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Modified Regression-Cum-Ratio Types Estimators of Population Coefficient of Variation in Survey Sampling

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Abstract

Enhancing the efficiency of estimators is one of the major concerns in sampling survey. Efficiency of estimators measures the precision of the estimator with respect to the corresponding parameters. In this paper, four methods of estimating the population coefficient of variation of the study variable using auxiliary information were suggested by modifying some existing estimators using unknown weight and power transformation techniques. The properties (Biases and MSEs) of the proposed estimators were derived up to first order of approximation using Taylor series approach. Numerical analysis of MSEs and PREs of the proposed and other related existing estimators considered in the study was conducted to justify the efficiencies of the proposed estimators and the results obtained revealed that the proposed estimators have minimum MSEs and higher PREs compared to competing estimators implying that the proposed estimators are more efficient than the existing estimators considered in the study.

Keywords: Auxiliary information, Efficiency, Coefficient of Variation, Mean Square Error, Percentage Relative Efficiency.

INTRODUCTION

Survey sampling deals with parameters estimation, sample size determination, data gathering for estimation (Singh, 2003). The use of auxiliary information is well known to improve the precision of the estimate of the population mean and other parameters of the study variable in survey sampling (Cochran, 1940). Coefficient of variation is a unit free tool used for making comparisons between two quantities with different units. For example, comparison of height and weight of individuals.

Many researchers like Ahmed et al. (2016), Audu and Adewara (2017), Audu et al. (2021a), Audu and Singh (2021), Muili et al. (2020), Sahai and Ray (1980), Srivastava and Jhajj (1981),

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Singh and Tailor (2005), Sisodia and Dwivedi (1981), Khoshnevisan et al. (2007), Singh and Solanki (2012), Adejumobi and Yunusa (2022) have worked extensively in the development of estimators using auxiliary information. The coefficient of variation is expressed in percentages to indicate the extent of variability percent in the data, the tool is used in the field of Economics, Sociology, Psychology, Business Studies, Education Biology, Physics, Chemistry Geography Computer science, Mathematics etc. The estimation of population coefficient of variation in the presence of auxiliary information can be found in the works of Rajyaguru and Gupta (2002), Archana and Rao (2014), Adichwal et al. (2016), Singh et al. (2018), Audu et al. (2021b), Yunusa et al. (2021), Singh and Kumari (2022), Yunusa et al. (2022). Having study the efficiency of the existing related estimator mentioned above, it is observed that their efficiency gained (PRE) is not significant and further transformation of the estimators can lead to additional efficiency gain of the estimators. Therefore, this study proposed modified improved estimators for the estimation of population coefficient of variation of the study variable in the presence of auxiliary information that will provide a more precise and efficiency estimate of the true parameter.

Consider a finite population $M=\left(M_1,M_2,......,M_N\right)$ of size N consisting of distinct and identifiable units. Let Y and X denotes the study and auxiliary variables and let Y_i and X_i be their values corresponding to i^{th} unit in the population $\left(i=1,2,.....,N\right)$. For the population observations, means, mean squares and covariance for the study and auxiliary variables, we define as

$$\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i,\ \bar{X}=\frac{1}{N}\sum_{i=1}^{N}X_i,\ S_Y^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^2,\ S_X^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2 \text{ and}$$

$$S_{YX}=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)\left(X_i-\bar{X}\right)\ \psi=n^{-1}-N^{-1},\ C_y=\frac{S_y}{\bar{Y}},\ C_x=\frac{S_x}{\bar{X}},\ \lambda_{rs}=\frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}},$$

$$\mu_{rs}=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^r\left(X_i-\bar{X}\right)^s$$

For the sample observations, means, mean squares and covariance for the study and auxiliary variables, we define as

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i,\ \bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i,\ s_y^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2,\ s_x^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2 \text{ and}$$

$$s_{yx}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)\left(x_i-\bar{x}\right),\ c_y=\frac{\sqrt{s_y^2}}{\bar{y}},\ c_x=\frac{\sqrt{s_x^2}}{\bar{x}}$$

Existing estimators in the literature

The usual unbiased estimator to estimate the population coefficient of variation is given by:

$$t_0=\hat{C}_y=\frac{s_y}{\bar{y}} \tag{1}$$

The mean square error (MSE) expression of the estimator t_0 is given by:

$$MSE\left(t_0\right)=\psi C_y^2\left(C_y^2+1/4\left(\lambda_{04}-1\right)-C_y\lambda_{30}\right) \tag{2}$$

Archana and Rao (2014) proposed ratio-type estimator for the population coefficient of variation, is given by

$$t_{AR}=C_y\left(\frac{\bar{X}}{\bar{x}}\right) \tag{3}$$

The mean square error indication of the estimator t_{AR} is given by:

$$MSE\left(t_{AR}\right)=\psi C_y^2\left(C_y^2+1/4\left(\lambda_{40}-1\right)-C_y\lambda_{30}-C_x\lambda_{21}+1/4\left(\lambda_{04}-1\right)\right) \quad (4)$$

Singh et al. (2018) proposed ratio-type, exponential ratio-type and difference-type estimators for coefficient of variation of the study variable Y using mean of the auxiliary variable and are given below with their MSEs as

$$t_1=\hat{C}_y\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha} \quad (5)$$

$$t_2=\hat{C}_y\exp\left\{\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\} \quad (6)$$

$$t_3=\hat{C}_y+d_1\left(\bar{X}-\bar{x}\right) \quad (7)$$

$$MSE\left(t_1\right)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\alpha^2C_x^2-C_y\lambda_{30}+2\alpha\rho_{yx}C_yC_x-\alpha C_x\lambda_{21}\right] \quad (8)$$

$$MSE\left(t_2\right)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\beta^2C_x^2}{4}-C_y\lambda_{30}+\beta\rho_{yx}C_yC_x-\frac{\beta}{2}C_x\lambda_{21}\right] \quad (9)$$

$$MSE\left(t_3\right)=\psi\left[C_y^2\left(C_y^2-C_y\lambda_{30}+\frac{\lambda_{40}-1}{4}\right)+d_1^2\bar{X}^2C_x^2+2d_1\bar{X}\rho_{yx}C_yC_x-d_1\bar{X}C_xC_y\lambda_{21}\right] \quad (10)$$

where, $\alpha=\dfrac{\lambda_{21}-2\rho_{yx}C_y}{2C_x}$, $\quad\beta=\dfrac{\lambda_{21}-2\rho_{yx}C_y}{C_x}$, $\quad d_1=\dfrac{C_y\lambda_{21}-2\rho_{yx}C_y^2}{2\bar{X}C_x}$

