https://dx.doi.org/10.4314/dujopas.v10i1b.12
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 10 No. 1b March 2024*

The Dynamical Equations of the Restricted Three-Body Problem with Poynting-Robertson Drag Force and Variable Masses

Tajudeen Oluwafemi Amuda [1] and Oni Leke[2]

[1]Department of Mathematics,
Faculty of Physical Science,
Airforce Institute of Technology Kaduna,
Kaduna State,
Nigeria

[2]Department of Mathematics,
College of Physical Science,
Joseph Sarwuan Tarka University
P.M.B. 2373, Makurdi,
Benue-State,
Nigeria

Email: teejaymath@gmail.com

Abstract

The restricted three-body problem (R3BP) is a formulation which defines the motion of a passively gravitating test particle having infinitesimal mass and moving in the gravitational environment of two bodies, called primaries. The R3BP is still an exciting and active research field that has been getting attention of scientists and astronomers because of its applications in dynamics of the solar and stellar systems, lunar theory, and artificial satellites. The equations of motion are usually the starting point in the investigations of the dynamical predictions of the infinitesimal mass. Therefore, in this paper, we examine the derivations of the dynamical equations of the R3BP with Poynting-Robertson (P-R) Drag force and variable masses. In this model formulation, both primaries are assumed to vary their masses under the combined Mestschersky law (CML) and they move in the frame of the Gylden-Mestschersky equation (GME). Further, the bigger primary is assumed to be emitting radiation force, which is a component of the radiation pressure and the P-R drag. The non-autonomous dynamical equations of the model are derived and converted into the autonomized equations with constant coefficients using the Mestschersky transformation (MT), the CML, the particular solutions of the GMP, and a transformation for the time dependent velocity of light. We observed that the P-R drag of the bigger primary depends on the mass parameter, radiation pressure, velocity of light and the mass variation constant κ*. The derived systems of equations with variable and constant coefficients can be used to model the long-term motion of satellites and planets in binary systems.*

Keywords: R3BP, Variable Masses, P-R Drag, Radiation Pressure, Test particle

**Author for Correspondence*

INTRODUCTION

The restricted three-body problem (R3BP) describes motion of an infinitesimal mass moving in the gravitational environment of two main masses, called primaries, which move in circular orbits around their center of mass on account of their common attraction and the infinitesimal mass not influencing the motion of the primaries (Szebehely 1967). There are several examples of the R3BP in space dynamics. One of these is the Sun-Earth-Moon combination which describes the motion of the moon. One of the main idea in space science is the creation of artificial bodies, which are required to move in the vicinity of two natural astronomic bodies, this is also analogous to the R3BP.

The classical R3BP considers masses in the R3BP to be constant. However, several celestial bodies change their masses during evolution. Hence, the phenomenon of absorption in stars made scientists to frame the R3BP with variable masses. The problem of the dynamics of celestial bodies with variable mass has lots of remarkable applications in galactic, planetary, and stellar dynamics. As an example, we could mention the motion of a satellite nearby a radiating star enclosed by a cloud and changing its mass as a result of the particles of the cloud. Also, comets loose part or all of their mass as they travel round the Sun (or other stars) due to their contact with the solar wind which blows off particles from their surfaces. Due to the inclusion of mass variations, many researchers such as Bekov (1988), Luk'yanov (1989), Singh & Leke (2010,2012,2013a, b, c), Ansari et. al (2019), Leke & Singh (2023), Leke & Shima (2023), Leke & Mmaju (2023) and more recently Leke & Orum (2024), have carried out various investigations to include mass variation of the primaries when the mass of the third body is kept constant while the formulation of the R3BP when the three masses vary with time, researchers such as El-Shaboury (1990), Bekov *et al*. (2005); Letelier & Da Silva (2011), and Singh & Leke (2013d), have investigated this formulation under different classifications.

