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Modified Variance Estimators in The Presence of Simultaneous Effects of Measurement Error and Non-Response Using Auxiliary Variable

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Abstract

The estimation of variance is crucial in recognizing the inherent diversity of natural items within any category. Survey sampling theories often assume that measurement error does not affect data collection observations, but in reality, deviations from true values lead to significant measurement errors, especially when respondents provide insufficient or no information – a situation known as nonresponse. This study investigates the simultaneous impact of measurement error and nonresponse on estimating finite population variance under simple random sampling (SRS) design, employing auxiliary variables. Proposed estimators for the study variable Y combine sample variance, ratio estimation, and exponential estimation based on various means. Through Taylor expansion, first-order mean square errors (MSEs) and approximate biases for the estimators are derived, and their efficiency is compared with existing methods. A numerical study is also conducted to confirm the efficacy of the proposed estimators.

Keywords; Non-response, Estimators, Measurement error, Efficiency

INTRODUCTION

Sampling is a critical aspect of research, influencing the accuracy of findings. Sampling theory aims to derive precise results about study variables through random samples. While assuming measurement errors don't impact observations during data collection, deviations from true values can lead to significant issues. Nonresponse occurs when respondents can't or won't provide enough information. The concept of modifying estimators using auxiliary information has been widely discussed by various authors, with Cochran (1940) and Kuk and Mak (1989) pioneering the use of auxiliary information for estimating population parameters. Many authors including Ahmed and Singh (2015), Audu *et al*. (2016a, b, c), Khalil *et al*. (2018,

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2019), Muili et al. (2018), Zahid and Shabbir (2019), Muili *et al*. (2019a), Muili *et al.* (2019b), Qureshi et al. (2019), Yadav *et al*. (2019), Sanaullah *et al*. (2020), Muil *et al*. (2020), Ishaq *et al*. (2020), Olayiwola *et al*. (2021), Yunusa *et al*. (2022) estimated the variance of population using auxiliary information.

Tariq et al. (2021) proposed variance estimators with auxiliary variables for measurement error under simple random sampling (SRS), drawing inspiration from Singh et al. (2011);

$$t_0 = \hat{s}_y^2 \quad (1.1)$$

$$t_1 = \hat{s}_y^2 \frac{S_x^2}{\hat{s}_x^2} \quad (1.2)$$

$$t_2 = \hat{s}_y^2 \exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right) \quad (1.3)$$

$$t_1^{AM} = \frac{t_0 + t_1}{2} = \frac{\hat{s}_y^2}{2\left(1 + S_x^2 \big/ \hat{s}_y^2\right)} \quad (1.4)$$

$$t_1^{GM} = \left(t_0\, t_1\right)^{1/2} = \hat{s}_y^2 \left(\frac{S_x^2}{\hat{s}_y^2}\right)^{1/2} \quad (1.5)$$

$$t_1^{AM} = \frac{2}{\left(\frac{1}{t_0} + \frac{1}{t_1}\right)} = \frac{2\hat{s}_y^2}{\left(1 + \frac{\hat{s}_y^2}{S_x^2}\right)} \quad (1.6)$$

$$MSE\left(t_1^{AM}\right) = MSE\left(t_1^{GM}\right) = MSE\left(t_1^{HM}\right) = \frac{S_y^4}{4n}\left(A_x - 4\left(A_y - (\delta - 1)\right)\right) \quad (1.7)$$

$$t_2^{AM} = \frac{t_0 + t_2}{2} = \frac{\hat{s}_y^2}{2}\left(1 + \exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right) \quad (1.8)$$

$$t_2^{GM} = \left(t_0\, t_2\right)^{1/2} = \hat{s}_y^2 \left(\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right)^{1/2} \quad (1.9)$$

$$t_2^{HM} = \frac{2}{\left(\frac{1}{t_0} + \frac{1}{t_2}\right)} = \frac{2\hat{s}_y^2}{\left(1 + \frac{1}{\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)}\right)}$$