Singh et al. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_1 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_4^{AM}=\frac{\hat{C}_y}{2}\left[1+\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}\right] \quad (11)$$

$$t_4^{GM}=\hat{C}_y\left[\frac{\bar{X}}{\bar{x}}\right]^{\alpha/2} \quad (12)$$

$$t_4^{HM}=2\hat{C}_y\left[1+\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}\right]^{-1} \quad (13)$$

$$MSE\left(t_4^r\right)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\alpha^2C_x^2}{4}-C_y\lambda_{30}+\alpha\rho_{yx}C_yC_x-\frac{\alpha}{2}C_x\lambda_{21}\right] \quad (14)$$

where, $\alpha=\dfrac{\lambda_{21}-2\rho_{yx}C_y}{2C_x}$, $\quad r=AM, GM\ \&\ HM$

Singh et al. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_5^{AM} = \frac{\hat{C}_y}{2}\left[1+\exp\left\{\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\}\right] \quad (15)$$

$$t_5^{GM} = \hat{C}_y \exp\left\{\frac{\beta}{2}\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\} \quad (16)$$

$$t_5^{HM} = 2\hat{C}_y\left[1+\exp\left\{\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\}\right]^{-1} \quad (17)$$

$$MSE\left(t_5^r\right) = \psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\beta^2 C_x^2}{16}-C_y\lambda_{30}+\frac{\beta}{2}\rho_{yx}C_yC_x-\frac{\beta}{4}C_x\lambda_{21}\right] \quad (18)$$

where, $\beta = \frac{2\left(\lambda_{21}-2\rho_{yx}C_y\right)}{C_x}$

Singh et al. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_1 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_6^{AM} = \frac{\hat{C}_y}{2}\left[\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}+\exp\left\{\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\}\right] \quad (19)$$

$$t_6^{AM} = \hat{C}_y\left[\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha/2}+\exp\left\{\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\}\right] \quad (20)$$

$$t_6^{HM} = 2\hat{C}_y\left[\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha}+\exp\left\{-\beta\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\right\}\right]^{-1} \quad (21)$$

$$MSE\left(t_6^r\right) = \psi C_y^2\left[\begin{array}{l} C_y^2+\frac{\lambda_{40}-1}{4}+\frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)^2 C_x^2-C_y\lambda_{30}+\left(\alpha+\frac{\beta}{2}\right)\rho_{yx}C_yC_x \\ -\frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)C_y\lambda_{21} \end{array}\right] \quad (22)$$

where, $\beta = 2\left(\frac{\lambda_{21}-2\rho_{yx}C_y}{C_x}-\alpha\right)$

Audu et al. (2021b) suggested the following two difference cum ratio type estimators of C_y utilizing the known $\bar{X}$ as:

$$t_{a1} = \left[\frac{\hat{C}_y}{2}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right)+w_1\left(\bar{X}-\bar{x}\right)+w_2\hat{C}_y\right]\left(\frac{\bar{X}}{\bar{x}}\right) \quad (23)$$

$$MSE\left(t_{a1}\right) = C_y^2\left(A+w_1^2B+w_2^2C+2w_1D-2w_2E-2w_1w_2F\right) \quad (24)$$

where,

$$A=\psi\left(C_x^2+C_y^2+2\rho_{yx}C_yC_x-C_x\lambda_{21}-C_y\lambda_{30}+\frac{(\lambda_{40}-1)}{4}\right),\qquad B=\psi\delta^2C_x^2,\qquad \delta=\frac{\bar{X}}{C_y}$$

$$C=1+\psi\left(3C_x^2+3C_y^2+4\rho_{yx}C_yC_x-2C_x\lambda_{21}-2C_y\lambda_{30}\right),\quad D=\psi\delta\left(C_x^2+\rho_{yx}C_yC_x-\frac{C_x\lambda_{21}}{2}\right)$$

$$E=\psi\left(\frac{3C_x\lambda_{21}}{2}-3\rho_{yx}C_yC_x-\frac{5C_x^2}{2}-2C_y^2+\frac{3C_y\lambda_{30}}{2}+\frac{(\lambda_{40}-1)}{8}\right),\ F=\psi\delta\left(\frac{C_x\lambda_{21}}{2}-\rho_{yx}C_yC_x-2C_x^2\right)$$

Adichwal et al. (2016) suggested the following estimator for estimating C_y using the known S_x^2 as,

$$t_7=\delta_1\left[\frac{(1-\eta)s_x^2+\eta S_x^2}{\eta s_x^2+(1-\eta)S_x^2}\right]\hat{C}_y+(1-\delta_1)\left[\frac{\eta s_x^2+(1-\eta)S_x^2}{(1-\eta)s_x^2+\eta S_x^2}\right] \tag{25}$$

Where δ_1 and η are the characterizing constants to be determined such that MSEs of the estimators t_7 is least.

The minimum MSEs of the estimator t_7 for the optimum values of these constants is,

$$MSE(t_7)=MSE(t_0)-\frac{1}{4}\psi\frac{\left[(\lambda_{22}-1)-2C_y\lambda_{12}\right]^2}{(\lambda_{04}-1)}C_y^4 \tag{26}$$

Singh et al. (2018) proposed ratio-type, exponential ratio-type and difference-type estimators for coefficient of variation of the study variable Y using mean of the auxiliary variable and are given below with their MSEs as

$$t_8=\hat{C}_y\left(\frac{S_x^2}{s_x^2}\right)^{\alpha} \tag{27}$$

$$t_9=\hat{C}_y\exp\left\{\beta\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right\} \tag{28}$$

$$t_{10}=\hat{C}_y+d_2\left(S_x^2-s_x^2\right) \tag{29}$$

$$MSE(t_8)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\alpha^2(\lambda_{04}-1)-C_y\lambda_{30}-\alpha(\lambda_{22}-1)+2\alpha C_y\lambda_{12}\right] \tag{30}$$

$$MSE(t_9)=\psi C_y^2\left[C_y^2+\frac{\lambda_{40}-1}{4}+\frac{\beta^2(\lambda_{04}-1)}{4}-C_y\lambda_{30}-\beta C_y\lambda_{12}+\frac{\beta}{2}(\lambda_{22}-1)\right] \tag{31}$$

$$MSE(t_{10})=\psi\left[\begin{array}{l}C_y^2\left(C_y^2-C_y\lambda_{30}+\frac{\lambda_{40}-1}{4}\right)+d_2^2S_x^4(\lambda_{04}-1)+2C_y^2d_2S_x^2\lambda_{12}\\-d_2C_yS_x^2(\lambda_{22}-1)\end{array}\right] \tag{32}$$

where, $\alpha=\dfrac{\lambda_{22}-1-2C_y\lambda_{21}}{2(\lambda_{04}-1)},\quad \beta=\dfrac{\lambda_{22}-1-2C_y\lambda_{21}}{(\lambda_{04}-1)},\quad d_2=\dfrac{C_y(\lambda_{22}-1)-2C_y^2\lambda_{12}}{2S_x^2(\lambda_{04}-1)}$