The model formulation of the classical R3BP did not characterize the primaries to be sources of radiation pressure. Radiation pressure acts as an orbital perturbation and can displace a dust grain from its position. Radzievskii (1950,1953) discussed the introduction of radiation pressure of one and both primaries and observed that their presence allows for the existence of additional EPs. In view of this, several investigations have been carried out when one or both primaries are emitters of radiation pressure. Notable among these are Singh & Ishwar (1999), Singh & Leke (2010, 2012, 2013a) and Singh & Sunusi (2020).

Further, characterization of the primaries which involves the force of radiation, is the effect of Poynting–Robertson (P–R) drag. This force is a part of the radiation force and can sweep dust grain particles of the solar system into the Sun at a cosmically fast rate. Several authors have conducted researches on the R3BP with P-R drag, amongst them are Ragos and Zafiropoulos (1995), Kushvah (2008), Das *et. al* (2009) Singh & Abdulkarim (2014), Singh & Amuda (2019), Amuda *et. al* (2021), and Amuda & Singh (2022).

In this current paper, we aim to formulate the dynamical equations of the R3BP with variable masses in which the bigger primary emit radiation pressure and P-R drag. The paper is an extension of the dynamical equations of motion given by Gelf'gat (1973), Bekov (1988) and Luk'yanov (1989) when the bigger primary is a radiation source with P-R drag component.

The setup of the paper, is in the following order. Section 2 gives the description of the dynamical equations. Sections 3 and 4, give the discussion and conclusion, respectively.

DESCRIPTIONS OF THE DYNAMICAL EQUATIONS

Gylden-Mestschersky Problem, Unified Mestschersky Law and Mestschersky Transformation

The absolute motion of the points is described by the Mestschersky (1902) equation for a point of variable mass,

$$\vec{F} = m\dot{\vec{v}} + \left(\vec{v} - \vec{u}\right)\dot{m} \tag{1}$$

where $\vec{F}$ is the sum of the forces acting on the body and $\vec{v}$ is the velocity, both of which is measured in an inertial coordinate system. Further, $\vec{u}$ is the velocity of the center of mass of the absorbed mass before its attachment with the body (or of the ejected mass after its ejection). The over dot depicts derivation with respect to the time $\vec{u}$.

The relative motion of mass m_2 about mass m_1 under the action of mutual gravitational force, was represented as the sum of the masses of these points as varying with time by a certain law Gylden (1884)

$$m_1 + m_2 = \mu\left(t\right) \tag{2}$$

Gylden (1884) wrote the differential equation of the formulation in the form

$$\ddot{\vec{r}} + \frac{\mu\left(t\right)}{r^3}\vec{r} = 0 \tag{3}$$

Later, Mestschersky (1902) revealed that the Gylden (1884) problem (3) is a special case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same.

Now, when the mass is expelled with the same velocity of the body at any time $\left(\vec{v} = \vec{u}\right)$, that is, mass discharge does not produce responsive forces. In this occasion, equation (1) reduces to the form

$$\vec{F} = m\dot{\vec{v}} \tag{4}$$

When this occurs, the relative motion of the problem of two bodies with variable masses is defined by the equation

$$\ddot{\vec{r}} = -G\frac{\left(m_1 + m_2\right)}{r^3}\vec{r} \tag{5}$$

Equation (5) is similar to the equation of the classical two-body problem with constant masses, with the alteration that now; the sum of the masses is a function of time and is referred to as the Gylden-Mestschersky equation (GME).

Mestschersky (1902), reduced the GMP through the introduction of new variables and “time” to the equations of the classical problem of two bodies with constant masses by a transformation, known as the Mestschersky transformation (MT) and is given as

$$x = \xi R(t),\ y = \eta R(t),\ z = \varsigma R(t),\ \frac{dt}{d\tau} = R^2(t)$$
$$r = \rho_{12}R(t),\ r_i = \rho_i R(t),\ (i = 1, 2), \tag{6}$$

where $R\left(t\right) = \sqrt{\alpha t^2 + 2\beta t + \gamma}$; ξ, η, ζ, τ are the new variables and, α, β, γ and ρ_{12} are constants.

