https://dx.doi.org/10.4314/dujopas.v10i4c.8
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 10 No. 4c December 2024*

The Performance Measures Analysis of Erlang Distribution in Solving Phase Type Distribution

[1]Agboola Sunday Olanrewaju [2]Ayoade Abayomi A. [3]Sebastine Fransis. [4]Dotia Ahmed K.

[1,4] Department of Mathematics,
Faculty of Natural and Applied Sciences,
Nigerian Army University Biu,
P. M. B. 1500 Biu,
Borno State,
Nigeria.

[2]Department of Mathematics,
Faculty of Sciences,
University of Lagos,
Nigeria.

[3]Department of Mathematics,
Olusegun Agagun University of Science & Technology
Okiti-Pupa,
Nigeria.

Email: agboolasunday70@gmail.com

Abstract
The importance of phase type distribution in modeling activities cannot be under emphasized when both a distribution's initial and second moments are accessible or when the sequence of data points for computing moments is the information available. In continuous time process for an absorbing finite state Markov chain, the phase-type distribution can be thought of as the distribution of the time until absorption and it is widely used in queueing theories and other fields of applied probabilities with the used of generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions. In this study, performance measures of phase type distribution using Erlang - r *distribution, and mix Erlang* $-(r-1)$ *with Erlang -* r *distributions have been looked into, in order to provide meaningful study into the probability function, mean,* k^{th} *moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution. We began from the tractability and memory less properties of exponential distribution, and since these properties are not enough, we examined the journey through a series of exponential phases to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for probability functions, Laplace Stieltjes transform, squared coefficient of variation,* k^{th} *moment, mean and variance for the phase type distribution. The result of its variation and mean value, as well as the likelihood that the waiting period will exceed 12 time units are obtained on the waiting time until the fourth arrival with Poisson arrival process. And we demonstrate that, using the Erlang-r distribution, the squared coefficient of variation might have a variety of values. Also, by increasing the number of phases (r) and setting the parameter at each phase to be* $r\mu$*. The variance goes to zero and the expectation stays at* $\frac{1}{\mu}$ *in the limit as* $r \to \infty$

Keywords: Coxian distribution, Erlang distribution, Hyper-exponential distribution, Hypo-exponential distribution, Phase type distribution.

INTRODUCTION

The exponential distribution is very important due to both its tractability and memory-less characteristics in performance modeling, but to overcome the model procedures these two properties may not be enough, and this makes the exponential distribution not sufficient. To model general distributions while sustaining the tractability property of the exponential, we make use of phase type distribution. Also, phase - type distributions is very useful when the distribution with known mean and variance is to be formed, and the name phase-type distributions came to be due to the fact that, processes can be seen as the movement via a series of exponential steps. Phase type distribution has it major applications in queueing theories and applied probabilities with the use of generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions. William (2009)

The useful technique in phase type distribution when the added network is represented by flow-equivalent servers instead of subsystems is found in Marie (1980). The principles, laws of phase type and most cited introductory article is Neuts (1981) and some important theoretical concepts on phase-type is established (Cumani, 1982; O'Cinneide, 1989). A reliable recursive method for calculating the probability vector of the steady state is suggested (Ramaswami *et al.*, 1980; Ramaswami, 1988). The Hessenberg matrix computation of the exponential in the evaluation of the Padé approximation for phase type (Aalen and Sidje, 1993). The simulation of phase type distribution in a unique way is found in William (2009) and, some new results on Markov chain connected to a distribution of phase types in Christian and Stephane, (2010)

Furthermore, the second order recurrence relation with constant coefficient, limiting behaviour and recursion process to arrive at performance measures is considered Agboola (2011). The introductory of phase type in survival analysis is analysed on how a phenomenon such as a disease, moves through different phases. Also, the calculation of hazard rates and densities of phase-type distributions using Markov chain to affirmed that hazard rates are asymptotically constant due to quasi-stationarity Aalen (2014). A phase-type distribution approximation function so as to Steer clear of inverse matrix calculations is introduced Belen *et al.(* 2020)

The direct equation approaches for the stationary distribution of Markov chains to yield a far more accurate response in less time once a predetermined number of clearly defined stages have been achieved Agboola (2021), and the likelihood of transitioning to one or more of the closed communicative classes from transitory states and the probability of the absorption matrix is established Agboola and Ayoade (2021).

