Optimizing Cost Policies in an Industrial Setting Using Duality Theory: A Case Study of Brewery Production Company

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Abstract

*The concept of duality is applied to achieve optimal revenue generation in a production. It provides optimal cost policies for selling available raw materials when production is not feasible. In this work, the duality theorem is applied to the primal linear programming (LP) model of a brewery production company by **Ekwonwune and Edebatu (2016) to achieve an optimal cost policy that satisfies the strong duality theorem**. The dual LP model of the brewery production problem is formulated which is aimed at minimizing costs of raw material sales when production is not feasible. The optimal cost obtained provides maximum revenue plans for the brewery as well as minimized cost of purchase for the buyer. Data used in the illustration of the dual LP model is secondary data which includes profit of cartons of drinks, quantity of raw materials used in the production of some selected drinks in a brewery and available quantity of raw materials in the production run. From the analysis of the dual LP model formulated, the optimal amount obtained from sales of raw materials is 43,196. This optimal sales amount corresponds to the optimal profit solution of the primal model when production holds which is also 43,196. This result therefore upholds the strong duality theorem.*

Keywords: Duality theory, Primal model, Dual model, multi-product company, minimum cost policy.

INTRODUCTION

The pursuit of optimal production management strategies is crucial for companies navigating the complexities of the evolving global financial landscape. (Omar and Bor 2022) Multi-product production companies strive towards maintaining optimal revenue generation in the face of the perpetual challenge of production bottlenecks (Okoroafor and Iwuji 2024). So as market production dynamics continue to shift amidst product competitors, production managers have to identify well-defined approaches to achieve effective risk management and revenue enhancement (Ahmed 2017). Linear programming, as a mathematical optimization technique, offers a structured approach to the optimal allocation of resources to achieve specific objectives in industries while adhering to raw materials and other constraints (Kunwar and Sapkota 2022). However, duality in linear programming has far-reaching

economic consequences in financial decision-making in industry. The Duality theory provides management with alternative courses of action in revenue generation when production is infeasible.(Okechukwu, Chukwunedum and Anetor 2018).

Duality concepts have been applied by researchers in solving problems in Industry as well as in other areas. Sarode (2017) applied the dual simplex method to solve the transportation problem with fuzzy objectives for a toy production company which transports its products from different warehouses to different supply points in different cities. Nyor *et al*. (2017) applied the dual LP model in maximizing flight combinations that were more acceptable than that of the primal model in an airline crew formulation problem. Vaidya (2017) applied the quick simplex method in the solution of linear programming problems using the dual simplex method. Marmolejo *et al.* (2017) used the primal-dual cross-decomposition technique to solve the design of a distribution network for a bottled drink company. Gao (2017) applied the duality theory to various optimization problems. They highlighted the importance of duality in understanding the relationship between primal and dual problems, offering insights into the efficiency of resource allocation and the impact of constraints on the objective function. Hu and Wang (2013), examined lagrangian duality for multi-objective programming problems in lexicographic order. They investigated a lexicographic saddle point of a vector-valued Lagrangean saddle point and the lexicographic solution to a multi-objective programming problem.

 Khatib and Khedr (2018) explored the application of linear programming with duality in optimizing production and inventory decisions. Their study emphasised the significance of duality in understanding the trade-offs between production levels, inventory costs, and customer demand in the industry. Zaman and Muhammad (2016) focused specifically on profit maximization in a paint company and presented an optimization model that considered factors such as resource allocation, production capacity, pricing, and market demand. Although it may not directly address duality, it offers insights into the practical implementation of linear programming techniques for profit optimisation in the paint industry. Darvishi and Nasseri (2020) extend grey system theory to optimisation by proposing a dual theory for primal grey linear programming. The work introduced and proved the complementary slackness theorem, presented a dual simplex method in a grey environment and introduced useful concepts related to grey linear programming. Jdid and Smarandache (2023) focused on the dualistic theory in linear programming, emphasizing that every linear model has a corresponding dual model. The study introduced the neutrosophic dual and the binary simplex algorithm, aiming to concurrently find optimal solutions for both the original and dual models. Despite the numerous literature on the application of the duality theory, this study explores how this duality framework can be applied in industry to enhance the decision-making process for a multi-product company, providing an alternative revenue generation means when production is infeasible and shedding light on the shadow prices associated with constraints.