Singh et al. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_2 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{11}^{AM}=\frac{\hat{C}_y}{2}\left[1+\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}\right] \tag{33}$$

$$t_{11}^{GM} = \hat{C}_y \left(\frac{S_x^2}{s_x^2} \right)^{\alpha/2} \tag{34}$$

$$t_{11}^{HM} = 2\hat{C}_y \left[1 + \left(\frac{S_x^2}{s_x^2} \right)^{\alpha} \right]^{-1} \tag{35}$$

$$MSE\left(t_{11}^{r}\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{\alpha^2\left(\lambda_{04}-1\right)}{4} - C_y\lambda_{30} + \alpha C_y \lambda_{12} - \frac{\alpha}{2}\left(\lambda_{22}-1\right) \right] \tag{36}$$

where, $\alpha = \frac{\lambda_{22} - 1 - 2C_y\lambda_{21}}{\lambda_{04}-1}$, $\quad r = AM, GM \,\&\, HM$

Rajyaguru and Gupta (2002) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_9 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{12}^{AM} = \frac{\hat{C}_y}{2} \left[1 + \exp\left\{ \beta \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right] \tag{37}$$

$$t_{12}^{GM} = \hat{C}_y \exp\left\{ \frac{\beta}{2} \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \tag{38}$$

$$t_{12}^{HM} = 2\hat{C}_y \left[1 + \exp\left\{ -\beta \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right]^{-1} \tag{39}$$

$$MSE\left(t_{12}^{r}\right) = \psi C_y^2 \left[C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{\beta^2\left(\lambda_{04}-1\right)}{16} - C_y\lambda_{30} + \frac{\beta}{2} C_y \lambda_{12} - \frac{\beta}{4}\left(\lambda_{22}-1\right) \right] \tag{40}$$

where, $\beta = \frac{2\left(\lambda_{22}-1\right) - 4C_y\lambda_{21}}{\left(\lambda_{04}-1\right)}$, $\quad r = AM, GM \,\&\, HM$

Singh et al. (2018) proposed arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_8 and t_9 estimators for estimating coefficient of variation of the study variable Y and are given below with their MSEs as:

$$t_{13}^{AM} = \frac{\hat{C}_y}{2} \left[\left(\frac{S_x^2}{s_x^2} \right)^{\alpha} + \exp\left\{ \beta \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right] \tag{41}$$

$$t_{13}^{AM} = \hat{C}_y \left(\frac{S_x^2}{s_x^2} \right)^{\alpha/2} \exp\left\{ \beta \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \tag{42}$$

$$t_{13}^{HM} = 2\hat{C}_y \left[\left(\frac{s_x^2}{S_x^2} \right)^{\alpha} + \exp\left\{ -\beta \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \right]^{-1} \tag{43}$$

$$MSE\left(t_{13}^{r}\right) = \psi C_y^2 \begin{bmatrix} C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{1}{4}\left(\alpha + \frac{\beta}{2}\right)^2 \left(\lambda_{40}-1\right) - C_y\lambda_{30} \\ + \frac{1}{2}\left(\alpha + \frac{\beta}{2}\right)\left(\lambda_{22}-1\right) + \left(\alpha + \frac{\beta}{2}\right) C_y \lambda_{12} \end{bmatrix} \tag{44}$$

where, $\beta = 2\left(\frac{(\lambda_{22}-1) - 2C_y\lambda_{12}}{(\lambda_{40}-1)} - \alpha \right)$

Audu et al. (2021b) suggested the following two difference cum ratio type estimators of C_y utilizing the known S_x^2 as:

$$t_{a2} = \left[\frac{\hat{C}_y}{2}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2} \right) + w_3\left(S_x^2 - s_x^2\right) + w_4\hat{C}_y \right]\left(\frac{S_x^2}{s_x^2} \right) \quad (45)$$

$$MSE\left(t_{a2}\right) = C_y^2\left(A + w_3^2 B_1 + w_4^2 C_1 + 2w_3 D_1 - 2w_4 E_1 - 2w_3 w_4 F_1\right) \quad (46)$$

where,

$$A_1 = \psi\left((\lambda_{04}-1) + C_y^2 + 2C_y\lambda_{12} - C_y\lambda_{30} - (\lambda_{22}-1) + \frac{(\lambda_{40}-1)}{4} \right), \quad B_1 = \psi\delta_1^2 C_x^2, \quad \delta_1 = \frac{S_x^2}{C_y}$$

$$C_1 = 1 + \psi\left(3(\lambda_{04}-1) + 3C_y^2 + 4C_y\lambda_{12} - 2(\lambda_{22}-1) - 2C_y\lambda_{30}\right), \; D_1 = \psi\delta_1\left((\lambda_{04}-1) + C_y\lambda_{12} - \frac{(\lambda_{22}-1)}{2} \right)$$

$$E_1 = \psi\left(\frac{3(\lambda_{22}-1)}{2} - 3C_y\lambda_{12} - \frac{5(\lambda_{04}-1)}{2} - 2C_y^2 + \frac{3C_y\lambda_{30}}{2} + \frac{(\lambda_{40}-1)}{8} \right)$$

$$F_1 = \psi\delta_1\left(\frac{(\lambda_{22}-1)}{2} - C_y\lambda_{12} - 2(\lambda_{04}-1) \right)$$

Yunusa et al. (2021) proposed log ratio type ratio estimator of C_y utilizing the known S_x^2 as:

$$t_{14} = \hat{C}_y\left(\frac{Ln\left(S_x^2\right)}{Ln\left(s_x^2\right)} \right) \quad (47)$$

The MSE of the estimator t_8, up to the first order of approximation is,

$$MSE\left(t_{14}\right) = \psi C_y^2\left[C_y^2 + \frac{\lambda_{40}-1}{4} + \frac{\lambda_{04}-1}{\left\{Ln\left(S_x^2\right)\right\}^2} - \frac{\left\{(\lambda_{22}-1) - 2C_y\lambda_{12}\right\}}{Ln\left(S_x^2\right)} - C_y\lambda_{30} \right] \quad (48)$$

Singh and Kumari (2022) proposed four estimators for coefficient of variation based on information on a single auxiliary variable.