Later, Mestschersky (1952) came up with a law which considers the masses and their sum to vary in the same proportion in such a way that

$$\mu\left(t\right) = \frac{\mu_0}{R\left(t\right)},\ \ \mu_1\left(t\right) = \frac{\mu_{10}}{R\left(t\right)},\ \mu_2\left(t\right) = \frac{\mu_{20}}{R\left(t\right)} \tag{7}$$

where $\mu_1\left(t\right) = Gm_1\left(t\right), \mu_2\left(t\right) = Gm_2\left(t\right)$, $\mu\left(t\right) = \mu_1\left(t\right) + \mu_2\left(t\right)$, μ_{10} and μ_{20} are constants.

The law (7) is called the combined Mestschersky law (CML) and it guarantees that the centre of the mass of the system moves initially.
Now, the GME has the particular solutions of the forms

$$\ddot{r} = \rho_{12} \frac{\left(\alpha\gamma - \beta^2\right)}{R^3\left(t\right)} \tag{8}$$

$$\omega\left(t\right) = \frac{\omega_0}{R^2\left(t\right)} \tag{9}$$

$$r\mu = \kappa C^2 \tag{10}$$

where

$$\kappa = \frac{\beta^2 - \alpha\gamma + \omega_0^2}{\omega_0^2}$$

(11)
Equation (10) is a particular integral of the GME and κ is a constant, and is such that $0 < \kappa < \infty$

Equations of Motion
The study of the motion of a satellite using the model of the R3BP under the condition that the motion of the variable-mass primaries is governed by the Gylden-Mestschersky problem with isotropic mass variation of the primaries varying in proportion to each other in accordance with the UML has been studied by Bekov (1988), Singh & Leke (2010, 2012, 2013a, b, c). Therefore, following the methodology, we consider a rotating frame of reference $0\,x\,y\,z$, where 0 is the origin. Further, we let m_1 and m_2 be the masses of the primary bodies and m_3 is the mass of the satellite. Also, let the distance from m_3 to m_1 be r_1 and from m_3 to m_2 be r_2 while the distance between the two primaries be r. Finally, we let ω be the angular velocity of revolution of the primaries. We consider same formulation by Bekov (1988) with further assumptions that the smaller primary is a radiation source having P-R drag force. In this premise, the kinetic energy in the rotating frame of reference $0xyz$ is given by (Szebehely, 1967)

$$T = \frac{1}{2} m_3 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m_3 \omega (x\dot{y} - y\dot{x}) + \frac{1}{2} m_3 (x^2 + y^2)\omega^2 \tag{12}$$

Now, let p_x, p_y and p_z be the generalized components of momentum, then

$$\rho_x = m_3\left(\dot{x} - \omega y\right), \quad \rho_y = m_3\left(\dot{y} + \omega x\right), \quad \rho_z = m_3 \dot{z} \tag{13}$$

Now using equations (13) in the Hamiltonian H and simplifying, gives

$$H = \frac{1}{2m_3}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega\left(\dot{y} p_x - x p_y\right) - U \tag{14}$$

Therefore, we have

$$\dot{p}_x = \omega p_y - \frac{\partial U}{\partial x}, \quad \dot{p}_y = -\omega p_x - \frac{\partial U}{\partial y}, \quad \dot{p}_z = -\frac{\partial U}{\partial z}$$

(15)
Now since the primaries move within the context of the GME and their masses change with time in agreement with the CML, then p_x, p_y, p_z and the angular velocity ω will all be time dependent. Thus, differentiating system p_x, p_y and p_z with respect to time, respectively, we get

$$\dot{p}_x = m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega} y\right), \qquad \dot{p}_y = m_3\left(\ddot{y} + \omega\dot{x} + \dot{\omega} x\right), \qquad \dot{p}_z = m_3 \ddot{z} \tag{16}$$