(1.10)

$$MSE\left(t_2^{AM}\right) = MSE\left(t_2^{GM}\right) = MSE\left(t_2^{HM}\right) = \frac{S_y^4}{4n}\left(4A_y + \frac{A_x}{4} - 2(\delta - 1)\right)$$

(1.11)

$$t_3^{AM} = \frac{t_1 + t_2}{2} = \frac{\hat{s}_y^2}{2}\left(\frac{S_x^2}{\hat{s}_x^2} + \exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right)$$

(1.12)

$$t_3^{GM} = \left(t_1\, t_2\right)^{1/2} = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right)^{1/2}$$

(1.13)

$$t_3^{HM} = \frac{2}{\left(\frac{1}{t_0} + \frac{1}{t_2}\right)} = \frac{2\hat{s}_y^2}{\left(\frac{1}{\frac{S_x^2}{\hat{s}_x^2}} + \frac{1}{\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)}\right)} \quad (1.14)$$

$$MSE\left(t_3^{AM}\right) = MSE\left(t_3^{GM}\right) = MSE\left(t_3^{HM}\right) = \frac{S_y^4}{4n}\left(4A_y + \frac{9A_x}{4} - 6(\delta - 1)\right) \quad (1.15)$$

$$t_4^{AM} = \frac{t_0 + t_1 + t_2}{3} = \frac{\hat{s}_y^2}{3}\left(1 + \frac{S_x^2}{\hat{s}_x^2} + \exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right)$$

(1.16)

$$t_4^{GM} = \left(t_0\, t_1\, t_2\right)^{1/2} = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)\right)^{1/3}$$

(1.17)

$$t_4^{HM} = \frac{2}{\left(\frac{1}{t_0} + \frac{1}{t_1} + \frac{1}{t_2}\right)} = \frac{3\hat{s}_y^2}{\left(1 + \frac{1}{\frac{S_x^2}{\hat{s}_x^2}} + \frac{1}{\exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right)}\right)} \quad (1.18)$$

$$MSE\left(t_{t5}^{AM}\right) = MSE\left(t_{t5}^{GM}\right) = MSE\left(t_{t5}^{HM}\right) = \frac{S_y^4}{n}\left(A_y + \frac{A_x}{4} - (\delta - 1)\right)$$

Finally, Tariq et al. (2021) suggest that their proposed estimators performed better than the existing estimators when the data is characterized by measurement error.

MATERIALS AND METHODS

Proposed Estimators

Having studied the work of Tariq et al. (2021) the following estimators were proposed

$$T_1^{AM} = \frac{\hat{s}_y^{*2}}{2}\left(1+\frac{S_x^2}{\hat{s}_x^{*2}}\right) \quad (2.1)$$

$$T_1^{GM} = \hat{s}_y^{*2}\left(\frac{S_x^2}{\hat{s}_x^{*2}}\right)^{1/2} \quad (2.2)$$

$$T_1^{HM} = \frac{2\hat{s}_y^{*2}}{\left(1+1/\frac{S_x^2}{\hat{s}_x^{*2}}\right)} \quad (2.3)$$

$$T_2^{AM} = \frac{\hat{s}_y^{*2}}{2}\left(1+\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right) \quad (2.4)$$

$$T_2^{GM} = \hat{s}_y^{*2}\left(\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right)^{1/2} \quad (2.5)$$

$$T_3^{AM} = \frac{\hat{s}_y^{*2}}{2}\left(\frac{S_x^2}{\hat{s}_x^{*2}}+\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right) \quad (2.6)$$

$$T_2^{HM} = \frac{2\hat{s}_y^{*2}}{\left(1+1/\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right)} \quad (2.7)$$

$$T_3^{GM} = \hat{s}_y^{*2}\left(\frac{S_y^2}{\hat{s}_x^{*2}}\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right)^{1/2} \quad (2.8)$$

$$T_3^{HM} = \frac{2\hat{s}_y^{*2}}{\left(1/\frac{S_x^2}{\hat{s}_x^{*2}}+1/\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right)} \quad (2.9)$$

