, *304*, 93-102.

Maleknejad, K., Lotfi, T., & Mahdiani, K. (2007). Numerical solution of first kind Fredholm integral equations with wavelets-Galerkin method (WGM) and wavelets precondition. *Applied Mathematics and computation*, *186*(1), 794-800.

Neggal, B., Boussetila, N., & Rebbani, F. (2016). Projected Tikhonov regularization method for Fredholm integral equations of the first kind. *Journal of Inequalities and*

Applications, *2016*, 1-21.

Ramadan, M., & Osheba, H. S. (2020). A new hybrid orthonormal Bernstein and improved block-pulse functions method for solving mathematical physics and engineering problems. *Alexandria Engineering Journal*, *59*(5), 3643-3652.

Tikhonov, A. N. (1977). Solutions of Ill-Posed Problems. *VH Winston and Sons*.

Temirbekov, N., & Temirbekova, L. (2022, November). Numerical solution of the first kind Fredholm integral equations by projection methods with wavelets as the basis functions. In *AIP Conference Proceedings* (Vol. 2483, No. 1). AIP Publishing.