From systems (15) and (16), we have

$$\begin{aligned}
&\ddot{x} - \omega\dot{y} - \dot{\omega}y = \omega p_y - \frac{\partial U}{\partial x} \\
&\ddot{y} + \omega\dot{x} - \dot{\omega}x = -\omega p_x - \frac{\partial U}{\partial y} \\
&\ddot{z} = -\frac{\partial U}{\partial z}
\end{aligned} \tag{17}$$

Next, the equations of motion of the CR3BP under effects of radiation pressure and P-R drag is written (Ragos *et al.*):

$$\vec{a} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{\mu_1 q_1 \vec{r}_1}{r_1^3} - \frac{\mu_2 \vec{r}_2}{r_2^3} - \frac{\mu_1(1-q_1)}{r_1^2}\left[\frac{\vec{r}_1}{r_1} - \left(\frac{(\dot{\vec{r}}_1 + \vec{\omega}\times\vec{r}_1).\vec{r}_1}{c_d r_1}\right)\frac{\vec{r}_1}{r_1} - \frac{(\dot{\vec{r}}_1 + \vec{\omega}\times\vec{r}_1)}{c_d}\right] \tag{18}$$

where q_1 is the radiation pressure of the bigger primary, c_d is the dimensionless velocity of light and

$$\begin{aligned}
&\vec{r} = x\vec{i} + y\vec{j} + z\vec{k},\ \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k},\ \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}, \\
&\vec{r}_1 = (x+\mu)\vec{i} + y\vec{j} + z\vec{k},\ \vec{r}_2 = (x+\mu-1)\vec{i} + y\vec{j} + z\vec{k},\ \vec{\omega}\times\vec{v} = -(\dot{y}\vec{i} - \dot{x}\vec{j}), \\
&\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -(x\vec{i} + y\vec{j}),\ \dot{\vec{r}}_2 + \vec{\omega}\times\vec{r}_2 = (\dot{x} - y)\vec{i} + [\dot{y} + (x+\mu-1)]\vec{j} + \dot{z}k, \\
&(\dot{\vec{r}}_2 + \vec{\omega}\times\vec{r}_2).\vec{r}_2 = +[(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}]
\end{aligned} \tag{19}$$

Comparing equations (17) and (18), and equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$, on both sides, we get

$$\begin{aligned}
&\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega}y - \frac{\mu_1 q_1 (x - x_1)}{r_1^3} - \frac{\mu_2 (x - x_2)}{r_2^3} - \frac{W_1}{r_1^2}\left[\frac{(x - x_1)}{r_1^2}\{(x - x_1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{x} - \omega y\right] \\
&\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega}x - \frac{\mu_1 q_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{W_1}{r_1^2}\left[\frac{y}{r_1^2}\{(x - x_1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{y} + \omega(x - x_1)\right] \\
&\ddot{z} = -\frac{\mu_1 q_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} - \frac{W_1}{r_1^2}\left[\frac{z}{r_1^2}\{(x - x_1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{z}\right]
\end{aligned} \tag{20}$$

where

$$W_1 = \frac{\mu_1(t)(1 - q_1)}{c_d(t)} \tag{21}$$

represents the P-R effect of the bigger primary and $c_d(t)$.is the velocity of light which depends on time, $\mu_i(t) = Gm_i(t)$, $r_i^2 = (x - x_i)^2 + y^2 + z^2$, $(i = 1,2)$

These equations describe the dynamics of the satellite in the gravitational environment of the variable mass primaries under radiation pressure and P-R effects of the bigger primary.

Now from the property of the center of mass μ_1 at $(x_1, 0, 0)$ and μ_2 at $(x_2, 0, 0)$ with the consideration that coordinates is barycentric, we have

$$x_1 = -\frac{\mu_2 r}{\mu_1 + \mu_2}, \quad x_2 = \frac{\mu_1 r}{\mu_1 + \mu_2} \tag{22}$$

The equations (22) unite the barycentric coordinates x_1 and x_2 with the common distance r.