The block iterative approach only needs one iteration to produce the solution for the stationary distribution of Markov chains Agboola and Ayinde (2022), and a partial characterization of the set of busy period durations which are presented by an r-phases Coxian distribution Osogami and Harchol (2002). The phase-type model is extended to accommodate competing risks using a few data points and the Coxian competing risks model Bo Henry (2022). The multi-state processes models are illustrated when the dimension of the state space is greater than one to obtain the proportional hazards specification. Martin (2023)

In addition, the general approach for the two-layer censored data, using the canonical form of a cyclic phase-type distributions (APHDs) and the expectation algorithm to compute the estimate by maximum likelihood Yudong and Zhi-Scheng (2023). A phase-type distribution that is not homogenous for the cumulative hazard rate, reliability function, and hazard rate using the maximum likelihood, and the characteristic function Acal *et al.* (2024). The phase-type distributions in coalescent models showing the states in the ancestral process are represented by stages in the phase-type distribution and concluded that a mathematical foundation for coalescent theory is made possible by phase-type distributions *Hobolth et al. (2024).*

However, this paper's primary goal is to examine how phase-type technique might be altered to incorporate performance measure of phase type distribution using Erlang – r exponential distributions, and the mix Erlang- $(r-1)$ with Erlang- r distributions in order to evaluate the mean, k^{th} moment, variance, Laplace Stieltjes transform and phase type distributions' squared coefficient of variation.

Nomenclatures
$E(Y)$, expected value of random variable Y; μ, service time parameter; σ_y^2, variance; $f_Y(y)$, density function for a random variable Y; $F_Y(y)$, distribution function for a random variable Y; $L_y(s)$, Laplace transform of random variable Y; p_k , likelihood that only the first k service phases will be executed before the process ends; $E[Y^k]$, k^{th} moment of a random variable Y; $\propto_i$, the likelihood of going from state i to state $(i+1)$; c_y^2, squared coefficient of variation for random variable Y; R_i, $i = 1, 2, 3, \dots, k$, initial probabilities; r_{ij}, $i, j = 1, 2, 3, \dots, k$, routine probability.

METHODOLOGY
The study area emphasized the analysis of one- exponential service stage distribution, two – exponential service phases processes or Erlang -2 distribution, Erlang-r distribution n, mix Erlangs distributions and general phase distribution, with the evaluation of Erlang-r distributions, mix Erlangs distributons and general phase distribution to arrive at performance parameters, mean, variance, k^{th} moment, Laplace transform and squared coefficient of variation for general stage type distribution .

Two Exponential Service Phase or Erlang-2 Distribution
We started by looking at a random variable Y that represents a customer's service time at a service center in order to analyze the exponential distribution, which has a single exponential phase. This is the amount of time that the client spends getting service; it excludes any waiting time that may have occurred. We assume that this service time is exponentially distributed with parameter $\mu > 0$. This is shown graphically in Figure 1, where a circle containing the exponential distribution's parameter represents the single exponential phase. In order to service customers, they enter the phase from the left, stay there for a period of time that is exponentially distributed with parameter μ, and then leave to the right. Assuming we have the random variable Y, which has an exponential distribution with mean $E(Y) = \frac{1}{\mu}$ and variance $\sigma_y^2 = \frac{1}{\mu^2}$, to represent the time it takes for clients to arrive at a service center.

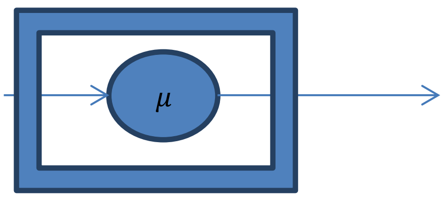

Figure 1: An Exponential Service Phase

The figure 1 indicates that, One exponential phase might be used to represent the service rendered to a user, while the figure 2 indicates that, the service time can be expressed by a second exponential phase.

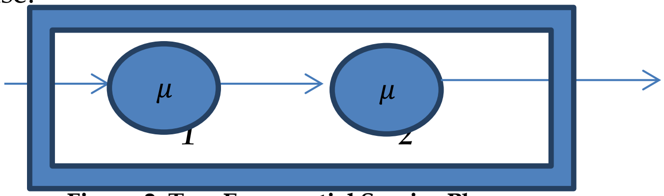

Figure 2: Two Exponential Service Phases

With random variable Y the customer receives service which is exponentially distributed with parameter μ as the customer enters the servicing process. At the completion of service's stage, the customer enters the second stage when the service time is exponentially distributed with parameter μ. At the second stage completion, another customer enter the phase when the service time is exponentially distributed with parameter μ. Since both service stages are the same exponentially distributed with parameter μ, and they are independent. Then, two independent servers are not containing in phases at the same time, but consist of one service provider operating in one or more stages at some point. In order to examine this instance, we will imagine that each phase's probability density function is given by

$$f_Y(y) = \mu \, e^{-\mu y}, \qquad (1)$$

such that

$$\text{Mean, } E(Y) = \frac{1}{\mu} \quad \text{and Variance,} \quad \sigma_y^2 = \frac{1}{\mu^2}.$$

