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Equilibrium Points and Locations in the Photogravitational Circular Robe's Restricted Three-Body Problem with Variable Masses

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Abstract

The restricted three-body problem (R3BP) defines the dynamics of an infinitesimal mass moving in the gravitational neighborhood of two primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. In this paper, we examine equilibrium points and their locations in the photogravitational circular Robe's restricted three-body problem (R3BP) with variable masses. The motion of the primaries and variation in masses of the primaries are governed by the Gylden-Mestschersky problem (GMP) and the unified Mestschersky law (UML), respectively, while the second primary is assumed to be a radiation emitter. The non-autonomous equations of the governing dynamical system are deduced and transformed using the Mestschersky transformation (MT), the UML and the particular solutions of the GMP, to a system of the autonomized equations with constant coefficients under the condition that there is no fluid inside the first primary. Next, the equilibrium points (EPs) of the autonomized system are explored using perturbation method and it is seen that axial EP which is defined by the mass parameter and the radiation pressure of the second primary exists. Further, a pair of non-collinear EPs which depends on the mass parameter, radiation pressure of the second primary and a constant of the mass variations of the primaries, is found. The EPs of the non-autonomous system are obtained using the MT and differ from those of the autonomized system by time t. The EPs may be used in different problems of stellar dynamics, and also in other astrophysical applications.

Keywords: Robe's R3BP; Equilibrium Points; Variable mass; Radiation pressure

INTRODUCTION

The restricted three-body problem (R3BP) defines the motion of an infinitesimal mass moving in the gravitational environment of two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The study of the R3BP is of great theoretical, practical, historical and educational relevance, and in its many modified forms, has had important implications in several scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics.

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A different model of the R3BP was worked by Robe (1977). This problem was later called the Robe's Restricted Three Body Problem (RR3BP). In this formulation, the first primary of mass m_1, is a rigid spherical shell, filled with homogenous, incompressible fluid of density ρ_1, with the second mass m_2 as a small point outside the shell and moving around the first primary mass in a Keplerian orbit. The third body of infinitesimal mass m_3 is taken as a small sphere of density ρ_3, moving inside the shell and is subject to the attraction of m_2 and the buoyancy force due to the fluid and that the radius of m_3 is assumed to be infinitesimal . Several researchers such as Shrivastava and Garain (1991), Plastino and Plastino (1995), Giordano *et al.* (1997), Hallan and Rana (2001a, b), Hallan and Mangang (2007), Singh and Sandah (2012), Singh and Laraba (2012), Kaur and Aggarwal (2012), Singh and Omale (2014),-Ansari et. al (2019), Abouelmagd et *al*. (2020), Kaur et. al (2020), Kaur et. al (2021), Kaur and Kumar (2021), Ansari (2021), Ansari and Sahdev (2022), Kaur et. al (2022) and, Leke and Ahile (2022) have studied the Robe's R3BP under different modifications.

Next, the formulation of the classical R3BP assumes that the masses of the participating bodies are constant. However, the occurrence of isotropic radiation or absorption in stars led scientists to formulate the R3BP when one or all the bodies are subject to mass variations. The study of the R3BP in the case where the primaries vary their masses under the unified Mestschersky Law (UML) and their motion described by the Gylden-Mestschersky problem (Gylden 1884; Mestschersky 1902), has been carried out by several authors under different characterizations. Among such authors are Gelf'gat (1973), Bekov (1988), Luk'yanov (1989), Singh and Leke (2010, 2012, 2013a, b, c,), Taura and Leke (2022), Leke and Singh (2023), Leke and Mmaju (2023), Leke and Shima (2023), Leke and Amuda (2024), Leke and Orum (2024) and Amuda and Leke (2024).

Singh and Leke (2013b) discussed the Robe's R3BP with variable masses which occurs in accordance with the UML. They found an EP at the center, which is stable and a pair of EPs on the $\xi\zeta-$ plane which are unstable. Motivated by this, our aim in this paper is to obtain the EPs and locations of the photogravitational circular Robe's (R3BP with variable masses) when the bigger primary is empty. Aside the three forces acting on the infinitesimal mass, we consider a fourth force which is due to the radiation pressure force of the smaller primary. The radiation pressure force has been shown to have strong destabilizing influence on dynamics of small particles in space. Hence, it is important to consider such an important perturbing force on locations of the EPs in the Robe's R3BP with variable masses.