Autonomization of the equations with variable coefficients

The derived dynamical equations (20) are not integrable and the solutions even for particular steady-state solutions –the EPs are difficult to seek directly from equation (20), because these equations contain unknown functions of time. In order to transform system (20), we use the MT (6); the CML (7); the particular integral (10) and solutions of the GME (8-9).

From the MT (6), we have

$$x = \xi R(t) \Rightarrow \quad x = \xi(\alpha t^2 + 2\beta t + \gamma)^{\frac{1}{2}}$$

We differentiate w.r.t.t and denotes differentiation w.r.t. τ by dashes, we get

$$\dot{x} = \frac{\xi(\alpha t + \beta)}{R(t)} + \frac{\xi'}{R(t)}, \dot{y} = \frac{\eta(\alpha t + \beta)}{R(t)} + \frac{\eta'}{R(t)}, \ \dot{z} = \frac{\varsigma(\alpha t + \beta)}{R(t)} + \frac{\varsigma'}{R(t)} \tag{23}$$

$$\ddot{x} = \frac{\xi(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\xi''}{R^3(t)}, \ \ddot{y} = \frac{\eta(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\eta''}{R^3(t)}, \quad \ddot{z} = \frac{\zeta(\alpha\gamma - \beta^2)}{R^2(t)} + \frac{\zeta''}{R^3(t)}$$

Also, from a particular solution (9) of the GMP, we get

$$\dot{\omega} = -\frac{2\omega_0(\alpha t + \beta)}{R^4(t)}$$

Additionally, we assume that the velocity of light varies in such a way that

$$c_d(t) = \frac{c_{d0}}{R(t)} \tag{24}$$

where c_{d0} is a constant velocity of light.

Substituting all the above in system (20) and simplifying, yields

$$\xi'' - 2\omega_0\eta' = \omega_0^2\xi - \xi(\alpha\gamma - \beta^2) - \frac{\mu_{10}q_1(\xi - \xi_1)}{\rho_1^3} - \frac{\mu_{20}(\xi - \xi_2)}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{(\xi - \xi_1)}{\rho_1^2}\{(\xi - \xi_1)\xi' + \eta\eta' + \zeta\zeta'\} + \xi' - \eta\right]$$

$$\eta'' + 2\omega_0\xi' = \omega_0^2\eta - \eta(\alpha\gamma - \beta^2) - \frac{q_1\mu_{10}\eta}{\rho_1^3} - \frac{\mu_{20}\eta}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{\eta}{\rho_1^2}\{(\xi - \xi_1)\xi' + \eta\eta' + \zeta\zeta'\} + \eta' + (\xi - \xi_1)\right]$$

$$\zeta'' = (\alpha\gamma - \beta^2)\zeta - \frac{\mu_{10}q_1\zeta}{\rho_1^3} - \frac{\mu_{10}\zeta}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{\zeta}{\rho_1^2}\{(\xi - \xi_1)\xi' + \eta\eta' + \zeta\zeta'\} + \zeta'\right] \tag{25}$$

where

$$W_{10} = \frac{\mu_{10}(1 - q_1)}{c_{d0}} \tag{26}$$

Performing same substitution on r_1^2 and r_2^2, and simplifying gives

$$\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2, \rho_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2 \tag{27}$$

$$\xi_1 = \frac{-\mu_{20}}{\mu_0}\rho_{12}, \quad \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12} \tag{28}$$

Next, we make choice for units of measurements such that at initial time t_0, we choose

$$\mu_0 = G \tag{29}$$

For the unit of time and length, we choose them, respectively, such that

$$\rho_{12} = 1 \tag{30}$$

We now introduce the mass parameter υ, expressed as

$$\frac{\mu_{10}}{\mu_0} = 1-\upsilon, \quad \frac{\mu_{20}}{\mu_0} = \upsilon \quad , \quad \text{where } 0 < \upsilon \le \frac{1}{2} \tag{31}$$

Substituting (29) in (31), we have

$$\mu_{10} = G(1-\upsilon) \quad , \quad \mu_{20} = G\upsilon \tag{32}$$

Now from the particular integral (10) and (29), we get

$$G = \kappa \tag{33}$$

If these measurements are substituted in equation (11), we get

$$(\kappa - 1)\omega_0^2 = \beta^2 - \alpha\gamma \tag{34}$$

where κ is the constant of integration of the GMP.