The time selected at random from $f_Y(y)$ is first spent by the customer. After the time completion, another amount of time chosen independently from $f_Y(y)$ is again spent. After the completion of this second time chosen, the new customer starts to receive service immediately the one in service departs. The customer's overall time distribution in the service is now looked into, and this is taking to be the sum of two identically distributed random variables, X which is independent exponential random variables.
Taking the randomly dependent exponentially distributed variable with parameter μ to be Y. Then

$$X = Y + Y. \qquad (2)$$

Therefore, using the convolution theorem relating to two random variables that are independent,
The convolution theorem states that

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) \, f_Y(x-y) dy$$

$$f_X(x) = \int_0^x \mu \, e^{-\mu y} \, \mu \, e^{-\mu(x-y)} \, dy$$

$$f_X(x) = \mu^2 e^{-\mu y} \int_0^x dy = \begin{cases} \mu^2 x e^{-\mu y}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad (3)$$

The equation (3) represents the frequency density function for an Erlang-2, E_2 distribution, while the equation (4) below represents its cumulative distribution

$$F_X(x) = 1 - e^{-\mu x} - \mu x e^{-\mu x} = 1 - e^{-\mu x}\{1 + \mu x\}, \;\; x \geq 0. \qquad (4)$$

The density function can equally be computed using Laplace transforms, by multiplying the Laplace transform of the various phases by the Laplace transform of the frequency density function for the entire service time.

Therefore, the Laplace transform to determine the overall duration of service distribution is

$$L_X(s) = \int_0^\infty e^{-sx} f_X(x)dx$$

and each stage of the exponential phases' Laplace transform is

$$L_y(s) = \int_0^\infty e^{-sy} f_Y(y)dy = \left(\frac{\mu}{s+\mu}\right)$$

Then

$$L_X(s) = E\left[e^{-s\{y_1+y_2\}}\right] = E[e^{-sy_1}] \times = E[e^{-sy_2}] = \left(\frac{\mu}{s+\mu}\right)^2 \quad (5)$$

To find the function of x whose transform is $\left(\frac{\mu}{s+\mu}\right)^2$, the Laplace inversion theorem is being used.

Since the Laplace transform of $\frac{1}{(s+a)^{r+1}}$, i..e. $L_X\left(\frac{1}{(s+a)^{r+1}}\right) = \frac{x^r}{r\,!} e^{-ax}$.

By representing a = μ and r = 1, we arrived at inversion of $L_X(s)$ to obtain

$$f_X(x) = \mu^2 x e^{-\mu y}, \qquad x \geq 0$$

Likewise, we may obtain the mean value and higher moments from the Laplace transform as

$$E[X^k] = (-1)^k \frac{d^k}{ds^k} L_X(s)\bigg|_{s=0}, \quad for\ k = 1, 2, \cdots$$

$$E[X] = \frac{d}{ds} L_X(s)\bigg|_{s=0} = -\mu^2 \frac{d}{ds}\left(\frac{1}{s+\mu}\right)^2 = \mu^2 \frac{d}{ds}(s+\mu)^{-2}\bigg|_{s=0} = \frac{2}{\mu}. \quad (6)$$

$$\sigma_X^2 = \left(\frac{1}{\mu}\right)^2 + \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}. \quad (7)$$

The Erlang-r Distribution

As shown in Figure 3, an Erlang - r, E_r distribution is a series of r distinct but similar exponential stages with parameter μ. A customer visiting a service facility with a single Erlang- r server, each exponentially dispersed with parameter μ, must wait r consecutive periods of time, and no other client is permitted to enter the service facility before the service is accomplished.