MATERIALS AND METHODS
The General Primal and Dual Linear Programming models
For given decision variables $x_1, x_2, \ldots, x_n$, the general LP problem involving n variables and m constraints ($m \geq n$), according to Gupta and Hira (2008), is presented in (1) – (4).

$$\text{Maximize } Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \qquad\qquad (1)$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \qquad\qquad (2)$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \qquad\qquad (3)$$

$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \qquad\qquad (4)$$

where $x_1, x_2, \ldots, x_n \geq 0$

with c_j as net contribution of decision variable x_j, b_j as total availability of ith resource and a_{ij} as ith resource per unit of variable x_j.

The associated dual LP model, for the primal LP model in (1) – (4), that minimizes the function W is given as

Minimize $W = b_1y_1 + b_2y_2 + \cdots + b_my_m \qquad\qquad (5)$

Subject to the restrictions

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$$

where $y_1, y_2, \ldots, y_m \geq 0$ are the dual variables.

Strong duality theorem

The fundamental theorem of duality, also called the strong duality theorem, states that if both the primal and dual problems have feasible solutions, then both have optimal solutions and $Max\, Z = Min\, W$.

Proof.

When both the primal and dual have feasible solutions, there is a lower bound on the minimum value of W as well as an upper bound on the maximum value of Z. (Pan 2013)

The symmetric primal-dual programmes (in matrix notation) are given as follows

Primal:

$max\, Z = cx$

Subject to $\quad Ax \leq b$

$\qquad\qquad x \geq 0$

Dual:

$min\, W = yb$

Subject to $\quad yA \geq c$

$\qquad\qquad y \geq 0$

Let the optimal solution to the primal be $x_B = B^{-1}b$. Then the corresponding optimal value of the primal objective function be $Z = c_Bx_B = c_B(B^{-1}b)$.

The corresponding conditions for optimality are

$[Z_j - c_j] \geq 0$ or $[c_BB^{-1}A - c, c_BB^{-1}\,] \geq 0$ or $c_BB^{-1}A - c \geq 0, c_BB^{-1} \geq 0$.

To verify that the associated optimal dual solution is $y_B = c_BB^{-1}$, observe that the above relation demonstrates that the feasibility conditions for the dual problem are satisfied.

The corresponding value of the dual objective function is given by

$\overline{W} = y_Bb = (c_BB^{-1})b = c_B(B^{-1}b) = c_Bx_B = max\, Z$

To show that $\overline{W}$ is optimal,

According to duality theory, the value of the objective function Z for any feasible solution of the primal is ≤ the value of the objective function W for any feasible solution of the dual.

i.e. $Z \leq W$

So $\quad max\, Z \leq min\, W$

$\qquad \overline{W} \leq W$

Since $\overline{W}$ cannot be less than $min\, W$

Therefore $max\, Z = min\, W \qquad\qquad (6)$

The Dual LP Model for Brewery Production Problem

The LP model for a brewery production problem was formulated by Ekwonwune and Edebatu (2016). Given six products of a brewery production company $x_1, x_2, x_3, x_4,$ and x_5 which represents the number of cartons of Golden Guinea larger (1x12), Golden Guinea larger (1x24), Bergedoff larger, Eagle stout and Bergedoff Malta produced during the production run. To maximize profit during a production run, the LP brewery production problem model under cost of raw material constraint is given by

Maximize (P) $= a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5$ (Objective function)   (7)

Subject to the constraint

$b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 + b_{15}x_5 \leq c_1$ (Sorghum raw material constraint)   (8)

$b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4 + b_{25}x_5 \leq c_2$ (Sugar raw Material Constraint)   (9)

$b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + b_{34}x_4 + b_{35}x_5 \leq c_3$ (Enzymes raw materials constraint)   (10)

$b_{41}x_1 + b_{42}x_2 + b_{43}x_3 + b_{44}x_4 + b_{45}x_5 \leq c_4$ (Hop's raw materials constraint)   (11)

$b_{51}x_1 + b_{52}x_2 + b_{53}x_3 + b_{54}x_4 + b_{55}x_5 \leq c_5$ (Brewing Sundries raw constraint)   (12)

$b_{61}x_1 + b_{62}x_2 + b_{63}x_3 + b_{64}x_4 + b_{65}x_5 \leq c_6$ (Bottling and washing machine raw materials constraint)   (13)

where $b_{11}, b_{12}, b_{13}, b_{14}, b_{15}$ is the quantity of Sorghum used in one carton of drink j

$b_{21}, b_{22}, b_{23}, b_{24}, b_{25}$ is the quantity of Sugar used in one carton of drink j

$b_{31}, b_{32}, b_{33}, b_{34}, b_{35}$ is the quantity of Enzyme used in one carton of drink j

$b_{41}, b_{42}, b_{43}, b_{44}, b_{45}$ is the quantity of Hops used in one carton of drink j

$b_{51}, b_{52}, b_{53}, b_{54}, b_{55}$ is the quantity of Brewing Sundries used in one carton of drink j

$b_{61}, b_{62}, b_{63}, b_{64}, b_{65}$ is the quantity of Bottling and Washing materials used in one carton of drink j

$c_1, c_2, c_3, c_4, c_5, c_6$ is the quantity of Sorghum, sugar, enzymes, hops, brewing sundries, bottling and wash machine available in the production run.