Therefore, substituting the units of measurement in equations of system (25), (26), (27) and (28), we get

$$\xi'' - 2\eta' = \kappa\xi - \frac{\kappa q_1(1-\upsilon)(\xi+\upsilon)}{\rho_1^3} - \frac{\kappa\upsilon(\xi+\upsilon-1)}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{(\xi+\upsilon)}{\rho_1^2}\{(\xi+\upsilon)\xi' + \eta\eta' + \zeta\zeta'\} + \xi' - \eta\right]$$

$$\eta'' + 2\xi' = \kappa\eta - \frac{\kappa q_1(1-\upsilon)\eta}{\rho_1^3} - \frac{\kappa\upsilon\eta}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{\eta}{\rho_1^2}\{(\xi+\upsilon)\xi' + \eta\eta' + \zeta\zeta'\} + \eta' + (\xi+\upsilon)\right] \tag{35}$$

$$\zeta'' = (\kappa-1)\zeta - \frac{\kappa q_1(1-\upsilon)\zeta}{\rho_1^3} - \frac{\kappa\upsilon\zeta}{\rho_2^3} - \frac{W_{10}}{\rho_1^2}\left[\frac{\zeta}{\rho_1^2}\{(\xi+\upsilon)\xi' + \eta\eta' + \zeta\zeta'\} + \zeta'\right]$$

$$\text{where } W_{10} = \frac{\kappa(1-\upsilon)(1-q_1)}{c_{d0}}, \; 0 < \kappa < \infty \tag{36}$$

$$\rho_1^2 = (\xi+\upsilon)^2 + \eta^2 + \zeta^2, \quad \rho_2^2 = (\xi+\upsilon-1)^2 + \eta^2 + \zeta^2 \tag{37}$$

Equations (35-37) gives the equations of motion of the autonomized system with constant coefficients, when the bigger primary emits radiation force.

DISCUSSION

The paper investigates derivations of the dynamical equations of an inactively gravitating satellite in the gravitational environment of two-variable mass primaries when the bigger primary emit radiation pressure and P-R drag. The equations of motion of the time-dependent system have been derived in equations (20) and thereafter converted to the autonomized forms with constant coefficients (35) with the help of the MT, the particular solutions of the GME, the CML. It is seen that the equations of motion of the non-autonomous system (20) is different from those of Bekov (1988, 2005), Luk'yanov (1989), Singh & Leke (2010, 2012, 2013a), Taura & Leke (2022), and, Leke & Singh (2023) due to the appearance of the P-R drag of the bigger primary. Also, the equations of motion (33) with constant coefficients are different from those of Kushvah (2008), Singh & Amuda (2014, 2017), Amuda et. al. (2021) and Amuda & Singh (2022) due to the presence of the parameter κ. Also, the P-R drag of our study depends on the mass variation constant κ.

The investigations of the R3BP is of immense historical, educational, theoretical and practical importance, and in its many modifications, has had significant implications in numerous scientific fields, comprising among others; chaos theory, celestial mechanics, galactic dynamics and molecular physics. The R3BP is still a stimulating and vigorous research field that has been receiving significant attention from scientists and astronomers because of its vast applications in stellar and solar systems dynamics, lunar theory, and artificial satellites. It can be hope that the derivations of the dynamical equations of this problem will provide more insight into further investigations of the R3BP with mass variations and P-R drag.