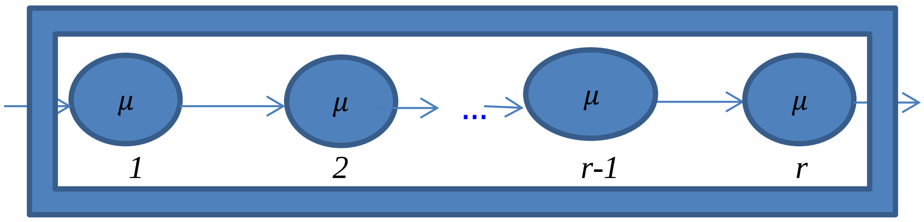

Figure 3: r – tandem exponential service stages

Because the density function can be used to determine how long a consumer spent in the first stage, we can then examine the situation as follow:

Given

$$f_y(y) = \mu\, e^{-\mu y}, \quad x \geq 0. \quad (7)$$

Considering each stage's mean value and variance as

$$E(y) = \frac{1}{\mu} \quad and \quad \sigma_y^2 = \frac{1}{\mu^2}, \tag{8}$$

Therefore,
The average and variation of a customer's overall service experience are

$$E(X) = r\left(\frac{1}{\mu}\right) = \frac{r}{\mu} \quad and \quad \sigma_y^2 = r\left(\frac{1}{\mu}\right)^2 = \left(\frac{r}{\mu^2}\right), \tag{9}$$

Suppose that the service time's Laplace transform is

$$L_X(s) = \left(\frac{\mu}{s+\mu}\right)^r. \tag{10}$$

Since the Laplace transform of

$$\left(\frac{1}{(s+a)^{r+1}}\right) \Leftrightarrow \frac{x^r}{r!}\ e^{-ax},$$

After applying the Laplace inversion theorem and inserting $a = u$, we obtained the frequency density function for the Erlang$-r$, E_r distribution of random variable X as

$$f_X(x) = \frac{\mu(\mu x)^{r-1}e^{-\mu x}}{(r-1)!}, \quad x \geq 0 \tag{11}$$

However, the associated cumulative distribution function is provided by

$$F_X(x) = 1 - e^{-\mu x}\sum_{I=0}^{r-1}\frac{(\mu x)^i}{i\,!}, \ x \geq 0, \quad r = 1, 2, \ldots \tag{12}$$

Assume that the number of arrivals throughout the service time period $[0, t]$ is represented by the random variable $N(t)$, which is a Poisson with parameter μt.
Therefore,
The probability of arrival of $(r - 1)$ number of customer is denoted by

$$prob\{N(t) \leq r - 1\} = \sum_{k=0}^{r-1}\frac{(\mu t)^k}{k\,!}\ e^{-\mu t}, \tag{13}$$

Let Z_r be the amount of time that will pass until the first r customers arrive.
We may first look at prob $\{Z_r > t\}$in order to determine prob $\{Z_r \leq t\}$.
Since the arrival of the r^{th} client will be more than t if $(r - 1)$ consumers arrive by time t
Prob $\{Z_r > t\}$ = Prob $\{N(t) \leq r - 1\}$

$$\text{Prob}\ \{Z_r \leq t\} = 1 - \text{Prob}\ \{Z_r > t\} = 1 - \sum_{k=0}^{r-1}\frac{(\mu t)^k}{k\,!}\ e^{-\mu t}, \ t \geq 0 \tag{14}$$

MIXING AN ERLANG- $(r - 1)$ DISTRIBUTION WITH ERLANG- r DISTRIBUTION

The distribution obtained by combining the Erlang-r and Erlang- (r-1) distributions is known as $E_{r-1,r}$. , and its squared coefficient of variation falls between $1/(r - 1)$ and $1/r$.

