https://dx.doi.org/10.4314/dujopas.v10i2c.25
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
**Dutse Journal of Pure and Applied Sciences (DUJOPAS),** *Vol. 10 No. 2c June 2024*

# Bounds of Substructures of Partially Ordered Multisets

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## Abstract

*Multiset theory naturally extends the classical set theory by considering the number of times an object occurs in a given collection. This work studies notions of bounds on an ordered multiset. The ordered multiset structure is obtained via the ordering induced by the underlying base set, which is assumed to be partially ordered. Analogous concepts of infimum and supremum are defined on substructures of the ordered multiset. In the sequel, some examples are outlined and new results are established on the ordered multiset structure.*

**Keywords**: Ordered set, multiset, partially ordered multiset, bounds, supremum, Infimum

## INTRODUCTION

Multisets have been described as mathematical models of entities with repetitions and are usually defined in relation to the Cantorian sets that support them. These non-classical sets are investigated using first-order predicate calculus; for instance, the formula $x \in^n y$ implies that an object $x$ occurs $n$ number of times in $y$ (Blizard, 1988). Multisets have applications in fields such as logic, linguistics, and computer science (Singh et al., 2007). In computer science, for instance, multisets are used to prove the termination of certain computer programs (Ruiz-Reina et al., 2003; Dershowitz & Manna, 1979). Multiset theory is usually studied as an extension of set theory; hence, under suitable assumptions and with the appropriate constructions, results from set theory can be extended to multiset theory (Ejegwa et al., 2023; Ifetoye et al., 2022; Balogun et al., 2020, 2021, 2022; Dang, 2014; Girish & Sunil, 2009; Blizard, 1988). In this article, the concept of bounds of a partially ordered set (see Schroder, 2003; Trotter, 1992, 1995 for expositions on ordered sets) is generalized via a partially ordered multiset structure (or pomset). For the purpose of this study, we will consider *finite multisets*. A multiset is finite if it contains a finite number of objects with finite multiplicities, it is infinite otherwise. The multisets used belong to the class of multisets that are defined over a partially ordered base set. The adopted multiset order is an extension of the partial order on the base set. This multiset order is consistent with the definition presented in Dershowitz & Manna (1979), often called the standard multiset ordering. The Dershowitz-Manna definition compares two multisets whereas the ordering in this work describes the hierarchy between the points of a finite multiset. Concepts of bounds, infimum and supremum of substructures of the ordered multiset are defined and some results are proved with examples. To make the

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article self-contained, section 2 is devoted to basic definitions and operations on multisets (Singh et al., 2007). The ordered multiset structure used in this study will also be discussed in section 2 (see Balogun et al., 2021 for details). In section 3, bounds of substructures of the ordered multiset structure will be presented. The concept of infimum and supremum of ordered submultisets is investigated in section 4 and new results are established. We conclude in section 5 with some recommendations.

## BASIC OPERATIONS AND MULTISET ORDER RELATIONS

In this section, preliminaries and basic operations on multisets are presented. We also discuss order relations on multiset structures.

### Multisets

A *multiset* $A$ is a function from a generic set $X$ into some set of numbers. $A(x_i)$ represents the number of times the object $x_i \in X$ occurs in $A$ for $i = 1,2, \ldots$. The number of copies of $x_i$ in $A$ is usually referred to as its *multiplicity*. The *root set* of $A$, denoted by $A^*$, is the set $\{x_i \in X | A(x_i) > 0\}$. The class of all multisets containing objects from $X$ is denoted by $\boldsymbol{A}(X)$. For distinction, the elements of the root set are referred to as *objects* while their individual occurrences make up the *elements* of the multiset. For instance, if a multiset $A$ is given by $[a_1, a_2, a_3, a_2, a_1, a_1, a_1, a_3, a_4, a_3, a_1, a_4]$, then $A^* = [a_1, a_2, a_3, a_4]$. Thus the objects of $A$ are the members of $A^*$. The axioms of **MST** (Blizard, 1988) will be assumed in this study. A *point* in a multiset $A$ represents an object $x_i$ in $A$ and its multiplicity, this will be denoted by $A(x_i)|x_i$, where $A(x_i) \in \mathbb{N}$ for all $i$ (see Felisiak et al., 2020 for a generalized theory). For any two multisets, $A$ and $B$ in $\boldsymbol{A}(X)$ the multiset $A$ is called a *submultiset* of $B$, if $A^* \subseteq B^*$ and $A(x_i) \leq B(x_i)$ for all $x_i \in X$. $A$ is a proper submultiset of $B$ if $A(x_i) < B(x_i)$ for at least one $x_i$. The multiset $B$ is called the *parent multiset* of $A$. A *whole submultiset* contains objects with the same multiplicity as they appear in the parent multiset i.e., $A$ is a whole submultiset of $B$ if $A(x_i) = B(x_i)$ whenever $x_i \in A \cap B$. A *full submultiset* contains all objects of the parent multiset, but not necessarily with the same multiplicities.