$$T_4^{AM} = \frac{\hat{s}_y^{*2}}{3}\left(1+\frac{S_x^2}{\hat{s}_x^{*2}}+\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right) \quad (2.10)$$

$$T_4^{GM} = \hat{s}_y^{*2}\left(\frac{S_x^2}{\hat{s}_x^{*2}}\exp\left(\frac{S_x^2-\hat{s}_x^{*2}}{S_x^2+\hat{s}_x^{*2}}\right)\right)^{1/3} \quad (2.11)$$

$$T_4^{HM} = \frac{3\hat{s}_y^{*2}}{\left(1 + 1/\frac{S_x^2}{\hat{s}_x^{*2}} + 1/\exp\left(\frac{S_x^2 - \hat{s}_x^{*2}}{S_x^2 + \hat{s}_x^{*2}}\right)\right)} \quad (2.12)$$

where

$$\hat{s}_y^{*2} = \frac{(n_1 - 1)\hat{s}_{y1}^2 + n_2\, \hat{s}_{y_{h2}}^2}{n_1 + n_2 - 1} \qquad \hat{s}_x^{*2} = \frac{(n_1 - 1)\hat{s}_{x1}^2 + n_2\, \hat{s}_{x_{h2}}^2}{n_1 + n_2 - 1}$$

With the effect of measurement error for both respondents at the initial stage and respondents after call-back;

$$\hat{s}_{y1}^2 = s_{y1}^2 - s_{u1}^2, \quad \hat{s}_{x1}^2 = s_{x1}^2 - s_{v1}^2, \quad \hat{s}_{y_{h2}}^2 = s_{y_{h2}}^2 - s_{u2}^2, \quad \hat{s}_{x_{h2}}^2 = s_{x_{h2}}^2 - s_{v2}^2, \quad s_{y1}^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(y_i - \bar{y}_1\right)^2$$

$$s_{x1}^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(x_i - \bar{x}_1\right)^2, \quad s_{y_{h2}}^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(y_i - \bar{y}_{h2}\right)^2, \quad s_{x_{h2}}^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(x_i - \bar{x}_{h2}\right)^2$$

$$\bar{Y}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} Y_i, \quad \bar{Y}_{h_2} = \frac{1}{N_{h_2}}\sum_{i=1}^{N_{h_2}} Y_{hi}, \quad \bar{y}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i, \quad \bar{y}_{h_2} = \frac{1}{h_2}\sum_{i=1}^{h_2} y_{hi}$$

Biases and Mean Squared Errors (MSEs) of the Proposed Estimators

To obtain the properties of T_i^{AM}, T_i^{GM} *and* T_i^{HM} $(i = 1, 2, 3)$ which are Biases and MSEs the following error terms are defined.

$$e_0 = \frac{\hat{s}_y^{*2} - S_y^2}{S_y^2}, \quad e_1 = \frac{\hat{s}_x^{*2} - S_x^2}{S_x^2},$$

such that $\hat{s}_y^{*2} = S_y^2(1 + e_0)$, $\hat{s}_x^{*2} = S_x^2(1 + e_1)$ respectively.

The expectation of error terms was obtained as;

$$\left.\begin{aligned}
E(e_0) &= E(e_1) = 0, \\
E(e_0^2) &= k_1\left[\gamma_{2y} + \gamma_u \frac{S_u^4}{S_y^4} + 2\left(1 + \frac{S_u^4}{S_y^4}\right)^2\right] + k_2\left[\gamma_{2y_2} + \gamma_{u_2} \frac{S_{u_2}^4}{S_{y_2}^4} + 2\left(1 + \frac{S_{u_2}^4}{S_{y_2}^4}\right)^2\right], \\
E(e_1^2) &= k_1\left[\gamma_{2x} + \gamma_v \frac{S_v^4}{S_x^4} + 2\left(1 + \frac{S_v^4}{S_x^4}\right)^2\right] + k_2\left[\gamma_{2x_2} + \gamma_{v_2} \frac{S_{v_2}^4}{S_{x_2}^4} + 2\left(1 + \frac{S_{v_2}^4}{S_{x_2}^4}\right)^2\right], \\
E(e_0 e_1) &= k_1\left[\frac{\mu_{22}(X, Y)}{S_x^2\, S_y^2}\right] + k_2\left[\frac{\mu_{22}(X, Y)_2}{S_{x_2}^2\, S_{y_2}^2}\right]
\end{aligned}\right\} \quad (2.13)$$