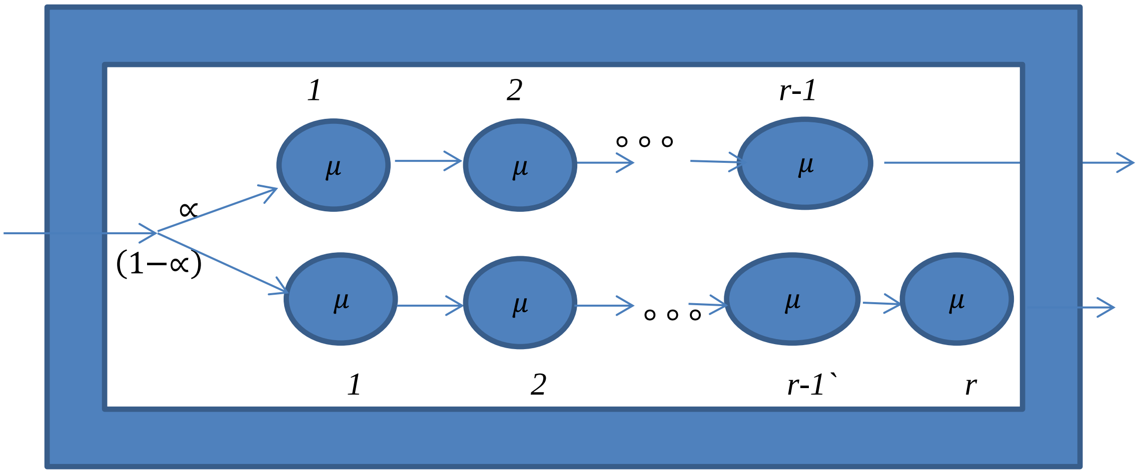

Figure 4: Mixed Erlang Representation

The mixed Erlang distribution is shown in Figure 4, where (1-∝) indicates the likelihood that the bottom series of r exponential phases will be chosen, and ∝ indicates the likelihood that the top series of $r-1$ exponential phases will be taken. Its probability function is provided by

$$f_Y(y) = \propto \mu\, e^{-\mu y} \frac{(\mu y)^{r-2}}{(r-2)!} + (1-\propto)\mu\, e^{-\mu y} \frac{(\mu y)^{r-1}}{(r-1)!} \quad , \quad y \geq 0. \qquad (15)$$

The mixed Erlang distribution's squared coefficient of variation is $1/(r-1)$ when ∝=1.
When $\propto = 0$, *it is equal to* $\frac{1}{r}$.
Intermediate values of C_Y^2 are produced for intermediate values of ∝.
The formulas provide the values of ∝ and μ to be utilized if the mean value E[y] and the squared coefficient of variation C_Y^2 fall within the range $\left[\frac{1}{r}, \frac{1}{r-1}\right]$.

$$\propto = \frac{1}{1+C_Y^2}\left[rC_Y^2 - \sqrt{r(1+C_Y^2) - r^2 C_Y^2}\right], \qquad (16)$$

And

$$\mu = \frac{r-\propto}{E[y]}. \qquad (17)$$

RESULTS
In order to determine the expectation, k^{th} moment, variance, and squared coefficient of variation of Z, along with its probability density function, the performance metrics for phase type distribution are presented in this section using the Erlang-r distribution.

Illustrative Example 1
Let Z_4 be the waiting time until the fourth arrival, and let $N(t)$ be a Poisson arrival process with rate $\mu = 0.5$. we are to determine Z_4's frequency density function and cumulative distribution function. Additionally, its variation and mean value, as well as the likelihood that the waiting period will exceed 12 time units.
Solution
In Equation (12), we substituted 0.5 for μ and 4 for r to obtain the cumulative distribution as

$$F_{Z_4}(t) = Prob\{Z_4 \leq t\} = 1 - Prob\{4 > t\} = 1 - e^{-\frac{t}{2}} \sum_{k=0}^{3} \frac{\left(\frac{t}{2}\right)^k}{k\,!}\,, \quad t \geq 0.$$

By directly entering the value into frequency density function of Equation (11) we obtain

$$f_{Z_4}(t) = \frac{\frac{1}{2}\left(\frac{t}{2}\right)^3 e^{-\left(\frac{t}{2}\right)}}{(3)!} = \frac{1}{96} t^3 \, e^{-\left(\frac{t}{2}\right)}, \quad t \geq 0.$$

Furthermore, the mean value and variance can be computed as

$$E[Z_4] = \frac{4}{1/2} = 8$$

$$\sigma[Z_4] = \sqrt{\frac{4}{0.25}} = 4$$

Where the variance

$$\sigma^2[Z_4] = 16.$$

$$Prob\{Z_4 > 12\} = 1 - Prob\{Z_4 \leq 12\} = e^{-6} \sum_{k=0}^{3} \frac{(6)^k}{k\,!} = e^{-6}\{1 + 6 + 18 + 36\} = 0.1512\,,$$