$$t_{p1} = \left[\frac{\hat{C}_y}{2}\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}} \right) + k_1\left(\bar{X} - \bar{x}\right) + k_2\hat{C}_y \right]\left\{ 2 - \left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}} \right) \right\} \quad (49)$$

$$t_{p2} = \left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}} \right) \right) + k_3\left(\bar{X} - \bar{x}\right) + k_4\hat{C}_y \right]\left\{ 2 - \left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}} \right) \right\} \quad (50)$$

$$t_{p3} = \left[\frac{\hat{C}_y}{2}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2} \right) + k_5\left(S_x^2 - s_x^2\right) + k_6\hat{C}_y \right]\left(2 - \frac{s_x^2}{S_x^2} \right) \quad (51)$$

$$t_{p4} = \left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2} \right) + \exp\left(\frac{s_x^2-S_x^2}{s_x^2+S_x^2} \right) \right) + k_7\left(S_x^2 - s_x^2\right) + k_8\hat{C}_y \right]\left(2 - \frac{s_x^2}{S_x^2} \right) \quad (52)$$

$$MSE\left(t_{p1}\right) = C_y^2\left(A_1 + k_1^2 B_1 + k_2^2 C_1 + 2k_1 D_1 - 2k_2 E_1 - 2k_1 k_2 F_1\right) \quad (53)$$

$$MSE\left(t_{p2}\right) = C_y^2\left(A_2 + k_3^2 B_2 + k_4^2 C_2 + 2k_3 D_2 - 2k_4 E_2 - 2k_3 k_4 F_2\right) \quad (54)$$

$$MSE\left(t_{p3}\right)=C_y^2\left(A_3+k_5^2B_3+k_6^2C_3+2k_5D_3-2k_6E_3-2k_5k_6F_3\right) \quad (55)$$

$$MSE\left(t_{p3}\right)=C_y^2\left(A_4+k_7^2B_4+k_8^2C_4+2k_7D_4-2k_8E_4-2k_7k_8F_4\right) \quad (56)$$

where,

$$A_1=\psi\left(C_y^2+\frac{9C_x^2}{4}+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+3C_{yx}-\frac{3C_x\lambda_{21}}{2}\right),\quad B_1=\psi g^2C_x^2, D_1=\psi g\left(\frac{3C_x^2}{2}+C_{yx}-\frac{C_x\lambda_{21}}{2}\right)$$

$$=1+\psi\left(3C_y^2+\frac{3C_x^2}{2}+6C_{yx}-2C_y\lambda_{30}-3C_x\lambda_{21}\right),\ g=\frac{\bar{X}}{C_y}, F_1=\psi g\left(\frac{C_x\lambda_{21}}{2}-C_{yx}-3C_x^2\right)$$

$$E_1=\psi\left(\frac{9C_x\lambda_{21}}{4}-\frac{9C_{yx}}{2}-\frac{19C_x^2}{8}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{4}\right), B_2=\psi g^2C_x^2,\quad g=\frac{\bar{X}}{C_y}$$

$$A_2=\psi\left(C_y^2+\frac{9C_x^2}{4}+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+3C_{yx}-\frac{3C_x\lambda_{21}}{2}\right),\quad D_2=\psi g\left(\frac{3C_x^2}{2}+C_{yx}-\frac{C_x\lambda_{21}}{2}\right)$$

$$C_2=1+\psi\left(3C_y^2+\frac{3C_x^2}{2}+6C_{yx}-2C_y\lambda_{30}-3C_x\lambda_{21}\right), F_2=\psi g\left(\frac{C_x\lambda_{21}}{2}-C_{yx}-3C_x^2\right)$$

$$E_2=\psi\left(\frac{9C_x\lambda_{21}}{4}-\frac{9C_{yx}}{2}-2C_x^2-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right), B_3=\psi g_1^2\left(\lambda_{04}-1\right),$$

$$A_3=\psi\left(C_y^2+(\lambda_{04}-1)+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+2C_y\lambda_{12}-(\lambda_{22}-1)\right),$$

$$C_3=1+\psi\left(3C_y^2+(\lambda_{04}-1)+4C_y\lambda_{12}-2C_y\lambda_{30}-\frac{3(\lambda_{22}-1)}{2}\right),\ D_3=\psi g_1\left((\lambda_{04}-1)+C_y\lambda_{12}-\frac{(\lambda_{22}-1)}{2}\right)$$

$$E_3=\psi\left(\frac{3(\lambda_{22}-1)}{2}-\frac{3(\lambda_{04}-1)}{2}-3C_y\lambda_{12}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right),\quad g_1=\frac{S_x^2}{C_y}$$

$$F_3=\psi g_1\left(\frac{(\lambda_{22}-1)}{2}-C_y\lambda_{12}-2(\lambda_{04}-1)\right)$$

$$A_4=\psi\left(C_y^2+(\lambda_{04}-1)+\frac{(\lambda_{40}-1)}{4}-C_y\lambda_{30}+2C_y\lambda_{12}-(\lambda_{22}-1)\right),\quad B_4=\psi g_1^2\left(\lambda_{04}-1\right),$$

$$C_4=1+\psi\left(3C_y^2+(\lambda_{04}-1)+4C_y\lambda_{12}-2C_y\lambda_{30}-\frac{3(\lambda_{22}-1)}{2}\right),\ D_4=\psi g_1\left((\lambda_{04}-1)+C_y\lambda_{12}-\frac{(\lambda_{22}-1)}{2}\right)$$

$$E_4=\psi\left(\frac{3(\lambda_{22}-1)}{2}-\frac{9(\lambda_{04}-1)}{8}-3C_y\lambda_{12}-2C_y^2+\frac{3C_y\lambda_{30}}{2}-\frac{(\lambda_{40}-1)}{8}\right),\quad g_1=\frac{S_x^2}{C_y}$$

$$F_4=\psi g_1\left(\frac{(\lambda_{22}-1)}{2}-C_y\lambda_{12}-2(\lambda_{04}-1)\right).$$

METHODOLOGY

Proposed Estimators

Having studied the estimators in the literature, four estimators for the coefficient of variation based on information on a single auxiliary variable were proposed as in (57)-(60)

$$t_{01}=\left[\frac{\hat{C}_y}{2}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right)+u_1\left(\bar{X}-\bar{x}\right)+u_2\hat{C}_y\right]\left\{2-\left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\}^a \quad (57)$$

$$t_{02}=\left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right)+u_3\left(\bar{X}-\bar{x}\right)+u_4\hat{C}_y\right]\left\{2-\left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\}^a \quad (58)$$

$$t_{03}=\left[\frac{\hat{C}_y}{2}\left(\frac{S_x^2}{s_x^2}+\frac{s_x^2}{S_x^2}\right)+u_5\left(S_x^2-s_x^2\right)+u_6\hat{C}_y\right]\left(2-\frac{s_x^2}{S_x^2}\right)^a$$

(59)

$$t_{04}=\left[\frac{\hat{C}_y}{2}\left(\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)+\exp\left(\frac{s_x^2-S_x^2}{s_x^2+S_x^2}\right)\right)+u_7\left(S_x^2-s_x^2\right)+u_8\hat{C}_y\right]\left(2-\frac{s_x^2}{S_x^2}\right)^a \quad (60)$$

Where a=1.3, real constant to minimize the mean square errors (MSEs) of the estimators.