METHODOLOGY

Dynamical Equations

In this model formulation, we assume that the second primary is a radiating body. Therefore, the
forces acting on m_3 are

i. The force of attraction of the radiating second primary, which is given by

$$\vec{F}_{m_2} = -\frac{Gm_2 q_2 m_3 \bar{r}_{23}}{r_{23}^3}$$

ii. The gravitational force $\vec{F}_A$ exerted by the fluid

$$\vec{F}_A = -\frac{4\pi G \Re^3 m_3 \delta_1 \vec{r}_{13}}{3r_{13}^3}$$

iii. The buoyancy force which is given by

$$\vec{F}_B = \frac{4G\pi\Re^3 m_3}{3r_{13}^3}\frac{\delta_1^2\vec{r}_{13}}{\delta_3}$$

iv. Radiation pressure q_2 of the second primary

where G is the gravitational constant, $\Re$ is the radius of the fluid, $\delta_i\left(i=1,3\right)$is the density of the fluid and the test particle, respectively, while $\bar{r}_{13}$ and $\bar{r}_{23}$ is the radius vector of the line joining the centers of the first and second primary to the test particle, respectively and q_2 is the radiation pressure factor of the second primary.

Now, the equation of motion of the third body in the inertial system, taking into account the combined forces acting on it:

$$\ddot{\vec{R}} = \frac{Gm_2 q_2\vec{r}_{23}}{r_{23}^3} - \frac{4\pi\Re^3}{3}\frac{G\delta_1}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)\vec{r}_{13} \tag{1}$$

where $\overrightarrow{OM}_3$ is the distance from the origin of center of mass to the infinitesimal mass

The equation of motion (1) in a synodic coordinates system *Oxyz* rotating with angular velocity $\vec{\omega}$ and origin at the center of mass, O, of the primaries; is

$$\frac{\partial^2\vec{R}}{\partial t^2} + 2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t} + \vec{R}\times\frac{\partial\vec{\omega}}{\partial t} + \vec{\omega}\times\left(\vec{\omega}\times\vec{R}\right) = -\frac{4\pi\Re^3}{3}\frac{G\delta_1}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)\left(x-x_1\right) - \frac{Gm_2 q_2\left(x-x_2\right)}{r_{23}^3}$$

where $\vec{\omega} = \omega\hat{k}$: $\hat{k}$ is a unit vector.

Following Singh and Leke (2013b), the equations of motion of the third body under the set-up of the Robe's CR3BP with variable masses in a Cartesian coordinate system, when the second primary is a radiation source, takes the form

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega}y - \frac{4\pi\Re^3}{3}\frac{G\delta_1}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)\left(x-x_1\right) - \frac{\mu_2 q_2\left(x-x_2\right)}{r_{23}^3}\\
\ddot{y} + 2\omega\dot{x} &= \omega^2 y - \dot{\omega}x - \frac{4\pi\Re^3}{3}\frac{G\delta_1 y}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right) - \frac{\mu_2 q_2 y}{r_{23}^3}\\
\ddot{z} &= -\frac{4\pi\Re^3}{3}\frac{G\delta_1 z}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right) - \frac{\mu_2 q_2 z}{r_{23}^3}
\end{aligned} \tag{2}$$

where $\mu(t) = \mu_1(t) + \mu_2(t)$: $\mu_1(t) = Gm_1$, $\mu_2(t) = Gm_2(t)$

$r_{13}^2 = \left(x-x_1\right)^2 + y^2 + z^2$, $r_{23}^2 = \left(x-x_2\right)^2 + y^2 + z^2$ and ω is the angular velocity of revolution of the primaries.

Now, system (2) is a non-autonomous system of equations and we have to transform it to the autonomized forms. However, it is impossible to carry out a complete transformation using the MT, the UML, the particular solutions and integral of the GMP when the density parameter is not zero. In view of this, we restrict ourselves to the case when the densities of the first and third body are equal $\left(\delta_1 = \delta_3\right)$. Hence, the system of equations (2) reduces to

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega}y - \frac{\mu_2 q_2\left(x-x_2\right)}{r_{23}^3}\\
\ddot{y} + 2\omega\dot{x} &= \omega^2 y - \dot{\omega}x - \frac{\mu_2 q_2 y}{r_{23}^3}
\end{aligned} \tag{3}$$

$$\ddot{z} = -\frac{\mu_2 q_2 z}{r_{23}^3}$$

$$r_{23}^2 = \left(x - x_2\right)^2 + y^2 + z^2$$

Equations (3) in the autonomized system with constant coefficients is given as

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \eta'' + 2\xi' = \frac{\partial \Omega}{\partial \eta}, \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{4}$$

where $$\Omega = \frac{\kappa\left(\xi^2 + \eta^2\right)}{2} + \frac{(\kappa - 1)\zeta^2}{2} + \frac{\kappa \upsilon q_2}{\rho_{23}} \tag{5}$$

$$\rho_{23}^2 = (\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2 \tag{6}$$

where υ is the mass parameter and is such that $0 < \upsilon < 1$, while κ is the random sum of the masses of the primaries in the autonomized system.