We can demonstrate that equation (12) is the distribution function with a matching density function provided by equation (11) by differentiating $F_Y(y)$ with respect to y. We have

$$f_Y(y) = \frac{d}{dy}\, F_Y(y) = \frac{\mu\, e^{-\mu y} \sum_{k=0}^{r-1} (\mu y)^k}{(k)!} - e^{-\mu y} \frac{\sum_{k=0}^{r-1} k\mu(\mu y)^{k-1}}{k\,!}$$

$$= \mu\, e^{-\mu y} + \frac{\mu\, e^{-\mu y} \sum_{k=1}^{r-1} (\mu y)^k}{(k)!} - e^{-\mu y} \frac{\sum_{k=1}^{r-1} k\mu(\mu y)^{k-1}}{k\,!}$$

$$= \mu\, e^{-\mu y} - \mu\, e^{-\mu y} \frac{\sum_{k=1}^{r-1}\{k(\mu y)^{k-1} - (\mu y)^k\}}{k\,!}$$

$$= \mu\, e^{-\mu y} \left\{1 - \frac{\sum_{k=1}^{r-1}\{k(\mu y)^{k-1} - (\mu y)^k\}}{k\,!}\right\}$$

$$= \mu\, e^{-\mu y} \left\{1 - \sum_{k=1}^{r-1} \left\{\frac{(\mu y)^{k-1}}{(k-1)!} - \frac{(\mu y)^k}{(k)!}\right\}\right\}$$

$$= \mu\, e^{-\mu y} \left\{1 - \left(1 - \frac{(\mu y)^{r-1}}{(r-1)!}\right)\right\} = \mu\, e^{-\mu y} \frac{(\mu y)^{r-1}}{(r-1)!}$$

To demonstrate that this density curve's area under the curve equals one

$$I_r = \int_0^\infty \mu\, e^{-\mu y} \frac{(\mu y)^{r-1}}{(r-1)!} dy = \int_0^\infty \frac{(\mu)^r (y)^{r-1}}{(r-1)!} e^{-\mu y} dy, \quad r = 1, 2, \ldots$$

Thus, by employing the integration by part technique, it is possible to demonstrate that I_r is the area under the exponential density curve. For example, by using $\int u dv = uv - \int v du,$ where

$$u = \frac{(\mu)^{r-1}(y)^{r-1}}{(r-1)!} \; and \;\; dv = \mu e^{-\mu y} dy$$

Therefore,

$$\int_0^\infty \frac{(\mu)^r (y)^{r-1}}{(r-1)!} e^{-\mu y} dy = \frac{-(\mu)^{r-1}(y)^{r-1}}{(r-1)!} e^{-\mu y} \Big|_{y=0}^{\infty} + \int_0^\infty \frac{(\mu)^{r-1}(y)^{r-2}}{(r-2)!} e^{-\mu y} dy$$

$$= 0 + I_{r-1}$$

This shows that

$$I_r = 1, \quad \forall\, r \geq 1.$$

For the Erlang-r distribution, the square coefficient of variation is provided as

$$C_Y^2 = \frac{r/\mu^2}{\left(r/\mu\right)^2} = \frac{1}{r} < 1, \;\; \forall\, r \geq 2.$$

The Erlang distribution's coefficient of variation is lower than the exponential distribution's, indicating that Erlang random variables are more regular than exponential ones.

Illustrative Example 2
Imagine a random variable Y that is represented by three successive exponential phases, each with the parameters $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 4$. Finding Y's mean value, variance, squared coefficient of variation, and probability density function are all of importance. Since the three exponential phases are independent of one another, the variance is thus equal to the sum of the variances of each phase, and the expectation of Y is simply equal to the sum of the expectations of each phase. Therefore

$$E(y) = \sum_{i=1}^{3} \frac{1}{\mu_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$\sigma^2(y) = \sum_{i=1}^{r} \frac{1}{\mu_I^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{61}{144}$$

$$C_y^2 = \frac{\sum_{i=1}^{r} \frac{1}{\mu_I^2}}{\left(\sum_{i=1}^{r} \frac{1}{\mu_i}\right)^2} = \frac{61}{144} \times \frac{144}{169} = \frac{61}{169} = 0.361 \leq 1$$

When $i = 1$,

$$\propto_1 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_1} = \frac{\mu_2}{\mu_2 - \mu_1} \times \frac{\mu_3}{\mu_3 - \mu_1} = \frac{3}{1} \times \frac{4}{2} = 6$$

When $i = 2$,

$$\propto_2 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_2} = \frac{\mu_1}{\mu_1 - \mu_2} \times \frac{\mu_3}{\mu_3 - \mu_2} = \frac{2}{-1} \times \frac{4}{1} = -8$$

When $i = 3$,

$$\propto_3 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_3} = \frac{\mu_2}{\mu_3 - \mu_3} \times \frac{\mu_1}{\mu_1 - \mu_3} = \frac{2}{-2} \times \frac{3}{-1} = 3$$

It follows then that

$$f_Y(y) = \sum_{i=1}^{r} \propto_i \mu_i\, e^{-\mu_i y} = 12e^{-2y} + 12e^{-2y} + 12e^{-2y} - 24e^{-3y} + 12e^{-4y}, y \geq 0.$$

If coefficients of variation larger than 1 are needed, neither the hypo-exponential distribution nor a combination of Erlang distributions can be applied, and instead of using phases in series as we have up to this point, we switch to phases in parallel.