Properties (Bias and MSE) of $t_{0k},\ k=1,2,3,4$

Let the sampling relative errors be given by:

$$\varepsilon_0=\frac{\bar{y}-\bar{Y}}{\bar{Y}},\quad \varepsilon_1=\frac{\bar{x}-\bar{X}}{\bar{X}},\quad \varepsilon_2=\frac{s_y^2-S_y^2}{S_y^2},\quad \varepsilon_2=\frac{s_x^2-S_x^2}{S_x^2}$$

Such that,

$$\left.\begin{array}{l}E(\varepsilon_0)=E(\varepsilon_1)=E(\varepsilon_2)=E(\varepsilon_3)=0\\ E(\varepsilon_0^2)=\psi C_y^2,\ E(\varepsilon_1^2)=\psi C_x^2,\ E(\varepsilon_2^2)=\psi(\lambda_{40}-1),\ E(\varepsilon_3^2)=\psi(\lambda_{04}-1)\\ E(\varepsilon_0\varepsilon_1)=\psi C_{yx}=\psi\rho_{yx}C_yC_x,\ E(\varepsilon_0\varepsilon_2)=\psi C_y\lambda_{30},\ E(\varepsilon_0\varepsilon_3)=\psi C_y\lambda_{12},\\ E(\varepsilon_1\varepsilon_2)=\psi C_x\lambda_{21},\ E(\varepsilon_1\varepsilon_3)=\psi C_x\lambda_{03},\ E(\varepsilon_2\varepsilon_3)=\psi(\lambda_{22}-1)\end{array}\right\} \quad (61)$$

The estimators $t_{0k},\ k=1,2,3,4$ in terms of errors can be expressed and simplified respectively as in (62), (63), (64) and (65)

$$t_{01}=\left[\begin{array}{l}\dfrac{S_y(1+\varepsilon_2)^{1/2}}{2\bar{Y}}\left(\dfrac{\bar{X}}{\bar{X}(1+\varepsilon_1)}+\dfrac{\bar{X}(1+\varepsilon_1)}{\bar{X}}\right)\\ +u_1\left(\bar{X}-\bar{X}(1+\varepsilon_1)\right)+\dfrac{S_y(1+\varepsilon_2)^{1/2}}{\bar{Y}(1+\varepsilon_0)}\end{array}\right]\left(2-\frac{\bar{X}(1+\varepsilon_1)}{\bar{X}}\exp\left(\frac{\bar{X}(1+\varepsilon_1)-\bar{X}}{\bar{X}(1+\varepsilon_1)+\bar{X}}\right)\right)^a \quad (62)$$

$$t_{02}=\left[\begin{array}{l}\dfrac{S_y(1+\varepsilon_2)^{1/2}}{2\bar{Y}(1+\varepsilon_0)}\left[\begin{array}{l}\exp\left(\dfrac{\bar{X}-\bar{X}(1+\varepsilon_1)}{\bar{X}+\bar{X}(1+\varepsilon_1)}\right)\\ +\exp\left(\dfrac{\bar{X}(1+\varepsilon_1)-\bar{X}}{\bar{X}(1+\varepsilon_1)+\bar{X}}\right)\end{array}\right]\\ +u_3\left(\bar{X}-\bar{X}(1+\varepsilon_1)\right)+u_4\dfrac{S_y(1+\varepsilon_2)^{1/2}}{\bar{Y}(1+\varepsilon_0)}\end{array}\right]\left(2-\frac{\bar{X}(1+\varepsilon_1)}{\bar{X}}\exp\left(\frac{\bar{X}(1+\varepsilon_1)-\bar{X}}{\bar{X}(1+\varepsilon_1)+\bar{X}}\right)\right)^a \quad (63)$$

$$t_{03}=\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}\left(1+\varepsilon_0\right)}\left[\begin{array}{l}\left(\frac{S_x^2}{S_x^2\left(1+\varepsilon_3\right)}+\frac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)\\+u_5\left(S_x^2-S_x^2\left(1+\varepsilon_3\right)\right)+u_6\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{array}\right]\left(2-\frac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)^a \quad (64)$$

$$t_{04}=\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{2\bar{Y}\left(1+\varepsilon_0\right)}\left[\begin{array}{l}\exp\left(\frac{S_x^2-S_x^2\left(1+\varepsilon_3\right)}{S_x^2+S_x^2\left(1+\varepsilon_3\right)}\right)+\exp\left(\frac{S_x^2\left(1+\varepsilon_3\right)-S_x^2}{S_x^2\left(1+\varepsilon_3\right)+S_x^2}\right)\\+u_7\left(S_x^2-S_x^2\left(1+\varepsilon_3\right)\right)+u_8\frac{S_y\left(1+\varepsilon_2\right)^{1/2}}{\bar{Y}\left(1+\varepsilon_0\right)}\end{array}\right]\left(2-\frac{S_x^2\left(1+\varepsilon_3\right)}{S_x^2}\right)^a \quad (65)$$

$$t_{01}=C_y\left[\begin{array}{l}\left(\begin{array}{l}1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\left(\theta_2+\frac{1}{2}\right)\varepsilon_1^2\\+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_1\frac{\bar{X}}{C_y}\left(\varepsilon_1-\theta_1\varepsilon_1^2\right)\\+u_2\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\theta_2\varepsilon_1^2+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (66)$$

$$t_{02}=C_y\left[\begin{array}{l}\left(\begin{array}{l}1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\left(\theta_2+\frac{1}{8}\right)\varepsilon_1^2\\+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_3\frac{\bar{X}}{C_y}\left(\varepsilon_1-\theta_1\varepsilon_1^2\right)\\+u_4\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\theta_2\varepsilon_1^2+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (67)$$

$$t_{03}=C_y\left[\begin{array}{l}\left(\begin{array}{l}1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\left(\theta_4+\frac{1}{2}\right)\varepsilon_3^2\\+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_5\frac{S_x^2}{C_y}\left(\varepsilon_3-\theta_3\varepsilon_3^2\right)\\+u_6\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\theta_4\varepsilon_3^2+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (68)$$

$$t_{04}=C_y\left[\begin{array}{l}\left(\begin{array}{l}1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\left(\theta_4+\frac{1}{8}\right)\varepsilon_3^2\\+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_7\frac{S_x^2}{C_y}\left(\varepsilon_3-\theta_3\varepsilon_3^2\right)\\+u_8\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\theta_4\varepsilon_3^2+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (69)$$