$a_1, a_2, a_3, a_4, a_5,$ is the profit of one carton of Golden Guinea larger (1x 12), Golden Guinea larger (1x24), Bergedoff larger, Eagle stout and Bergedoff Malta respectively.

Let us assume that production is not feasible due to maybe breakdown of the machine, and the management of the brewery wants to sell the quantities of raw materials c_1, c_2, c_3, c_4, c_5 and c_6 instead of them being kept to waste. The problem will be what quantity of the raw materials will be sold to minimize the cost to the buyer while making sure that the revenue from sales is the same as revenue from profit if production takes place. This implies getting a solution that upholds the strong duality theory presented in equation (6). This problem can be solved by solving the dual model of the primal linear programming model above.

Hence, the dual linear programming model for the brewery raw material minimum cost policy problem is given by:

Minimize W $= c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5 + c_6y_6$ (objective function)   (14)

The subject of the Constraints

$b_{11}y_1 + b_{21}y_2 + b_{31}y_3 + b_{41}y_4 + b_{51}y_5 + b_{61}y_6 \geq a_1$ (15)

$b_{12}y_1 + b_{22}y_2 + b_{32}y_3 + b_{42}y_4 + b_{52}y_5 + b_{62}y_6 \geq a_2$ (16)

$b_{13}y_1 + b_{22}y_2 + b_{33}y_3 + b_{43}y_4 + b_{53}y_5 + b_{63}y_6 \geq a_3$ (17)

$b_{14}y_1 + b_{24}y_2 + b_{34}y_3 + b_{44}y_4 + b_{54}y_5 + b_{64}y_6 \geq a_4$ (18)

$b_{15}y_1 + b_{25}y_2 + b_{33}y_3 + b_{44}y_4 + b_{55}y_5 + b_{65}y_6 \geq a_5$ (19)

where

y_1, y_2, y_3, y_4, y_5 are the selling price per unit of sorghum, sugar, enzymes, hops, brewing sundries, bottling and washing machine.

$b_{11}, b_{22}, b_{31}, b_{41}, b_{51}, are$ the Sorghum, Enzymes, Hops, Brewing Sundries, Bottling and Wash Machine

$b_{12}, b_{22}, b_{32}, b_{42}, b_{52}, b_{62}$ are the Sorghum, Enzymes, Hops, Brewing Sundries, Bottling and Washing Machine

$b_{13}, b_{23}, b_{33}, b_{43}, b_{53}, b_{63}$ are the Sorghum, Enzymes, Hops, Brewing Sundries, Bottling and Washing Machine

$b_{14}, b_{24}, b_{34}, b_{44}, b_{54}, b_{64}$ are the Sorghum, Enzymes, Hops, Brewing Sundries, Bottling and Washing Machine

$b_{15}, b_{25}, b_{35}, b_{45}, b_{55}, b_{65}$ are the Sorghum, Sugar, Enzymes, Hops, Brewing Sundries, Bottling and Washing Machine.

$c_1, c_2, c_3, c_4, c_5, c_6$ is the cost of a unit of sorghum, sugar, enzymes, hops, brewing sundries, bottling and washing machine to purchase.

$a_1, a_2, a_3, a_4, a_5,$ is the Minimum selling price of raw material used in producing one carton of Golden Guinea larger (1x 12), Golden Guinea larger (1x24), Bergedoff larger, Eagle stout, Bergedoff Malta.

The objective function of the dual is to minimize the total selling price of the raw materials $c_1, c_2, c_3, c_4, c_5, c_6$. The constraint of the dual ensures that the selling price of raw material used in producing each drink is greater than or equal to the supposed profit from the sale of the different produced drinks if production had taken place.

Data Illustration of Dual LP Model of Brewery Optimal Cost Problem
The data used in this work is secondary data collected from the work by Ekwonwune and Edebatu (2016). Table 1 presents data of unit profit contribution of the products- Golden Guinea larger (1x 12), Golden Guinea larger (1x24), berged off larger (BDF), Eagle stout (E/S), Berged off Malta (B/M). Table 2 shows data on the total cost of raw materials.