The *union* of two multisets $A \cup B$ is the multiset defined by $A(x_i) \cup B(x_i) = \max\,(A(x_i), B(x_i))$ for each $x_i \in X$. The minimum is taken whenever the maximum does not exist, as in the case of infinite multisets. The *intersection* of $A$ and $B$ is the multiset given by $A(x_i) \cap B(x_i) = \min\,(A(x_i), B(x_i))$ for each $x_i \in X$. The *additive union* of $A$ and B is the multiset given by $A(x_i) \uplus B(x_i) = A(x_i) + B(x_i)$ for each $x_i \in X$. For these three operations, the properties of commutativity, associativity, and idempotency hold. Also, the multiset operation $\uplus$ distributes over $\cup$ and $\cap$.

### Multiset Order Relations

Let $A$ be a multiset and $A(x_i)|x_i$, $A(x_j)|x_j$ be any two points in $A$. A submultiset $R$ of $A \times A$ is called a multiset relation on $A$ if every pair $(A(x_i)|x_i, A(x_j)|x_j)$ of $R$ have multiplicity $A(x_i)\, A(x_j)$. The domain and range of $R$ are given by

$Dom\ R = \{A(x_i)|x_i \in A; \exists A(x_j)|x_j \in A\ such\ that\ A(x_i)|x_i\ R\ A(x_j)|x_j\}$ and

$Ran\ R = \{A(x_j)|x_j \in A; \exists A(x_i)|x_i \in A\ such\ that\ A(x_i)|x_i\ R\ A(x_j)|x_j\}$, respectively (details on multiset relations are presented in Girish & Sunil, 2009).

For comparability and incomparability, the following ordering is assumed on the points of a finite multiset $A$ (see Balogun & Tella, 2018 for an alternative definition of a multiset ordering induced by the generic set):

**Definition 2.1**
Let $A \in \boldsymbol{A}(X)$. For any pair of points $A(x_i)|x_i, A(x_j)|x_j \in A$, where $i, j \in \{1,2, \dots, n\}$, $A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j$ if and only if $x_i \preccurlyeq x_j$ in $X$. The two points coincide if $x_i = x_j$.
*Remark:* The points $A(x_i)|x_i$ and $A(x_j)|x_j$ are *comparable* if $(A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j) \vee (A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i)$, otherwise they are *incomparable*, this will be denoted by $A(x_i)|x_i||A(x_j)|x_j$.

**Definition 2.2**
A partially ordered multiset (pomset) is a multiset with a reflexive, antisymmetric, and transitive multiset relation defined on it.
*Remark:* For any multiset $A \in \boldsymbol{A}(X)$, the pair $(A, \preccurlyeq\leq)$ with '$\preccurlyeq\leq$' as defined in definition 2.1 is a pomset. This will also be denoted by $\mathcal{P}$. The ordering on $A$ is a linear multiset order if it is a partial multiset order, and $(A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j) \vee (A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i)$ holds for any pair of points $A(x_i)|x_i, A(x_j)|x_j \in A$. The strict ordering $A(x_i)|x_i \prec< A(x_j)|x_j$ on $A$ implies $A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j$ and $A(x_i)|x_i \neq\neq A(x_j)|x_j$.

**BOUNDS OF SUBSTRUCTURES OF ORDERED MULTISETS**
This section presents bounds of subpomsets of an ordered multiset with some results.

**Definition 3.1**
Let $A(x_i)|x_i$ be an arbitrary point of a multiset $M$. Then $A(x_i)|x_i$ is *minimal* (resp. *maximal*) in the pomset $\mathcal{P} = (A, \preccurlyeq\leq)$ if for any other point $A(x_j)|x_j \in A$ with $A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i$ (resp. $A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j$), we have that $A(x_i)|x_i == A(x_j)|x_j$. A unique minimal (resp. maximal) point, if it exists, is called a *minimum or least* (resp. *maximum* or *greatest*) point.