Equations (4) admits the Jacobian integral

$$2\Omega(\xi, \eta, \zeta) - \left(\xi'^2 + \eta'^2 + \zeta'^2\right) = C \tag{7}$$

2.2. Locations of Equilibrium Points

The positions of the EPs are found by solving the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$. That is, we need to solve the system of equations

$$\begin{gathered}
k\left[\xi - \frac{q_2 \upsilon(\xi + \upsilon - 1)}{\left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}}\right] = 0 \\
k\eta\left[1 - \frac{q_2 \upsilon}{\left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}}\right] = 0 \\
\zeta\left\{(k - 1) - \frac{q_2 \kappa \upsilon}{\left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}}\right\} = 0
\end{gathered} \tag{8}$$

We search for the EPs on the $\xi\eta\zeta -$ plane of motion.

Clearly, from the last equation in (8), if $\kappa = 1$, we'll have

$$\frac{q_2 \upsilon \zeta}{\left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} = 0$$

If $\zeta \neq 0$, then

$$\frac{q_2 \upsilon}{\left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} = 0$$

which is not possible, since $\upsilon \neq 0, q_2 \neq 0$, so we must have $\zeta = 0$ when $\kappa = 1$.

Now, if in equations (8), we suppose that $\zeta = 0$, we have

$$k\left[\xi - \frac{q_2 \upsilon(\xi + \upsilon - 1)}{\left[(\xi + \upsilon - 1)^2 + \eta^2\right]^{\frac{3}{2}}}\right] = 0$$

$$\kappa\eta\left[1-\frac{q_2\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2\right]^{\frac{3}{2}}}\right]=0 \quad \eta\neq 0 \tag{9}$$

From the first equation of (9), we get

$$\kappa\left[1-\frac{q_2\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2\right]^{\frac{3}{2}}}\right]\xi+\frac{q_2\kappa\upsilon(1-\upsilon)}{\left[(\xi+\upsilon-1)^2+\eta^2\right]^{\frac{3}{2}}}=0 \tag{10}$$

Using the second equation of (9) in (10), we get

$$\frac{q_2\kappa\upsilon(1-\upsilon)}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}=0$$

But $\upsilon(1-\upsilon)\neq 0$, since $0<\upsilon<1$, hence, a contradiction and so we must have $\eta=0$.

Hence, there is no solution on the $\xi\eta-$plane and so we search for the solutions on the $\xi\zeta-$plane only.

2.2.1 Axial Equilibrium Point

The axial EP is the solution of equations (8) when $\eta=\zeta=0$. Meaning this point lie only on the $\xi-$plane of motion. Therefore, equations (8) is reduced to the equation

$$k\xi+\frac{q_2k\upsilon}{(1-\upsilon-\xi)^2}=0 \tag{11}$$

Now let

$$f(\xi)=k\xi+\frac{q_2k\upsilon}{(1-\upsilon-\xi)^2}=0 \tag{12}$$

Now $f(\xi)>0$ for $\xi<1-\upsilon$ therefore in the open interval $(-\infty,1-\upsilon)$ as $\xi\to-\infty$ $f(\xi)\to-\infty$ and as $\xi\to 1-\upsilon$, $f(\xi)\to+\infty$. Consequently $f(\xi)$ vanishes only once in the interval $(-\infty,1-\upsilon)$. Hence equation (12), has only one root in this interval.

To get the axial EP, we shall use perturbation method, by first ignoring the radiation pressure of the second primary. Thus, we suppose that $q_2=1$ then equation (11) becomes

$$\xi+\frac{\upsilon}{(1-\upsilon-\xi)^2}=0 \tag{13}$$

Following Singh and Leke (2013b), we get

$$\xi_1=-\upsilon \tag{14}$$

Hence, the only solution of equation (13) is equation (14) and so we can assume the solution of (11) to be

$\xi=-\upsilon+\chi$ (15)

where $\chi<<1$

Substituting (15) into equation (11), gives

$$-\upsilon\left(1-2\chi+\chi^2\right)+\chi\left(1-2\chi+\chi^2\right)+q_2\upsilon=0$$