CONCLUSION

The R3BP illustrates motion of a test particle having infinitesimal mass and travelling in the gravitational environment of two massive bodies, called primaries. The equations of motion are usually the starting point in the investigations of the dynamical predictions of the infinitesimal mass. Therefore, in this paper, we examined the dynamical equations of the R3BP with Poynting-Robertson (P-R) drag force and mass variations of the primaries. We assumed both primaries to vary their masses in agreement with the combined Mestschersky law (CML) and they move under the context of the Gylden-Mestschersky equation (GME). Further, the bigger primary is assumed to be an emitter of radiation force, which is a component of the radiation pressure and the P-R drag. The non-autonomous equations of the dynamical frameworks are derived and converted into the autonomized equations with constant coefficients with the aid of the Mestschersky transformation (MT), the CML, the particular solutions of the GME. We found that the P-R drag of the bigger primary is defined by mass variation constant κ, mass parameter, radiation pressure of the bigger primary and the velocity of light. The derived systems of equations with variable and constant coefficients can be used to model the long-term motion of satellites and planets in binary systems.

REFERENCES

Amuda, T.O. and Singh, J., (2022). Instability of Triangle Equilibrium points in the restricted three-body problem under effects of circumbinary disc, radiation pressure and P-R drag. Earth, Moon and Planets 126: (1)

Amuda, T.O., Leke, O., and AbdulRaheem, A. (2021). Unveiling Perturbing Effects of P-R Drag on Motion around Triangular Lagrangian Points of the Photogravitational Restricted Problem of Three Oblate Bodies. *CJAST*. 40(1):10-31

Ansari, A.A., Kellil, R., Ziyad, A. A., Wasi,m U. D, (2019). Effect of variation of charge in the circular restricted three-body problem with variable masses Journal of Taibah University for Science. Vol 13:1

Bekov, A. A. (1988). Libration points of the restricted problem of Three Bodies with variable Mass. *Soviet Astronomy Journal, 33*, 92-95

Bekov, A.A., Beysekov, A.N. and Aldibaeva, L.T. (2005). On the dynamics of non-stationary binary stellar systems with non-isotropic mass flow. *Astronomical and Astrophysical Transactions*, *24*, 311–316

Das, M.K, et al. 2009. On the out of plane equilibrium points in photogravitational restricted three body problem. Journal of Astronomy and Astrophysics. 30:177.

El-Shaboury, S.M. (1990). Existence of particular solutions in the generalized restricted problem of three bodies with variable masses. *Astrophysics and Space Science 174*, 151-154.

Gelf'gat, B.E. (1973). *Current Problems of Celestial Mechanics and Astrodynamics*, Nauka, Moscow

Gylden, H.: (1884). Die Bahnbewegungen in Einem Systeme von zwei Körpern in dem Falle, dass die Massen Ver Nderun- Gen Unterworfen Sind, Astronomische Nachrichten. *109*, 1-6.

Kushvah, B. S., (2008). The effect of radiation pressure on the equilibrium point in the generalized photogravitational restricted three-body problem. Astrophysics and Space Science. 315:231.

Leke, O, Mmaju, C, (2023) Zero velocity curves of a dust grain around equilibrium points under effects of radiation, perturbations and variable Kruger 60. Phys. Astron. Int. J. 7, 280-285

Leke, O., Orum, S., (2024). Motion and zero velocity curves of a dust grain around collinear libration points for the binary IRAS 11472-0800 and G29-38 with a triaxial star and variable masses. New Astronomy, 108, 102177

Leke, O., Shima, S. (2023). The Dynamical Equations of a Test Particle in the Restricted Three- Body Problem with a Triaxial Primary and Variable Masses, Dutse Journal of Pure and Applied Sciences, DUJOPAS 9, 49-61

Leke, O., Singh, J., (2023). Out-of-plane equilibrium points of extra-solar planets in the central binaries PSR B1620-26 and Kepler-16 with cluster of material points and variable masses. New Astronomy, 99, 101958