Illustrative Example 3
Imagine a Coxian distribution with the initial probabilities vector $R = (1,0,0,0)$ that is represented by four successive phases with rates $\mu_1 = 1, \mu_2 = 2, \mu_3 = 4, \mu_4 = 8$ and $\mu_4 = 8$. At the completion of phase $i = 1,2,3$, the process moves to phase $i + 1$ with probability 0.5 or enters the sink phase with probability 0.5.
The performanc measures are obtained as follows:

Solution:
Given the initial probabilities vector $R = (1,0,0,0)$

$$\boldsymbol{R^I} = (\mathbf{1} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}) = (\boldsymbol{R} \quad \mathbf{0})$$

While the routine probabilities, r_{ij}, are the elements of the matrix

$$M = \begin{pmatrix} 0 & \propto_1 & 0 & \cdots & 0 \\ 0 & 0 & \propto_2 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \propto_{k-1} \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

Where α_i, is the probability that after receiving service at phase i, the process, move to a phase $(i+1)$ to continue service.

The terminal probabilities are $\varphi = \big((1-\alpha_1), (1-\alpha_2),\ (1-\alpha_3), \ldots, (1-\alpha_{k-1}),\ 1\big)$
Therefore,

$$M = \begin{pmatrix} -\mu_1 & 0.5\mu_1 & 0 & 0 & 0.5\mu_1 \\ 0 & -\mu_2 & 0.5\mu_2 & 0 & 0.5\mu_2. \\ 0 & 0 & -\mu_3 & 0.5\mu_3 & 0.5\mu_3 \\ 0 & 0 & 0 & -\mu_4 & \mu_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0.5 & 0 & 0 & 0.5 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -4 & 2 & 2 \\ 0 & 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} S & S^0 \\ 0 & 0 \end{pmatrix}$$

Given that

$$S^{-1} = \begin{pmatrix} -1 & -0.25 & -0.0625 & -0.015625 \\ 0 & -0.5 & -0.125 & -0.03125 \\ 0 & 0 & -0.25 & -0.0625 \\ 0 & 0 & 0 & -0.125 \end{pmatrix}$$

Where $-RS^{-1} = 1 \quad 0.25 \quad 0.0625 \quad 0.1563$.
Therefore, the mean time to absorption in the Markov chain is given as

$$-RS^{-1}e = 1.3281$$

In addition, the mean time spent in

$$\text{Phase 1: } \frac{1}{\mu_1} = 1 = y_1$$

$$\text{Phase 2: } \frac{0.5}{\mu_2} = 0.25 = y_2$$

$$\text{Phase 3: } \frac{0.5 \times 0.5}{\mu_3} = 0.0625 = y_3$$

$$\text{Phase 4: } \frac{(0.5)^3}{\mu_4} = 0.015625 = y_4$$

DISCUSSION

Example 1 show that by using the Erlang-r distribution, the squared coefficient of variation might have a variety of values, including

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{1}{4}, \ldots$$

This implies a mean of approximating a constant distribution; we produce a distribution with decreasing variance by increasing the number of phases (r) and setting the parameter at each phase to be $r\mu$. The variance goes to zero and the expectation stays at $\frac{1}{\mu}$ in the limit as $r \to \infty$. Example three determined the mean time to absorption and mean spent in 4 phases of the illustrated example

CONCLUSION

In this study, performance measures of phase type distribution using Erlang $-r$ distribution and mix Erlang distributions have been looked into, in order to provide meaningful study into the probability function, mean, k^{th} moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution. We begin from the tractability and memory less properties of exponential distribution, and since these properties are not enough, we examined the journey through a series of exponential phases by the use of matrix and vector operations to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for probability functions, Laplace Stieltjes transform, squared coefficient of variation, k^{th} moment, mean and variance for the phase type distribution

REFERENCES

Aalen, O.O. (2014). Phase Type Distribution in Survival Analysis, *Encyclopedia of Biostatistics* © John Wiley & Sons, **https://doi.org/10.1002/9781118445112.stat06048.**

Acal, C., Juan E.R., David, M. and Juan, B. (2024). One Cut Point Phase Type Distribution in Reliability. An Application to Resistive Random Variable. *Mathematics* 9(21), DOI:10.3390/math9212734.