Using Binomial and ignoring second and higher powers of χ, we get

$$\chi=\frac{\upsilon(1-q_2)}{(1+2\upsilon)}$$

Therefore, equation (15) becomes

$$\xi=-\upsilon\left[1-\frac{(1-q_2)}{1+2\upsilon}\right] \tag{16}$$

This gives an EP on the line joining the centers of the primaries away from the center of the first primary but located inside it and is defined by the mass parameter and radiation factor of the second primary. When there is no radiation from the second primary, the point fully coincides with that in Robe (1977), Hallan and Rana (2001a), and, Singh and Leke (2013b)

2.2.2 Non-collinear Equilibrium Points

The existence of the non-collinear EPs was not pointed out in the Robe (1977) problem. The non-collinear EPs of the autonomized system are the solutions of system (8), when $\eta = 0$, $\zeta \neq 0$. That is, they are obtained by solving the equations

$$k\left[\xi - \frac{q_2\upsilon(\xi+\upsilon-1)}{\left[(\xi+\upsilon-1)^2+\zeta^2\right]^{\frac{3}{2}}}\right] = 0$$

$$(k-1) - \frac{q_2\kappa\upsilon}{\left[(\xi+\upsilon-1)^2+\zeta^2\right]^{\frac{3}{2}}} = 0 \tag{17}$$

$$\rho_{23}^2 = (\xi+\upsilon-1)^2+\zeta^2 \tag{18}$$

From the second equation of (17), we have

$$\rho_{23} = \left(\frac{q_2\kappa\upsilon}{k-1}\right)^{\frac{1}{3}} \tag{19}$$

Substituting equation (19) in the first equation of (17), at once gives

$$\xi = -(1-\upsilon)(\kappa-1) \tag{20}$$

This gives the abscissa of the non-collinear EPs.

Next, we substitute equation (19) in equation (18), to get

$$\left(\frac{q_2\kappa\upsilon}{k-1}\right)^{\frac{2}{3}} = (\xi+\upsilon-1)^2+\zeta^2$$

Using equation (20), yields

$$\zeta = \pm\sqrt{\left(\frac{q_2\kappa\upsilon}{k-1}\right)^{\frac{2}{3}} - \kappa^2(1-\upsilon)^2}$$

And so we have the coordinates of the non-collinear EPs is represented by the equations

$$\xi = -(1-\upsilon)(\kappa-1) \text{ and } \zeta = \pm\sqrt{\left(\frac{q_2\kappa\upsilon}{k-1}\right)^{\frac{2}{3}} - \kappa^2(1-\upsilon)^2} \tag{21}$$

Equations (21) give the position of a pair of EPs $(\xi, 0, \zeta)$ which exist for $\kappa > 1$ and lies in the $\xi\zeta-$plane. We call these points, non-collinear EPs and they depend on the mass ratio, mass variation parameter and radiation pressure of the second primary.

For the non-autonomous system, the EP near the center of the shell and the non-collinear points are sought using the MT. These points differ from those of the autonomized system with constant coefficient only by the function $R(t)$. We express them as

$$x^{(1)}(t) = \xi^{(1)}R(t),\ x^{(2)}(t) = \xi^{(2)}R(t),\ z^{(2,3)}(t) = \zeta^{(2,3)}R(t) \tag{22}$$

where $x^{(1)}$ is the axial EP which varies with time while $x^{(2)}$ and $z^{(2,3)}$ are the non-collinear EPs of the non-autonomous systems varying with time. $\xi^{(1)}$, and $\xi^{(2)}$, $\zeta^{(2,3)}$ are the axial and non-collinear EPs, of the autonomized system, respectively.

3Numerical Exploration

In this section, we present the numerical results of the obtained EPs and we shall consider the third body to be an artificial satellite under the gravitational influence of two primaries. We carry out all numerical exploration with the help of the software *Mathematica* (Wolfram 2015). Throughout, we select the radiation pressure of the second primary to be $q_2 = 0.99996$.

3.1 Numerical computation of the axial equilibrium point

The axial EP has been obtained in equation (16) and depends on the mass parameter and the radiation pressure of the second primary. We now compute numerically locations of the axial EP in Table 1 when $0 < \upsilon < 1$ and $q_2 = 0.99996$.

Table 1: Axial EPs for $0 < \upsilon < 1$ and $q_2 = 0.99996$.