Let;

$$A = k_1\left[\gamma_{2y} + \gamma_u \frac{S_u^4}{S_y^4} + 2\left(1 + \frac{S_u^4}{S_y^4}\right)^2\right] + k_2\left[\gamma_{2y_2} + \gamma_{u_2} \frac{S_{u_2}^4}{S_{y_2}^4} + 2\left(1 + \frac{S_{u_2}^4}{S_{y_2}^4}\right)^2\right], \quad (2.14)$$

$$B = k_1\left[\gamma_{2x} + \gamma_v \frac{S_v^4}{S_x^4} + 2\left(1 + \frac{S_v^4}{S_x^4}\right)^2\right] + k_2\left[\gamma_{2x_2} + \gamma_{v_2} \frac{S_{v_2}^4}{S_{x_2}^4} + 2\left(1 + \frac{S_{v_2}^4}{S_{x_2}^4}\right)^2\right], \quad (2.15)$$

$$C = k_1 \frac{\mu_{22}(X,Y)}{S_x^2 S_y^2} + k_2 \frac{\mu_{22}(X,Y)_2}{S_{x_2}^2 S_{y_2}^2} \tag{2.16}$$

where

$$k_1 = \frac{1}{n} - \frac{1}{N}, \quad k_2 = \frac{N_2(k-1)}{Nn} = \frac{N_2\left(\frac{n_2}{h_2} - 1\right)}{Nn} \quad and\ k = \frac{n_2}{h_2}$$

Expressing the proposed estimators T_1^{AM}, T_1^{GM}, T_1^{HM} in term of error terms, the following are obtained.

$$T_1^{AM} = \frac{S_y^2(1+e_0)}{2}\left(2 + - e_1 + e_1^2\right) = S_y^2\left(1 - \frac{e_1}{2} + \frac{e_1^2}{2} + e_0 - e_0 e_1\right) \tag{2.17}$$

$$T_1^{GM} = S_y^2(1+e_0)\left(1 - \frac{1}{2}e_1 + \frac{3}{8}e_1^2\right) = S_y^2\left(1 - \frac{1}{2}e_1 + \frac{3}{8}e_1^2 + e_0 + e_0 e_1\right) \tag{2.18}$$

$$T_1^{HM} = S_y^2(1+e_0)\left(1 - \frac{1}{2}e_1 + \frac{1}{4}e_1^2\right) = S_y^2\left(1 - \frac{1}{2}e_1 + \frac{1}{4}e_1^2 + e_0 - \frac{1}{2}e_0 e_1\right) \tag{2.19}$$

By subtracting S_y^2 from (2.16), (2.17) and (2.18), and taking the expectation we have;

$$Bias\left(T_1^{AM}\right) = \frac{S_y^2}{2}\left(2E(e_0) - E(e_1) + E(e_1^2) - E(e_0 e_1)\right)$$

$$Bias\left(T_1^{AM}\right) = \frac{S_y^2}{2}\{B - C\} \tag{2.20}$$

$$Bias\left(T_1^{GM}\right) = \frac{S_y^2}{8}\left(8E(e_0) - 4E(e_1) + 3E(e_1^2) - 4E(e_0 e_1)\right)$$

$$Bias\left(T_1^{GM}\right) = \frac{S_y^2}{8}\{B - 4C\} \tag{2.21}$$

$$Bias\left(T_1^{HM}\right) = \frac{S_y^2}{4}\left(4E(e_0) - 2E(e_1) + E(e_1^2) - 2E(e_0 e_1)\right)$$