**Proposition 3.2**
In a linearly ordered multiset, minimal and minimum points are the same.

*Proof:*
Let $\mathcal{P} = (A, \preccurlyeq\leq)$ be a linearly ordered multiset. Then $(A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j) \vee (A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i)$ holds for any arbitrary pair of points $A(x_i)|x_i, A(x_j)|x_j \in A$. Suppose $A(x_i)|x_i$ and $A(x_j)|x_j$ are the minimal and minimum points, respectively, in $\mathcal{P}$. By definition, $A(x_i)|x_i$ is minimal implies

$A(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j$ in $\mathcal{P}$ for all other points $A(x_j)|x_j \in A$ (1)

But $A(x_j)|x_j$ is a minimum point, hence

$A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i$ (2)

By antisymmetry we get $A(x_i)|x_i == A(x_j)|x_j$ □

**Definition 3.3**
A restriction of the ordering $\preccurlyeq\leq$ to any submultiset of $A$ in a pomset $\mathcal{P} = (A, \preccurlyeq\leq)$ is called a *suborder* of $\preccurlyeq\leq$.
*Remark:* Any suborder of a partial order is a partial order since the properties of reflexivity, antisymmetry, and transitivity are preserved by suborders. The submultiset structures obtained via the action in definition 3.3 will be called a subpomsets of $\mathcal{P}$.

**Definition 3.4**
A multiset chain (or connected subpomset) is a subpomset of $\mathcal{P}$ with a linear order on it. A multiset antichain is a subpomset of $\mathcal{P}$ with no comparable points.

**Definition 3.5**
A subpomset of $\mathcal{P}$ is maximal if it is not strictly contained in any other subpomset of $\mathcal{P}$. It is minimal if it does not contain any other subpomset.
**Definition 3.6**
A pomset is well-ordered if every subpomset has a minimum (or least) point.

**Definition 3.7**
Let $\mathcal{P} = (A, \preccurlyeq\leq)$ be a pomset and $B \subseteq A$. A point $A(x_i)|x_i \in A$ bounds $B$ from below if for all $B(x_j)|x_j \in B$ we have, $A(x_i)|x_i \preccurlyeq\leq B(x_j)|x_j$, where $i, j \in \{1,2, \dots, n\}$.
Upper bounds are defined analogously.
*Remark:* A subpomset does not necessarily contain its lower (resp. upper) bound(s).

**Proposition 3.8**
If the lower bound of a minimal subpomset exists, it is contained in it.

*Proof:*
Let $\mathcal{H} = (B, \preccurlyeq\leq_{\mathcal{H}})$ be a minimal subpomset of a pomset $\mathcal{P} = (A, \preccurlyeq\leq)$, where $\preccurlyeq\leq_{\mathcal{H}}$ is a suborder of $\preccurlyeq\leq$ and $B \subset A$. We will consider the following cases:
Case 1: If the minimal subpomset $\mathcal{H} = (B, \preccurlyeq\leq_{\mathcal{H}})$ is connected, then $\mathcal{H}$ contains a minimum(or least) point, say $B(x_i)|x_i$. Since $\mathcal{H}$ is minimal, either $B(x_i)|x_i \preccurlyeq\leq A(x_j)|x_j$ or $A(x_j)|x_j || B(x_i)|x_i$ for all other points $A(x_j)|x_j \in A$. Thus $A(x_j)|x_j \not\prec\not< B(x_i)|x_i$ for any $j$. Consequently, $\mathcal{H}$ contains its lower bound.
Case 2: Suppose the minimal subpomset $\mathcal{H}$ contains incomparable point(s). If $\mathcal{H}$ has only one point say $B(x_i)|x_i$, since $\mathcal{H}$ is minimal, $B(x_i)|x_i \in \mathcal{H}$ is a lower bound for $\mathcal{H}$. Next, assume that $\mathcal{H}$ has incomparable points say $B(x_1)|x_1, \dots, B(x_k)|x_k$. If $\mathcal{H}$ has a minimum point, this point is its lower bound, otherwise the lower bound does not exist i.e., $\nexists A(x_j)|x_j \in A$ such that $A(x_j)|x_j \preccurlyeq\leq B(x_i)|x_i$ for all $i = 1, .., k$. □