Mass Ratio (υ)	$\xi(q_2 = 1)$	-2υ	$\xi(q_2 = 0.99996)$	Deviation due to effect of radiation
0.000000001	-0.000000001	-0.000000002	-9.9996*10^-10	4*10^-14
0.00001	-0.00001	-0.000020000	-0.00000999	0.00000001
0.001	-0.001	-0.002000000	-0.00099996	0.00000004
0.012	-0.012	-0.024000000	-0.01199950	0.0000005
0.1	-0.1	-0.200000000	-0.09999670	0.0000033
0.2	-0.2	-0.400000000	-0.19999400	0.0000060
0.3	-0.3	-0.599985000	-0.29999300	0.0000070
0.4	-0.4	-0.799982000	-0.39999100	0.0000090
0.5	-0.5	-1.000000000	-0.49999000	0.0000100
0.6	-0.6	-1.200000000	-0.59998900	0.0000110
0.7	-0.7	-1.400000000	-0.69998800	0.0000120
0.8	-0.8	-1.600000000	-0.79998800	0.0000120
0.9	-0.9	-1.800000000	-0.89998700	0.0000130
0.9999	-0.9999	-1.999800000	-0.99988700	0.0000300

The location of the axial EP has been explored numerically for all mass ratio under effect of the radiation pressure of the second primary. It is seen from Table 1, that every point lie inside the first primary for all mass ratio since $-2\upsilon < \xi < 0$ throughout. When the second primary is not a radiating body, the EP reduces to that at the centre of the first primary. The effect of the radiation pressure of the second primary on the deviations of the axial point at the centre is shown on the last column on Table 1. It is seen that the points deviate to the right of the centre towards the origin. The points have been plotted in Figure 1 using the numerical values in Table 1. The axial point designated with a red spot is when the mass parameter is $\upsilon = 0.00001$ and the corresponding value is $\xi = -0.00000999$, while the points designated with a blue spot is when $\xi = -0.01199950$ and correspond to the case when $\upsilon = 0.012$. Also, the point designated with a green spot is the location of the axial point at $\xi = -0.29999300$ which correspond to the case when $\upsilon = 0.3$. The points designated with black and yellow spots are respectively when $\xi = -0.69998800$ and $\xi = -0.99988700$, respectively. These correspond to the case when $\upsilon = 0.7$ and $\upsilon = 0.9999$, respectively. All these points lie on the line inside the first primary. The points differ from those of Robe (1977), Shrivastava and Garain (1991), Hallan and Rana (2001a), Singh and Leke (2013b, c)

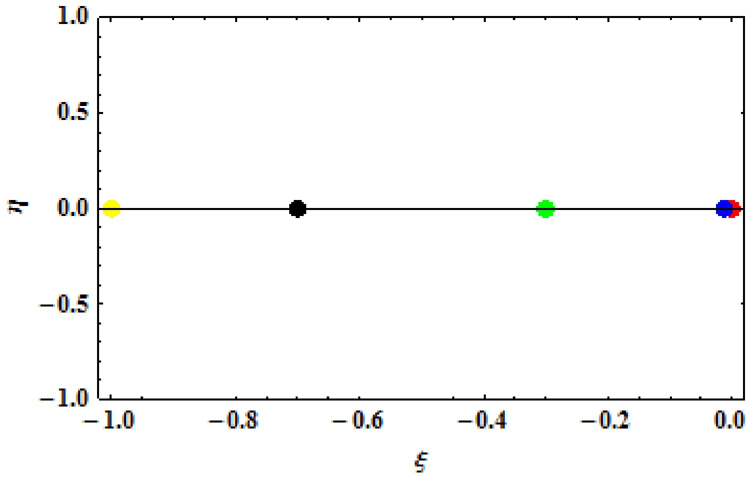


Fig 1: Axial EPs for $\upsilon = 0.00001$ (Red), $\upsilon = 0.012$ (Blue), $\upsilon = 0.3$ (Green), $\upsilon = 0.7$ (Black) and $\upsilon = 0.9999$ (Yellow).

Numerical computations of the non-collinear equilibrium points

Equations (21) give the positions of a pair of EPs $(\xi, 0, \zeta)$ which exist for $\kappa > 1$ and lies in the $\xi\zeta-$plane. We call these points, non-collinear EPs and they depend on the mass ratio and a constant of a particular integral of the GMP and radiation pressure of the second primary. The locations of the non-collinear EPs have been computed numerically in Table 2to Table 9using equation (21). We have indicated under the remark column whether the points exist or do not exists. The remark that they exist means the points are located inside the first primary, while they do not exist when the points are located outside the first primary.