$$Bias\left(T_1^{HM}\right) = \frac{S_y^2}{4}\{B - 2C\} \tag{2.22}$$

By subtracting S_y^2 from (2.17), (2.18) and (2.19), and square it to first order approximation then take the expectation we obtained;

$$MSE\left(T_1^{AM}\right) = E\left(S_y^2\left(1 - \frac{e_1}{2} + \frac{e_1^2}{2} + e_0 - e_0 e_1\right) - S_y^2\right)^2$$

$$MSE\left(T_1^{AM}\right) = \frac{S_y^4}{4}\{B + 4(A - C)\} \tag{2.23}$$

$$MSE\left(T_1^{GM}\right) = E\left(S_y^2\left(1 - \frac{1}{2}e_1 + \frac{3}{8}e_1^2 + e_0 + e_0 e_1\right) - S_y^2\right)^2$$

$$MSE\left(T_1^{GM}\right) = \frac{S_y^4}{4}\{B + 4(A - C)\} \tag{2.24}$$

$$MSE\left(T_1^{HM}\right)=E\left(S_y^2\left(1-\frac{1}{2}e_1+\frac{1}{4}e_1^2+e_0-\frac{1}{2}e_0e_1\right)-S_y^2\right)^2$$

$$MSE\left(T_1^{HM}\right)=\frac{S_y^4}{4}\left\{B+4\left(A-C\right)\right\} \tag{2.25}$$

This implies that $MSE\left(T_1^{AM}\right)=MSE\left(T_1^{GM}\right)=MSE\left(T_1^{HM}\right)$

Also, by expressing T_2^{AM}, T_2^{GM} *and* T_2^{HM} in terms of e_0 *and* e_1;

$$T_2^{AM}=\frac{S_y^2\left(1+e_0\right)}{2}\left(1+\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)=\frac{S_y^2\left(1+e_0\right)}{2}\left(1+\exp\left(\frac{-e_1}{2+e_1}\right)\right) \tag{2.26}$$

$$T_2^{GM}=S_y^2\left(1+e_0\right)\left(1+\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)^{\frac{1}{2}}=S_y^2\left(1-\frac{1}{4}e_1+\frac{5}{32}e_1^2+e_0-\frac{1}{4}e_0e_1\right) \tag{2.27}$$

$$T_2^{HM}=\frac{2S_y^2\left(1+e_0\right)}{\left(1+1/\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)}=S_y^2\left(1-\frac{1}{4}e_1+\frac{3}{16}e_1^2+e_0-\frac{1}{4}e_0e_1\right) \tag{2.28}$$

By subtracting S_y^2 from (2.26), (2.27) and (2.28), and taking the expectation we have;

$$Bias\left(T_2^{AM}\right)=\frac{S_y^2}{16}\left\{B-8C\right\} \tag{2.29}$$

$$Bias\left(T_2^{GM}\right)=\frac{S_y^2}{4}\left\{\frac{5}{8}B-C\right\} \tag{2.30}$$

$$Bias\left(T_2^{HM}\right)=\frac{S_y^2}{4}\left\{\frac{3}{4}B-C\right\} \tag{2.31}$$

Also, by subtracting S_y^2 from (2.26), (2.27) and (2.28), and squaring it to first order approximation then take the expectation we obtained;

$$MSE\left(T_2^{AM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{1}{4}B-2C\right\} \tag{2.32}$$

$$MSE\left(T_2^{GM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{1}{4}B-2C\right\} \tag{2.33}$$

$$MSE\left(T_2^{HM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{1}{4}B-2C\right\} \tag{2.34}$$

This also implies that $MSE\left(T_2^{AM}\right)=MSE\left(T_2^{GM}\right)=MSE\left(T_2^{HM}\right)$.