Subtracting C_y from both sides of (66), (67), (68) and (69) to obtain,

$$t_{01}-C_y=C_y\left[\begin{array}{l}\left(\begin{array}{l}\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\left(\theta_2+\frac{1}{2}\right)\varepsilon_1^2\\+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_1\frac{\bar{X}}{C_y}\left(\varepsilon_1-\theta_1\varepsilon_1^2\right)\\+u_2\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\theta_2\varepsilon_1^2+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (70)$$

$$t_{02}-C_y=C_y\left[\begin{array}{l}\left(\begin{array}{l}\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\left(\theta_2+\frac{1}{8}\right)\varepsilon_1^2\\+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_3\frac{\bar{X}}{C_y}\left(\varepsilon_1-\theta_1\varepsilon_1^2\right)\\+u_4\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_1\varepsilon_1+\varepsilon_0^2+\theta_2\varepsilon_1^2+\theta_1\varepsilon_0\varepsilon_1-\frac{\theta_1\varepsilon_1\varepsilon_2}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (71)$$

$$t_{03}-C_y=C_y\left[\begin{array}{l}\left(\begin{array}{l}\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\left(\theta_4+\frac{1}{2}\right)\varepsilon_3^2\\+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_5\frac{S_x^2}{C_y}\left(\varepsilon_3-\theta_3\varepsilon_3^2\right)\\+u_6\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\theta_4\varepsilon_3^2+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (72)$$

$$t_{04}-C_y=C_y\left[\begin{array}{l}\left(\begin{array}{l}\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\left(\theta_4+\frac{1}{8}\right)\varepsilon_3^2\\+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\end{array}\right)+u_7\frac{S_x^2}{C_y}\left(\varepsilon_3-\theta_3\varepsilon_3^2\right)\\+u_8\left(1+\frac{\varepsilon_2}{2}-\varepsilon_0-\theta_3\varepsilon_3+\varepsilon_0^2+\theta_4\varepsilon_3^2+\theta_3\varepsilon_0\varepsilon_3-\frac{\theta_3\varepsilon_2\varepsilon_3}{2}-\frac{\varepsilon_0\varepsilon_2}{2}-\frac{\varepsilon_2^2}{8}\right)\end{array}\right] \quad (73)$$

Squaring and taking expectations on both sides of (70), (71), (72), (73), the MSEs of the suggested estimators were obtained as:

$$MSE\left(t_{01}\right)=C_y^2\left(A_1^*+u_1^2B_1^*+u_2^2C_1^*-2u_1D_1^*+2u_2E_1^*-2u_1u_2F_1^*\right) \quad (74)$$

$$MSE\left(t_{02}\right)=C_y^2\left(A_2^*+u_3^2B_2^*+u_4^2C_2^*-2u_3D_2^*+2u_4E_2^*-2u_3u_4F_2^*\right) \quad (75)$$

$$MSE\left(t_{03}\right)=C_y^2\left(A_3^*+u_5^2B_3^*+u_6^2C_3^*-2u_5D_3^*+2u_6E_3^*-2u_5u_6F_3^*\right) \quad (76)$$

$$MSE\left(t_{04}\right)=C_y^2\left(A_4^*+u_7^2B_4^*+u_8^2C_4^*-2u_7D_4^*+2u_8E_4^*-2u_7u_8F_4^*\right) \quad (77)$$

where,

$$\theta_1=\frac{3a}{2},\theta_2=\frac{9a^2-12a}{2},\theta_3=a,\theta_4=\frac{a^2-a}{2},$$

$$A_1^* = \psi\left[\frac{(\lambda_{40}-1)}{4} - \frac{3a}{2}C_x\lambda_{21} - C_y\lambda_{30} + \frac{9a}{4}C_x^2 + 3aC_{yx} + C_y^2\right], \quad B_1^* = \psi\left[\frac{\bar{X}}{C_y}\right]^2 C_x^2$$

$$C_1^* = 1 + \psi\left[3C_y^2 + \left(3a - \frac{9a^2}{2}\right)C_x^2 + 6aC_{yx} - 2C_y\lambda_{30} - 3aC_x\lambda_{21}\right], \; D_1^* = \psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2} - \frac{3a}{2}C_x^2 - C_{yx}\right]$$

$$E_1^* = \psi\left[2C_y^2 - \left(\frac{12a - 27a^2 - 4}{8}\right)C_x^2 + \frac{9a}{2}C_{yx} - \frac{9a}{4}C_x\lambda_{21} - \frac{3C_y\lambda_{30}}{2} + \frac{(\lambda_{40}-1)}{8}\right],$$

$$F_1^* = \psi\left[\frac{\bar{X}}{C_y}\right]\left[\frac{C_x\lambda_{21}}{2} - C_{yx} - 3aC_x^2\right]$$

$$A_2^* = A_1^*, \; B_2^* = B_1^*, C_2^* = C_1^*, \; D_2^* = D_1^*, F_2^* = F_1^*$$

$$E_2^* = \psi\left[2C_y^2 - \left(\frac{12a - 27a^2 - 1}{8}\right)C_x^2 + \frac{9a}{2}C_{yx} - \frac{9a}{4}C_x\lambda_{21} - \frac{3C_y\lambda_{30}}{2} + \frac{(\lambda_{40}-1)}{8}\right],$$

$$A_3^* = \psi\left[C_y^2 + \frac{(\lambda_{40}-1)}{4} + \theta_3^2(\lambda_{04}-1) - C_y\lambda_{30} + \theta_3(\lambda_{22}-1) + 2\theta_3C_y\lambda_{12}\right], \quad B_3^* = \psi\left[\frac{S_x^2}{C_y}\right]^2(\lambda_{04}-1)$$

$$C_3^* = 1 + \psi\left[3C_y^2 + (\theta_3^2 + 2\theta_3)(\lambda_{04}-1) + 4\theta_3C_y\lambda_{12} - 2\theta_3(\lambda_{22}-1) - 2C_y\lambda_{30}\right],$$

$$D_3^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - C_y\lambda_{12} - \theta_3(\lambda_{04}-1)\right]$$

$$E_3^* = \psi\left[2C_y^2 - \frac{3\theta_3(\lambda_{22}-1)}{2} - \frac{3C_y\lambda_{30}}{2} + \left(\theta_3^2 + \theta_3 + \frac{1}{2}\right)(\lambda_{04}-1) + 3\theta_3C_y\lambda_{12} + \frac{(\lambda_{40}-1)}{8}\right],$$

$$F_3^* = \psi\left[\frac{S_x^2}{C_y}\right]\left[\frac{(\lambda_{22}-1)}{2} - C_y\lambda_{12} - 2\theta_3(\lambda_{04}-1)\right]$$

$$A_4^* = A_3^*, \quad B_4^* = B_3^*, C_4^* = C_3^*, D_4^* = D_3^*, F_4^* = F_3^*,$$

$$E_3^* = \psi\left[2C_y^2 - \frac{3\theta_3(\lambda_{22}-1)}{2} - \frac{3C_y\lambda_{30}}{2} + \left(\theta_3^2 + \theta_3 + \frac{1}{8}\right)(\lambda_{04}-1) + 3\theta_3C_y\lambda_{12} + \frac{(\lambda_{40}-1)}{8}\right],$$