Table 2: Non-collinear EPs for $\upsilon = 0.000000001,\ q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	ζ	Remarks
1.000000001	-0.000000001	-0.0000133607	Exists
1.00001	-0.00001	-0.498933	Exists
1.01	-0.01	-0.510039	Exists
1.1	-0.1	-0.604998	Exists
2	-1	-2	Exists
5	-4	-12.5	Does not Exists
10	-9	-50	Does not Exists
20	-19	-200	Does not Exists
50	-49	-1250	Does not Exists
100	-99	-500	Does not Exists
1000	-999	-500000	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 3: Non-collinear EPs for $\upsilon = 0.00001,\ q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	ζ	Remarks
1.000000001	-9.9999*10^-10	231.573	Does not Exists
1.00001	-0.0000099	-0.0000100001	Exists
1.01	-0.0099999	-0.505007	Exists

1.1	-0.099999	-0.60384	Exists
2	-0.99999	-1.99959	Exists
5	-3.99996	-12.4995	Does not Exists
10	-8.99991	-49.9988	Does not Exists
20	-18.9998	-199.996	Does not Exists
50	-48.9995	-1249.97	Does not Exists
100	-98.999	-4999.9	Does not Exists
1000	-998.99	-499990.	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 4: Non-collinear EPs for $\upsilon = 0.012$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	ζ	Remarks
1.000000001	-9.88*10^-10	26206.2	Does not Exists
1.00001	-9.88*10^-6	55.973	Does not Exists
1.01	-0.00988	0.0704821	Exists
1.1	-0.0988	-0.460946	Exists
2	-0.988	-1.91069	Exists
5	-3.952	-12.1714	Does not Exists
10	-8.892	-48.7791	Does not Exists
20	-18.772	-195.202	Does not Exists
50	-48.412	-1220.15	Does not Exists
100	-97.812	-4880.69	Does not Exists
1000	-987.012	-488072.	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 5: Non-collinear EPs for $\upsilon = 0.5$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	$\pm\zeta$	Remarks
1.000000001	-5.*10^-10	314972.	Does not Exists
1.00001	0.000005	678.466	Does not Exists
1.01	-0.005	6.70351	Does not Exists
1.1	-0.05	1.40663	Exists
2	-0.5	-0.0000133334	Exists
5	-2	-2.75951	Lies on the circle
10	-4.5	-12.1621	Does not Exists
20	-9.5	-49.6741	Does not Exists
50	-24.5	-312.181	Does not Exists
100	-49.5	-1249.68	Does not Exists
1000	-499.5	-125000.	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 6: Non-collinear EPs for $\upsilon = 0.7$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	$\pm\zeta$	Remarks
1.000000001	-3.*10^-10	394176	Does not Exists
1.00001	-3.*10^-6	849.188	Does not Exists
1.01	-0.003	8.50289	Does not Exists
1.1	-0.03	1.89518	Exists
2	-0.3	0.44571	Exists
5	-1.2	-0.66759	Exists
10	-2.7	-4.07714	Does not Exists
20	-5.7	-17.5921	Does not Exists
50	-14.7	-112.100	Does not Exists
100	-29.7	-449.603	Does not Exists
1000	-299.7	-44999.6	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 7: Non-collinear EPs for $\upsilon = 0.8$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	$\pm\zeta$	Remarks
1.000000001	-2.*10^-10	430875.	Does not Exists
1.00001	0.000002	928.279	Does not Exists
1.01	-0.002	9.32431	Does not Exists
1.1	-0.02	2.10695	Exists

2	-0.2	0.60397	Exists
5	-0.8	-0.00001	Exists
10	-1.8	-1.53777	Exists
20	-3.8	-7.55414	Does not Exists
50	-9.8	-49.5633	Does not Exists
100	-19.8	-199.566	Does not Exists
1000	-199.8	-19999.6.	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 8: Non-collinear EPs for $\upsilon = 0.9$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	$\pm\zeta$	Remarks
1.000000001	-1.*10^-10	466072.0	Does not Exists
1.00001	0.000001	1004.120	Does not Exists
1.01	-0.001	10.10300	Does not Exists
1.1	-0.01	2.299190	Exists
2	-0.1	-0.719844	Exists
5	-0.4	0.415829	Exists
10	-0.9	-0.000013	Exists
20	-1.9	-1.517710	Lies on the circle
50	-4.9	-12.02760	Does not Exists
100	-9.9	-49.53080	Does not Exists
1000	-99.9	-4999.530	Does not Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

Table 9: Non-collinear EPs for $\upsilon = 0.9999$, $q_2 = 0.99996$ and $1 < \kappa < \infty$

κ	ξ	$\pm\zeta$	Remarks
1.000000001	-5.*10^-13	499953.0	Does not Exists
1.00001	0.000000001	1077.120	Does not Exists
1.01	-0.000001	10.84290	Does not Exists
1.1	-0.00001	2.472810	Exists
2	-0.0001	0.793626	Exists
5	-0.0004	0.580144	Exists
10	-0.0009	0.536332	Exists
20	-0.0019	0.517343	Exists
50	-0..0049	0.506720	Exists
100	-0.0099	0.503264	Exists
1000	-0.0999	0.495287	Exists
$\kappa \to \infty$	$\kappa \to -\infty$	$\kappa \to -\infty$	Does not Exists