Also, by expressing T_3^{AM}, T_3^{GM} *and* T_3^{HM} in terms of e_0 *and* e_1;

$$T_3^{AM}=\frac{S_y^2\left(1+e_0\right)}{2}\left(\frac{S_x^2}{S_x^2\left(1+e_1\right)}+\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)=S_y^2\left(1-\frac{3}{4}e_1+\frac{11}{16}e_1^2+e_0-\frac{3}{4}e_0e_1\right) \tag{2.35}$$

$$T_3^{GM} = S_y^2\left(1+e_0\right)\left(\frac{S_y^2}{S_x^2\left(1+e_1\right)}\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)^{1/2} = S_y^2\left(1-\frac{3}{4}e_1+\frac{21}{32}e_1^2+e_0-\frac{3}{4}e_0e_1\right) \quad (2.36)$$

$$T_3^{HM} = \frac{2S_y^2\left(1+e_0\right)}{\left(1/\left(\frac{S_x^2}{S_x^2\left(1+e_1\right)}\right)+1/\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)} = S_y^2\left(1-\frac{3}{4}e_1+\frac{5}{8}e_1^2+e_0-\frac{3}{4}e_1\right) \quad (2.37)$$

By subtracting S_y^2 from (2.35), (2.36) and (2.37), and taking the expectation we have;

$$Bias\left(T_3^{AM}\right)=\frac{S_y^2}{8}\left(\frac{11}{2}B-6C\right) \quad (2.38)$$

$$Bias\left(T_3^{GM}\right)=\frac{S_y^2}{4}\left\{\frac{21}{8}B-3C\right\} \quad (2.39)$$

$$Bias\left(T_3^{HM}\right)=\frac{S_y^2}{4}\left\{\frac{5}{2}B-3C\right\} \quad (2.40)$$

Also, by subtracting S_y^2 from (2.35), (2.36) and (2.37), and squaring it to first order approximation then take the expectation we obtained;

$$MSE\left(T_3^{AM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{9}{4}B-6C\right\} \quad (2.41)$$

$$MSE\left(T_3^{AM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{9}{4}B-6C\right\} \quad (2.42)$$

$$MSE\left(T_3^{HM}\right)=\frac{S_y^4}{4}\left\{4A+\frac{9}{4}B-6C\right\} \quad (2.43)$$

This also implies that $MSE\left(T_3^{AM}\right)=MSE\left(T_3^{GM}\right)=MSE\left(T_3^{HM}\right)$

Also, by expressing T_3^{AM}, T_3^{GM} *and* T_3^{HM} in terms of e_0 *and* e_1;

$$T_4^{AM} = \frac{S_y^2\left(1+e_0\right)}{3}\left(1+\frac{S_x^2}{S_x^2\left(1+e_1\right)}+\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right) = S_y^2\left(1-\frac{1}{2}e_1+\frac{11}{24}e_1^2+e_0-\frac{1}{2}e_0e_1\right) \quad (2.44)$$

$$T_4^{GM} = S_y^2\left(1+e_0\right)\left(\frac{S_x^2}{S_x^2\left(1+e_1\right)}\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)^{1/3} = S_y^2\left(1-\frac{1}{2}e_1+\frac{3}{8}e_1^2+e_0-\frac{1}{2}e_0e_1\right) \quad (2.45)$$

$$T_4^{HM} = \frac{3S_y^2\left(1+e_0\right)}{\left(1+1/\frac{S_x^2}{S_x^2\left(1+e_1\right)}+1/\exp\left(\frac{S_x^2-S_x^2\left(1+e_1\right)}{S_x^2+S_x^2\left(1+e_1\right)}\right)\right)} = S_y^2\left(1-\frac{1}{2}e_1+\frac{7}{24}e_1^2+e_0-\frac{1}{2}e_0e_1\right) \quad (2.46)$$

By subtracting S_y^2 from (2.44), (2.45) and (2.46), and taking the expectation we have;

$$Bias\left(T_4^{AM}\right)=\frac{S_y^2}{2}\left\{\frac{11}{12}B-C\right\} \quad (2.47)$$

$$Bias\left(T_4^{GM}\right) = \frac{1}{2}S_y^2 \left\{\frac{3}{4}B - C\right\} \quad (2.48)$$

$$Bias\left(T_4^{HM}\right) = \frac{S_y^2}{2}\left\{\frac{7}{12}B - C\right\} \quad (2.49)$$