Differentiating (74) partially with respect u_1 and u_2, equate the results to zero and solve for u_1 and u_2 simultaneously, to obtain $u_1 = \frac{E_1^*F_1^* - C_1^*D_1^*}{F_1^{*2} - B_1^*C_1^*}$ and $u_2 = \frac{B_1^*E_1^* - D_1^*F_1^*}{F_1^{*2} - B_1^*C_1^*}$. Substituting the results in (74), the minimum mean square error of t_{01} denoted by $MSE(t_{01})_{\min}$ is obtained as in (78)

$$MSE(t_{01})_{\min} = C_y^2\left[A_1^* + \frac{C_1^*D_1^{*2} - B_1^*E_1^{*2} - 2D_1^*E_1^*F_1^*}{F_1^{*2} - B_1^*C_1^*}\right] \quad (78)$$

Differentiating (75) partially with respect u_3 and u_4, equate the result to zero and solve for u_3 and u_4 simultaneously, to obtain $u_3 = \frac{E_2^*F_2^* - C_2^*D_2^*}{F_2^{*2} - B_2^*C_2^*}$ and $u_4 = \frac{B_2^*E_2^* - D_2^*F_2^*}{F_2^{*2} - B_2^*C_2^*}$. Substituting the results in (75), the minimum mean square error of t_{02} denoted by $MSE\left(t_{02}\right)_{\min}$ is obtained as in (79)

$$MSE\left(t_{02}\right)_{\min}=C_y^2\left[A_2^*+\frac{C_2^*D_2^{*2}-B_2^*E_2^{*2}-2D_2^*E_2^*F_2^*}{F_2^{*2}-B_2^*C_2^*}\right] \quad (79)$$

Differentiating (76) partially with respect u_5 and u_6, equate the result to zero and solve for u_5 and u_6 simultaneously, to obtain $u_5=\frac{E_3^*F_3^*-C_3^*D_3^*}{F_3^{*2}-B_3^*C_3^*}$ and $u_6=\frac{B_3^*E_3^*-D_3^*F_3^*}{F_3^{*2}-B_3^*C_3^*}$. Substituting the results in (76), the minimum mean square error of t_{03} implied by $MSE\left(t_{03}\right)_{\min}$ is as obtained in (80)

$$MSE\left(t_{03}\right)_{\min}=C_y^2\left[A_3^*+\frac{C_3^*D_3^{*2}-B_3^*E_3^{*2}-2D_3^*E_3^*F_3^*}{F_3^{*2}-B_3^*C_3^*}\right] \quad (80)$$

Differentiating (77) partially with respect u_7 and u_8, equate the result to zero and solve for u_7 and u_8 simultaneously, to obtain $u_7=\frac{E_4^*F_4^*-C_4^*D_4^*}{F_4^{*2}-B_4^*C_4^*}$ and $u_8=\frac{B_4^*E_4^*-D_4^*F_4^*}{F_4^{*2}-B_4^*C_4^*}$. Substituting the results in (77), the minimum mean square error of t_{04} denoted by $MSE\left(t_{04}\right)_{\min}$ is obtained as in (81)

$$MSE\left(t_{04}\right)_{\min}=C_y^2\left[A_4^*+\frac{C_4^*D_4^{*2}-B_4^*E_4^{*2}-2D_4^*E_4^*F_4^*}{F_4^{*2}-B_4^*C_4^*}\right] \quad (81)$$

RESULTS AND DISCUSSION

In this section, numerical analysis to justify the performance of suggested estimators t_{amk}, $k=1,2,3,4$ with respect to some existing estimators using two (2) datasets was conducted

Dataset 1: Murthy [20]

X: Area under wheat in 1963 Y: Area under wheat in 1964

$N=34,\ n=15,\ \bar{X}=208.88,\ \bar{Y}=199.44,\ C_x=0.72,\ C_y=0.75,\ \rho=0.98,\ \lambda_{21}=1.0045,$
$\lambda_{12}=0.9406,\ \lambda_{40}=3.6161,\ \lambda_{04}=2.8266,\ \lambda_{30}=1.1128,\ \lambda_{03}=0.9206,\ \lambda_{22}=3.0133$

Dataset 2: Singh [21]

X: Number of fish caught in year 1993 Y: Number of fish caught in year 1995

$N=69,\ n=40,\ \bar{X}=4591.07,\ \bar{Y}=4514.89,\ C_x=1.38,\ C_y=1.38,\ \rho=0.96,\ \lambda_{21}=2.19,$
$\lambda_{12}=2.30,\ \lambda_{40}=7.66,\ \lambda_{04}=9.84,\ \lambda_{30}=1.11,\ \lambda_{03}=2.52,\ \lambda_{22}=8.19$

Table 1: MSEs and PREs of the proposed and existing estimators considered in the study

Estimators	Dataset 1		Dataset 2	
	MSE	PRE	MSE	PRE
Auxiliary Information: $\bar{X}$, $\bar{x}$				
t_0	0.008003575	100	0.03808821	100
t_{AR}	0.027115658	29.47	0.7645918	49.82
t_1	0.006868341	116.53	0.03731461	102.07
t_2	0.006868341	116.53	0.03731461	102.07
t_3	0.006868341	116.53	0.03731461	102.07
t_4^r	0.006868341	116.53	0.03731461	102.07
t_5^r	0.006868341	116.53	0.03731461	102.07
t_6^r	0.006868341	116.53	0.03731461	102.07
t_{a1}	0.006737495	118.79	0.03404568	111.87
t_{p1}	0.006033	132.66	0.036522	104.29
t_{p2}	0.005659	141.43	0.035795	106.41
Proposed estimators $(t_{0k},\ k=1,2)$ *a=1.3*				
t_{01}	**0.004322121**	**185.18**	**0.032886721**	**115.82**
t_{02}	**0.003662027**	**218.56**	**0.03122314**	**121.99**
Auxiliary Information: S_x^2, s_x^2				
t_7	0.00696301	114.94	0.037568	101.38
t_8	0.006962763	114.95	0.037568156	101.38
t_9	0.006962763	114.95	0.037568156	101.38
t_{10}	0.006962763	114.95	0.037568156	101.38
t_{11}^r	0.006962763	114.95	0.037568156	101.38
t_{12}^r	0.006962763	114.95	0.037568156	101.38
t_{13}^r	0.006962763	114.95	0.037568156	101.38
t_{14}	0.00712551	112.32	0.0375686	101.38
t_{a2}	0.006013652	133.09	0.02810758	135.51
t_{p3}	0.006417	124.72	0.036308	104.90
t_{p4}	0.004996	160.19	0.029721	128.15
Proposed estimators $(t_{0k},\ k=3,4)$ *a=1,3*				
t_{03}	**0.003773914**	**212.08**	**0.02276788**	**167.29**
t_{04}	**0.0006149313**	**1301.54**	**0.004143642**	**919.20**