Table 1: Total and unit costs of production and profit contribution

Product (in carton)	Unit Profit Contribution (per carton)
Golden Guinea larger (1x12)	66.62
Golden Guinea larger (1x24)	96.48
Bergedoff larger	60.51
Eagle stout	218.73
Bergedoff Malta	104.42

Source: Ekwonwune, E.N. & Edebatu, D.C. (2016)

Table 2. Total cost of raw materials

Raw Material	G.G(1x12)	G.G(1x24)	BDF	E/S	B/M
Sorghum	21.144768	0.071741	7.721884	4.0005	20.968132
Sugar	0	0	0	0	0.300000
Enzymes	7.218816	0.024493	4.966576	2.544372	1.245227
Hops	0.005190	0.017612	3.421288	0.900576	3.536607
Brewing Sundries	7.935048	0.026933	2.990746	0.342768	11.068012
Bottling and Washing Materials	22.84128	0.119559	11.700076	2.857844	20.551603

Source: Ekwonwune, E.N. & Edebatu, D.C. (2016)

The dual LP model of the data above was solved with the objective function given by equation (14) which is to minimize the total selling price of the raw materials subject to the constraint of the dual, equations (15) to (19).

Table 3: Optimal selling plan of raw material for dual LP model.

Dual variables	Value	Objective Coefficient	Objective Value Contribution
y_1(Sorghum):	0.00	53.91	0.00
y_2(Sugar):	0.00	0.30	0.00
y_3(Enzymes):	5468.48	7.88	43098.62
y_4(Hops):	0.00	22.36	0.00
y_5(brewing sundries):	1.67	58.07	96.98
y_6(Bottling&Washing Machine):	0.00	432.59	0.00

Min. W = objective value = 43,195.6 ≈ 43,196

The solution of the dual LP model is presented in Table 3. The solution presents the raw material selling plan that minimizes the total selling price of the raw materials to the buyer as well as gives optimal revenue to the company. The objective value contribution of each raw material is the product of its objective coefficient and achieved value. Meanwhile, Table 4 shows the optimal solution of the product profit maximization production plan solved using the primal LP model for the brewery production problem data. This solution of the primal LP model presents the optimal profit if production took place while the solution of the dual LP model shows the minimum cost of selling the raw materials to a purchaser if production couldn't take place. From the results, the minimum cost of selling the raw materials to a purchaser, which is 43,196 as shown in Table 4 (from the sale of enzymes and brewing sundries) is equal to the profit margin that would be achieved had production taken place (i.e. 43,196) as shown in Table 4.

Table 4: Optimal profit maximization production plan for primal LP model.

Primal variables	Value	Objective Coefficient(Profit per carton)	Objective Value Contribution
x_1	0.20	66.60	13.32
x_2	447.49	96.50	43182.79
x_3	0.00	60.50	0.00
x_4	0.00	218.70	0.00
x_5	0.00	104.40	0.00

Max. Z = profit achieved = 43,196.11 ≈ 43,196

Hence the solution of the dual LP model of the brewery optimal cost policy problem with that of the primal profit maximization solution upholds the strong duality theory as $max\ Z\ (43,196) = min\ W\ (43,196)$.

CONCLUSION
The application of the Duality theory to the LP model of Nigeria Breweries PLC's revenue problem presented in this study has further portrayed the effectiveness of the duality theory in providing optimal decision-making strategies in an industry setting. This study provides a solution for optimal revenue problems in the brewery industries as well as for other multi-product industries in instances where production is unfeasible. So the dual LP model steps in as an alternative model to minimize the costs of raw materials sold to purchasers without minimum optimal profit. This ensures that, even in the absence of production, the brewery maximizes profitability through the use of the efficient dual LP model strategy. This work presents the application of the dual LP model on a brewery's optimal cost policy problem when production is infeasible. The dual LP model is used to optimize cost of purchasing the raw materials (i.e. Sorghum, Sugar, Enzymes, Hops, Brewing Sundries, Bottling and Washing machines) of the buyer such that profit from sales for the breweries management equals revenue that would have been accrued from profits of sales of products (Golden Guinea larger

(1x 12), Golden Guinea larger (1x24), Bergedoff larger, Eagle stout, Bergedoff Malta) supposed production. This means actualizing the strong duality theorem. After the solution of the dual LP model, the minimum cost for the sale of raw materials is 43,196. This is equal to the maximum profit solution of the primal LP model which is 43,196. Therefore, the solution of the dual LP solution of this brewery production problem satisfies the duality theorem.

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