**Lemma 3.9**
A submultiset containing distinct lower bounds of minimal subpomsets has no comparable points.

*Proof:*
Let $\mathcal{H}_1, \dots, \mathcal{H}_k$ be the minimal subpomsets of a pomset $\mathcal{P}$. Let $\mathcal{A} = [M(x_1)|x_1, \dots, M(x_n)|x_n]$ be the submultiset containing distinct lower bounds of $\mathcal{H}_1, \dots, \mathcal{H}_k$. From Proposition 3.8, we know that if a minimal subpomset has a lower bound, it must be contained in it. We need to show that $M(x_i)|x_i || M(x_j)|x_j$ for distinct $i, j \in 1, \dots, n$. Suppose $(M(x_i)|x_i \prec< M(x_j)|x_j) \vee (M(x_j)|x_j \prec< M(x_i)|x_i)$ holds for some distinct $i, j$. Then either $M(x_i)|x_i$ or $M(x_j)|x_j$ is not a minimum point in any of the subpomsets $\mathcal{H}_1, \dots, \mathcal{H}_k$, and hence cannot be a lower bound. Therefore $M(x_i)|x_i || M(x_j)|x_j$ for all $\neq j$ . □

**Example 3.10**
Let $\mathcal{P} = (A, \preccurlyeq\leq)$ be a pomset where
$A = [2|x_1, 5|x_2, 3|x_3, x_4, 3|x_5, 4|x_6, 2|x_7]$
and the ordering on $A$ is defined as follows:
$2|x_1 \preccurlyeq\leq 3|x_3 \preccurlyeq\leq 2|x_7$, $5|x_2 \preccurlyeq\leq x_4 \preccurlyeq\leq 2|x_7$, $3|x_5 \preccurlyeq\leq 4|x_6$
Consider the following subpomsets of $\mathcal{P}$:
$\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$ given by, $N(x_1)|x_1 \preccurlyeq\leq_{\mathcal{N}} N(x_3)|x_3$, $N(x_2)|x_2 \preccurlyeq\leq_{\mathcal{N}} x_4$;
$\mathcal{L} = (L, \preccurlyeq\leq_{\mathcal{L}})$ given by, $L(x_2)|x_2 \preccurlyeq\leq_{\mathcal{L}} x_4$, $L(x_5)|x_5 \preccurlyeq\leq_{\mathcal{L}} L(x_6)|x_6$,
where $N$ and $L$ are subsets of $A$ with $N_i(x_i) \leq A_i(x_i)$ and $L_i(x_i) \leq A_i(x_i)$ for all $i \in \{1, \dots, 7\}$

Based on definition 2.1 we have the following:

i. the subpomset $\mathcal{N}$ has one upper bound i.e. $2|x_7$. $\mathcal{N}$ has no lower bounds.
ii. the subpomset $\mathcal{L}$ has neither an upper bound nor a lower bound.

## INFIMUM AND SUPREMUM OF SUBSTRUCTURES OF THE POMSET $\boldsymbol{\mathcal{P}}$

Analogous concepts of infimum and supremum of submultisets of $\mathcal{P}$ are presented in this section with example. Lastly, new results are established.

**Definition 4.1**

The *greatest lower bound* (glb), also known as *infimum* (inf), of a subpomset $\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$ of $\mathcal{P}$ is a lower bound of $\mathcal{N}$ that is greater than or the same with any other lower bound of $\mathcal{N}$. Hence, it is the greatest point of $\mathcal{P}$ that is less than or equal to any point in $N$. Thus if $\mathcal{K} = \{A(x_1)|x_1, \dots, A(x_n)|x_n\}$ is the set of lower bounds of the subpomset $\mathcal{N}$, then a point $A(x_i)|x_i$ in $\mathcal{K}$ is the glb of $\mathcal{N}$ if $A(x_j)|x_j \preccurlyeq\leq A(x_i)|x_i$ for all $x_j \neq x_i$.

**Definition 4.2**

The *least upper bound* (lub) also known as *supremum* (sup) of a subpomset $\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$ of $\mathcal{P}$ is an upper bound of $\mathcal{N}$ that is less than or the same with any other upper bound of $\mathcal{N}$. Hence, it is the least point of $\mathcal{P}$ that is greater than or equal to any point in $N$. Thus if $\mathcal{J} = \{A(x_1)|x_1, \dots, A(x_k)|x_k\}$ is the set of upper bounds of the subpomset $\mathcal{N}$, then a point $A(x_i)|x_i$ in $\mathcal{J}$ is the lub of $\mathcal{N}$ if $A(x_j)|x_j \geq\succcurlyeq A(x_i)|x_i$ for all $x_j \neq x_i$.
*Remark*: If the infimum (resp. supremum) exists, it will be unique.