Letelier, P. S. and Da Silva, T. A. (2011). Solutions to the restricted three-body problem with variable mass. *Astrophysics and Space Science*., *332*, 325–329

Luk'yanov, L. G. (1989). Particular solutions in the restricted problem of three bodies with variable masses. *Astronomical Journal of Academy of Sciences of USSR, 66, 180-187*

Mestschersky, I.V.: (1902). Ueber die Integration der Bewegungs- gleichungen im Probleme zweier Körper von ver nderli- cher Masse, Astronomische Nachrichten. 159, 229-242.

Mestschersky, I.V.: (1952) Works on the mechanics of bodies of variable mass, GITTL, Moscow, p. 205.

Radzievskii, V. V., (1950), *Astron.Zh (USSR) 27, 250.*

Radzievskii, V. V., (1953), *Astron.Zh (USSR) 30, 265.*

Ragos O, Zafiropoulos F.A.(1995). A numerical study of the influence of the study of the Poynting-Robertson effect on the equilibrium points of the photogravitational restricted three-body problem. Astronomy & Astrophysics. 300:568.

Singh, J. and Abdulkarim, A., (2014). Instability of triangular libration points in the perturbed photogravitational R3BP with Poynting-Robertson (P-R) drag. Astrophys Space Sci.351:473-482.

Singh, J. and Amuda, T. O. (2017). Effects of Poynting-Robertson (P-R) drag, Radiation, and Oblateness on motion around the $L_{4,5}$ equilibrium points in the CR3BP. Journal of Dynamical Systems and Geometric Theories (Tylor and Francis). Vol. 15 (2), 177-200.

Singh, J. and Amuda, T. O., (2014). Poynting-Robertson (P-R) drag and oblateness effects on motion around the triangular equilibrium points in the photogravitational R3BP. Astrophys Space Sci.350:119-126.

Singh, J. and Leke, O., (2010). Stability of the photogravitational restricted three-body problem with variable masses. Astrophys Space Sci. 326, 305- 314.

Singh, J. and Leke, O., (2012). Equilibrium points and stability in the restricted three- body problem with oblateness and variable masses. Astrophys Space Sci. 340: 27-41.

Singh, J. and Leke, O., (2013a). Effects of oblateness, perturbations, radiation and varying masses on the stability of equilibrium points in the restricted three-body problem. Astrophys. & Space Sci 344: 51-61.

Singh, J. and Leke, O., (2013b). Existence and stability of equilibrium points in the Robe's restricted three-body problem with variable masses. International J. of Astron. and Astrophys. 3: 113–122.

Singh, J. and Leke, O., (2013c). Robe's restricted three-body problem with variable masses and perturbing forces. ISRN Astron. and Astrophys. 2013, Article ID 910354

Singh, J. and Leke, O. (2013d) "On Robe's Circular Restricted Problem of Three Variable Mass Bodies" *Journal of Astrophysics 2013, Article ID 898794, Hindawi*

Singh, J. and Amuda, T.O., (2019). Stability of triangular equilibrium points in restricted three- body problem under effects of circumbinary disc, radiation and drag forces. Journal Astronomy and Astrophysics. 40:5.

Singh, J. and Ishwar, B., (1999). Stability of triangle points in the generalized photogravitational restricted three-body problem. Bulletin of Astronomical Society of India. 27, 415-424.

Singh, J. and Sunusi, H. (2020). Motion around triangular points in the restricted three-body problem with radiating heterogeneous primaries surrounded by a belt. Scientific reports, 10, 18861.

Szebehely, V.G. (1967): Stability of the points of equilibrium in the restricted problem. Astronomical Journal, 72, 7-9.

Taura, J, J and Leke, O, (2022). Derivation of the dynamical equations of motion of the R3BP with Variable masses and Disk. FUDMA JOURNAL OF SCIENCES 6 (4), 125-133