The computational formula for Percentage Relative Efficiency (PRE) is given by:

$$PRE(T) = \frac{MSE(t_0)}{MSE(T)} \times 100$$

where T is any estimator and t_0 is sample coefficient of variation

Table 1 indicate the mean square errors and percentage relative efficiencies of the suggested and other existing related estimators considered in the study using two datasets. Results obtained from the table revealed that proposed estimators under the use of auxiliary information, has minimum MSEs and higher PREs compared to other competing existing estimators, it could also be seen that the estimator t_{04} is highly efficient than other proposed estimators. This shows that the suggested estimators are more efficient than their counterparts and have higher chances to produce estimates closer to the true values of means for any population of interest.

CONCLUSION

In this study, four improved estimators for estimating the population coefficient of variation of the study variable Y using the information of auxiliary variables were suggested. The variants of the proposed estimators were derived up to first order of approximation using Taylor series techniques. From the empirical study, the results show that the proposed estimator are better than the existing estimators considered in this study. Hence we recommend that the proposed estimators should be used in practice.

Competing Interests

Authors have declared that no competing interests exist.

REFERENCES

Adejumobi, A. and M. A. Yunusa, M. A. (2022). Some Improved Class of Ratio Estimators for Finite Population Variance with the Use of Known Parameters," *LC International Journal of Stem,* 3, 2708-7123

Ahmed A., Adewara A. A. snd Singh R. V. K. (2016). Class of ratio estimators with known functions of auxiliary variable for estimating finite population variance. *Asian Journal of Mathematics and Computer Research*, 12(1):63-70.

Adichwal, N. K, Singh, R., Mishra, P., Singh, P. and Yan, Z. (2016). A two parameter ratio-product-ratio type estimator for population coefficient of variation based on simple random sampling without replacement. *J. Adv. Res. Appl. Math. Stat*., 1(3 & 4), 1–5.

Archana, V. and Rao, K. A. (2014). Some improved estimators of co-efficient of variation from bivariate normal distribution: a Monte Carlo comparison. *Pakistan Journal of Statistics and Operation Research,* 10(1):87–105.

Audu, A. and Adewara, A. A, (2017). Modified factor-type estimators under two phase sampling. *Punjab Journal Mathematics*, 49(2) 59-73.

Audu A, Singh R, Khare S, Dauran NS. (2021a). Almost unbiased estimators for population mean in the presence of non-response and measurement error. *Journal of Statistics & Management Systems*, 24(3), 573-589. DOI: 10.1080/09720510.2020.1759209

Audu, A. and Singh, R. V. K. (2021). Exponential-type regression compromised imputation class of estimators. *Journal of Statistics and Management Systems*, 24(6), 1253-1266. DOI: 10.1080/09720510.2020.1814501

Audu, A., Yunusa, M. A., Ishaq, O. O., Lawal, M. K., Rashida, A., Muhammad, A. H., Bello, A. B., Hairullahi, M. U. and Muili, J. O. (2021b). Difference-cum-ratio estimators for estimating finite population coefficient of variation in simple random sampling. *Asian Journal of robability and Statistics*, 13(3), 13–29.

Cochran, W.G. (1940). The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *Journal of Agricultural Science*, 30, 262-275.

Khoshnevisan, M., Singh, R., Chauhan, P., Sawan, N. and Smarandache, F. (2007). A general family of estimators for estimating population means using known value of some population parameter(s). *Far East Journal of Theoretical Statistics*, 22(2), 181-191.

Muili, J. O., Agwamba, E. N., Erinola, Y. A., Yunusa, M. A., Audu, A. and Hamzat, M. A. (200). Modified ratio-cum-product estimators of finite population variance. *International Journal of Advances in Engineering and Management*; 2(4), 309-319. DOI: 10.35629/5252-0204309319.

Murthy, M. N. (1967). Sampling theory and methods. 2nd Edition, Statistical Publishing Society, Calcutta

Rajyaguru, A. and Gupta, P. (2002). On the estimation of the co-efficient of variation from finite population. *Model Assisted Statistics and Application*, 36(2), 145-56.

Sahai, A. and Ray, S. K. (1980). And efficient estimator using auxiliary information, *Metrika*, 27(4), 271-275.

Singh, H. P. and Tailor, R. (2005). Estimation of finite population mean with known coefficient of variation of an auxiliary character. *Statistica*, 65(3), 301-313.

Singh, H. P, and Solanki, R. S. (2012). An efficient class of estimators for the population mean using auxiliary information in systematic sampling. *Journal of Statistical Theory and Practice*, 6(2), 274-285.

Singh, R., Mishra, M., Singh, B. P., Singh, P. and Adichwal, N. K. (2018). Improved estimators for population coefficient of variation using auxiliary variables. *Journal of Statistics and Management Systems*, 21(7), 1335-1355.

Singh, S. (2003). *Advanced sampling theory with applications. How Michael "Selected" Amy*. Springer Science and Media.

Singh, R, and Kumari, A. (2022). Improved Estimators of Population Coefficient of Variation under Simple Random Sampling. *Asian Journal of Probability and Statistics*, 19(4), 22-36. DOI: 10.9734/AJPAS/2022/v19i430474

Sisodia, B. V. S. and Dwivedi, V. K. (1981). Modified ratio estimator using coefficient of variation of auxiliary variable. *Journal-Indian Society of Agricultural Statistics*, 33:13-18.

Srivastava, S. K. and Jhajj, H. A. (1981). A class of estimators of the population mean in survey sampling using auxiliary information. *Biometrika*, 68(1):341-343.

Yunusa, M. A., Audu, A. and Adejumobi, A. (2022). Logarithmic product-cum-ratio type estimator for estimating finite population coefficient of variation. *Oriental Journal of Physical science*, 7(2):82–87.

Yunusa, M. A., Audu, A., Musa, N., Beki, D. O., Rashida, A., Bello, A. B. and Hairullahi, M. U. (2021). Logarithmic ratio-type estimator of population coefficient of variation. Asian *Journal of Probability and Statistics*, 14(2), 13–22.