The points differ from those in Singh and Leke (2013b) due to the radiation pressure of the second primary. These EPs do not have an analogy in the Robe (1977) problem when the spherical shell is empty. They are also different from those of the R3BP of Bekov (1988), Luk'yanov (1989, 1990), Singh and Leke (2010) and Leke and Singh (2023).When $\kappa = 1$, we have $\xi = 0$ which lies outside the shell and since $\zeta = \pm\infty$, it is seen that infinite remote EPs $L\pm\infty$ do not exist for any value of $0 < \upsilon < 1$.

The non-collinear EPs have been drawn in Fig 5 to Fig 7 using the numerical data in Tables 2 to 9. In particular, Fig 2 is the location of the non-collinear points when $\upsilon = 0.000000001$ and $\kappa = 1.000000001$ (Red), $\kappa = 1.01$ (Blue) and $\kappa = 1.1$ (Green), while Fig 3 is non-collinear EPs for $\upsilon = 0.012$ when $\kappa = 1.01$ (Red), $\kappa = 1.1$ (Blue) and $\kappa = 2$ (Green). Also, Figures 4a-c are the non-collinear EPs for $\upsilon = 0.7$, $\upsilon = 0.8$ and $\upsilon = 0.9999$, respectively, when $\kappa = 1.1$ (Red), $\kappa = 2$ (Blue) and $\kappa = 5$ (Green)., while the variations in the positions of the non-collinear points corresponding to when $\upsilon = 0.7$ (red) and $\upsilon = 0.8$ (Blue plot) have been drawn in Fig 4d.

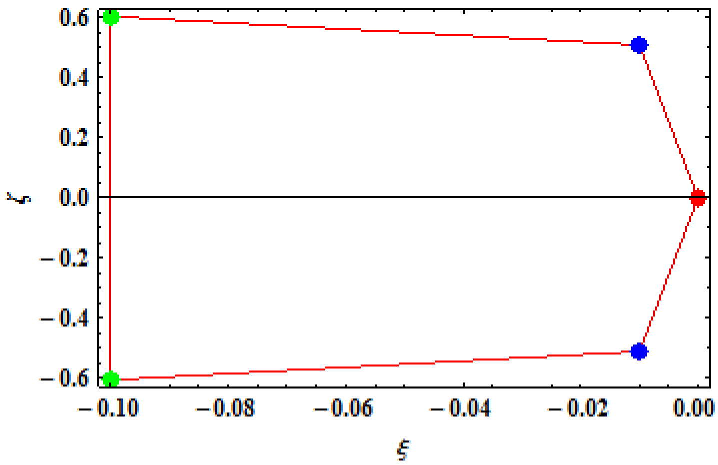


Fig 2 Non-collinear EPs for $\upsilon = 0.000000001$ when $\kappa = 1.000000001$ (Red), $\kappa = 1.01$ (Blue) and $\kappa = 1.1$ (Green)