Agboola, S.O. (2010). The Analysis of Markov Inter-arrival Queues Model with K – Server under Various Service Point. Unpublished M.Sc. (Statistics) thesis submitted to Department of Mathematics, Obafemi Awolowo University Ile – Ife, Nigeria.

Agboola, S.O. (2021). Direct Equation Solving Methods Algorithms Compositions of Lower-Upper Triangular Matrix and Grassman- TaksarHeyman for the Stationary Distribution of Markov Chain. *Interantional Journal of Applied Science and Mathematics*, 8(6): 87 – 96.

Agboola, S.O. and Ayinde S.A. (2022). On the Application of Succesive Over-relaxation Algorithmic and Block Numerical Iterative Solutions for the Stationary Distribution in Markov Chain. *Nigerian Journal of Pure and Applied Sciences*, 35 (1): 4263 -4272.

Agboola, S.O. and Ayoade, A.A. (2021). Analysis of Reachability Matrix and Absorption Probabilities for Close and Open Classification Group of States in Markov Chain. *Nigerian Journal of Scientific Research (NJSR),* 20 (5): 634 – 639.

Asger H. I., Rivas, G., Mogens, B. and Andreas F. (2024). Phase –Type Distributions in Mathematical Population Genetics. *Theoretical Population Biology*, 157(4). DOI:10.1016/j.tpb.2024.03.001.

Belen, G.O., Cristina, S. and Gregorio, A. (2020). A Phase Type Distribution for the Sum of Two Concatenated Markov Processes Application to Analysis Survival in Bladder Cancer. *Mathematics*, *8*(12), https://doi.org/10.3390/math8122099..

Henry B.O. and Lindqvist, A. (2022). Phase Type Models for Competing Risks with Emphasis on Identifiability issues. *Springer Nature Link*, 29: 318 – 341.

Christian, C. and Stéphane, M. (2010). Phase Type Distributions and Representations. Some Results and Open Problems for System Theory. *International Journal of Control*, 76(6): 566 – 580.

Cumani, O. (1982). On the Canonical Representation of Homogeneous Markov Processes Modeling Failure Time Distribution. Institute Elettronoco Nasionale, *Galileo Ferraria Strada Delle Cacce* 91 –I- 10135.

Marie, R. (1980). Calculating Equilibrium Probabilities for $\lambda(n)/C_k/1/N$ Queues. *ACM Sigmetrics Performance Evaluation Review*, 9(2): 117–125.

Martin, Bladt. (2022). Phase Type Distribution for Claim Severity Regression Modelling. *Astin Bulletin*, 52(2): 1 – 32.

Neuts, M.F. (1981). Matrix Geometric Solutions in Stochastic Models—An Algorithmic Approach. Johns Hopkins University Press, Baltimore, MD.

Neuts, M.F. (1989). Structured Stochastic Matrices of M/G/1 Type and Their Applications. Marcel Dekker, New York.

O'Cinneide, C. (1989). A continuous multivariate exponential distribution that is multivariate phase type, *Statistics and Probability Letters*, 7(4): 323 – 325.

Osogami, T. and Harchol-Balter, M. (2002). Necessary and Sufficient Conditions for Representing General Distributions by Coxians. *Carnegie Mellon University Technical Report, CMU-CS-02-178.* DOI:10.1007/978-3-540-45232-4_12.

Philippe, B. and Sidje, B. (1993). .Transient Solution of Markov Processes by Krylov Subspaces. Technical Report, IRISA—Campus de Beaulieu, Rennes, France.

Ramaswami, V. (1988). A Stable Recursion for the Steady State Vector in Markov chains of M/G/1 type. *Commun. Statist. Stochastic Models*, 4: 183–188.

Ramaswami, V. and Neuts, M.F. (1980). Some explicit formulas and computational methods for infinite server queues with phase type arrivals, *Journal of Applied Probability*, 17: 498–514.

Wiliam, S. (2009). Probability, Markov chain, Queueing and Simulation. Princeton University Press, Princeton and Oxford.

Yudong, Wang and Zhi, Scheng. (2023). Phase Type Distributions for Product Return Data with Two Layer Censoring'. *Journal of Royal Statistical Society Series C, Applied Statistics*, 72(5): 1475 – 1492. DOI:10.1093/jrsssc/qlad079.