**Example 4.3**

Consider the multiset $A = [5|x_1, 3|x_2, 2|x_3, 2|x_4, 2|x_5, x_6, 4|x_7, 3|x_8, 2|x_9, 2|x_{10}, |x_{11}, 4|x_{12}]$ with the following submultisets:
$L = [2|x_2, x_6, x_8, 2|x_{10}]$
$M = [x_2, 2|x_4, x_6]$
$N = [x_2, 2|x_3]$
Where the ordering $\preccurlyeq$ on the generic set is defined as follows: $x_i \preccurlyeq x_j$ if and only if $i$ divides $j$. The multiset ordering $\preccurlyeq\leq$ is induced by $\preccurlyeq$, thus by definition 2.1 we would have the following subpomsets:
$\mathcal{L} = (L, \preccurlyeq\leq_{\mathcal{L}})$: $2|x_2 \preccurlyeq\leq_{\mathcal{L}} x_6$, $2|x_2 \preccurlyeq\leq_{\mathcal{L}} x_8$, $2|x_2 \preccurlyeq\leq_{\mathcal{L}} 2|x_{10}$
with lower bounds: $\{5x_1, x_2\}$, and infimum: $x_2$, $\mathcal{L}$ has no upper bounds
$\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$: $x_2 \preccurlyeq\leq_{\mathcal{M}} 2|x_4$, $x_2 \preccurlyeq\leq_{\mathcal{M}} x_6$
with lower bounds: $\{5x_1, x_2\}$ and infimum: $x_2$, Upper bound: $4x_{12}$, and supremum: $4x_{12}$
$\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$: $x_2, 2|x_3$
with lower bound: $5x_1$ and infimum: $5x_1$, Upper bounds: $\{x_6, x_{12}\}$, and supremum: $x_6$

**Proposition 4.4**

Every maximal multiset chain of a pomset $\mathcal{P}$ contains its supremum.

*Proof:*
Let $\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$ be a maximal multiset chain in a pomset $\mathcal{P} = (A, \preccurlyeq\leq)$. Suppose $N(x_i)|x_i$ is a maximal point in $\mathcal{N}$, where $N(x_i) \leq A(x_i)$ for all $i$. By proposition 3.2, $N(x_i)|x_i$ is a maximum point. Since $\mathcal{N}$ is a maximal subpomset of $\mathcal{P}$, $\nexists A(x_j)|x_j \in A$ with $N(x_i)|x_i \prec\!< A(x_j)|x_j$ for $x_i \neq x_j$. Therefore $\mathcal{N}$ is bounded above by $N(x_i)|x_i$. The point $N(x_i)|x_i$ is also a unique upper bound for $\mathcal{N}$ (since $N(x_i)|x_i$ is a maximum point). Hence $\mathcal{N}$ contains its supremum.

**Lemma 4.5**
A minimal point of a subpomset $\mathcal{N}$ is its infimum if and only if it is a least point.

*Proof:*
Let $\mathcal{N} = (N, \preccurlyeq\leq_{\mathcal{N}})$ be a subpomset of a pomset $\mathcal{P} = (A, \preccurlyeq\leq)$, and $N(x_i)|x_i \in N$ a minimal point in $\mathcal{N}$. Suppose $N(x_i)|x_i$ is the infimum of $\mathcal{N}$. Clearly, $N(x_i)|x_i$ is a lower bound for $\mathcal{N}$ and $N(x_i)|x_i \prec\prec_{\mathcal{N}} N(x_j)|x_j$ for all $x_i \neq x_j$. Therefore $N(x_i)|x_i$ is the least point of $\mathcal{N}$. Assume conversely that $N(x_i)|x_i$ is the least point of $\mathcal{N}$. Then $N(x_i)|x_i \prec\prec_{\mathcal{N}} N(x_j)|x_j$ for all $x_i \neq x_j$. Hence, $N(x_i)|x_i$ is a lower bound for $\mathcal{N}$. To show that $N(x_i)|x_i$ is the greatest lower bound, let $A(x_k)|x_k \in A$ be any other lower bound for $\mathcal{N}$. Since $N(x_i)|x_i$ is the least point in $\mathcal{N}$, any other distinct lower bound of $\mathcal{N}$ cannot be a point in $N$ and hence must be smaller.
Thus, $A(x_k)|x_k \preccurlyeq\leq N(x_i)|x_i$. Hence $N(x_i)|x_i$ is the greatest lower bound. □

**CONCLUSION**
Analogous notions of bounds of an ordered set were presented using an ordered multiset structure. The supremum and infimum of subpomsets of the ordered multiset structure were also defined with examples, and new results were established. The concepts of meet and join of a lattice could be extensively studied, with possible applications, in the context of the ordered multiset structure described in this work (Anusuya & Vimala, 2018). Also, the analogous notion of a semilattice (upper and lower) can be investigated for ordered multisets.

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