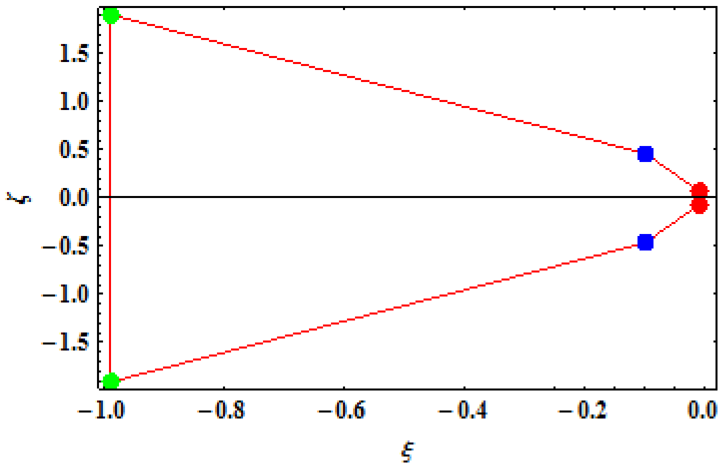
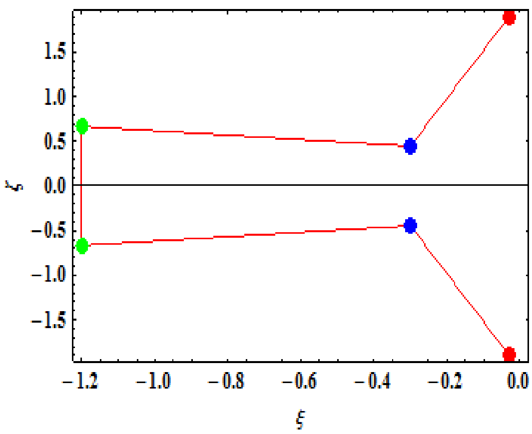
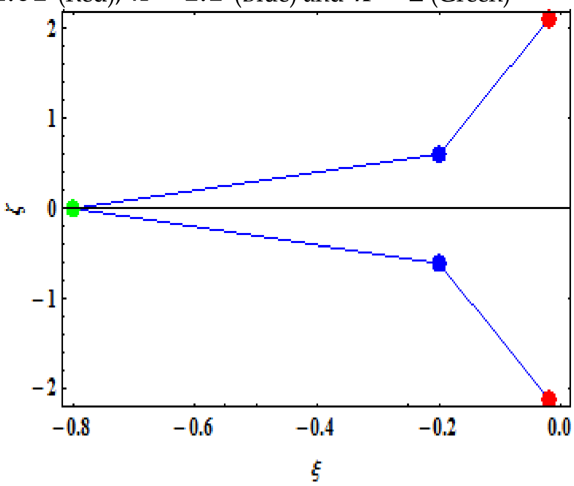


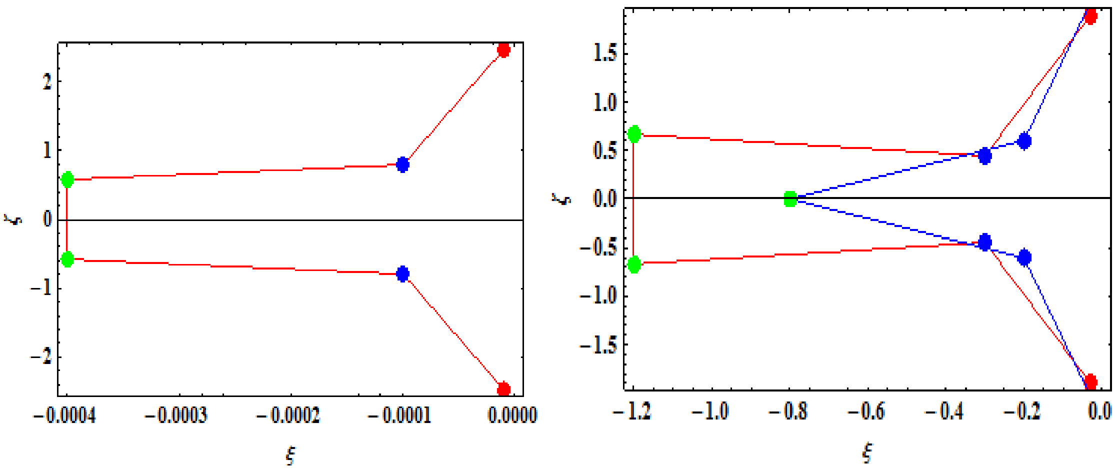
Fig 3: Non-collinear EPs for $\upsilon = 0.012$ when $\kappa = 1.01$ (Red), $\kappa = 1.1$ (Blue) and $\kappa = 2$ (Green)



(a)



(b)



(c) (d)

Fig 4: Non-collinear EPs for $\kappa = 1.1$ (Red), $\kappa = 2$ (Blue) and $\kappa = 5$ (Green) when (a) $\upsilon = 0.7$ (b) $\upsilon = 0.8$ (c) $\upsilon = 0.9999$ (d) a and b

CONCLUSION

The equilibrium points and locations in the photogravitational circular Robe's R3BP with variable masses, has been investigated. The primaries are assumed to move under the Gylden-Mestschersky problem while their masses vary with time in accordance with the UML and the second primary was a radiation emitter. The non-autonomous equations of the governing dynamical system were deduced and transformed to a system of the autonomized equations with constant coefficients under the condition that the first primary had no fluid. Next, the EPs of the autonomized system were explored and was observed that axial EP which depends on the mass parameter and the radiation pressure of the second primary exists. Further, a pair of non-collinear EPs which were defined by the mass parameter, radiation pressure of the second primary and the parameter kappa, were found. The EPs of the non-autonomous systems were obtained using the MT and differ from those of the autonomized system by the function $R(t)$.

The obtained EPs may be used in different problems of stellar dynamics, and also in other astrophysical applications.

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