Also, by subtracting S_y^2 from (2.44), (2.45) and (2.46), and squaring it to first order approximation then take the expectation we obtained;

$$MSE\left(T_4^{AM}\right) = S_y^4\left\{A + \frac{1}{4}B - C\right\} \quad (2.50)$$

$$MSE\left(T_4^{GM}\right) = S_y^4\left\{A + \frac{1}{4}B - C\right\} \quad (2.51)$$

$$MSE\left(T_4^{HM}\right) = S_y^4\left\{A + \frac{1}{4}B - C\right\} \quad (2.52)$$

This also implies that $MSE\left(T_4^{AM}\right) = MSE\left(T_4^{GM}\right) = MSE\left(T_4^{HM}\right)$

Efficiency Comparison

In this section, the efficiency conditions of the proposed estimators over some existing estimators considered in this study were established.

Comparing sample, ratio, and exponential variance estimators with proposed estimators; $MSE\left(t_{i\bullet}\right)$, $i\bullet = 0, 1, 2.$ with $MSE\left(T_i^J\right)$, where $J = AM, GM, HM$ .and $i = 1, 2, 3, 4$

$$MSE\left(T_1^J\right) - MSE\left(t_0\right) < 0$$

$$S_y^4\{A + B - 2C\} - S_y^4 A < 0 \qquad if\ B < 4C \quad (3.1)$$

$$MSE\left(T_2^J\right) - MSE\left(t_0\right) < 0$$

$$\frac{S_y^4}{4}\left\{4A + \frac{1}{4}B - 2C\right\} - S_y^4 A < 0 \qquad if\ B < 8C \quad (3.2)$$

$$MSE\left(T_3^J\right) - MSE\left(t_0\right) < 0$$

$$\frac{S_y^4}{4}\left\{4A + \frac{1}{4}B - 2C\right\} - S_y^4 A < 0 \qquad if\ B < \frac{8}{3}C \quad (3.3)$$

$$MSE\left(T_4^J\right) - MSE\left(t_0\right) < 0$$

$$\frac{S_y^4}{4}\left\{4A + \frac{1}{4}B - 2C\right\} - S_y^4 A < 0 \qquad if\ B < 4C \quad (3.4)$$

$$MSE\left(T_1^J\right) - MSE\left(t_1\right) < 0$$

$$S_y^4\{A + B - 2C\} - S_y^4\{A + B - 2C\} < 0 \quad if\ B > \frac{4}{3}C \quad (3.5)$$

$$MSE\left(T_2^J\right) - MSE\left(t_1\right) < 0$$

$$\frac{S_y^4}{4}\left\{4A + \frac{1}{4}B - 2C\right\} - S_y^4\{A + B - 2C\} < 0 \qquad if\ B > \frac{8}{5}C \quad (3.6)$$

$$MSE\left(T_3^J\right)-MSE\left(t_1\right)<0$$
$$\frac{S_y^4}{4}\left\{4A+\frac{1}{4}B-2C\right\}-S_y^4\left\{A+B-2C\right\}<0 \quad if\ B>\frac{8}{7}C \tag{3.7}$$
$$MSE\left(T_4^J\right)-MSE\left(t_1\right)<0$$
$$\frac{S_y^4}{4}\left\{4A+\frac{1}{4}B-2C\right\}-S_y^4\left\{A+B-2C\right\}<0 \quad if\ B>\frac{4}{3}C \tag{3.8}$$
$$MSE\left(T_1^J\right)-MSE\left(t_2\right)<0$$
$$S_y^4\left\{A+B-2C\right\}-\frac{S_y^4}{4}\left\{B+4\left(A-C\right)\right\}<0 \quad if\ B>\frac{8}{3}C \tag{3.9}$$

Empirical Study

In this section, empirical studies were conducted to assess the performance of proposed estimators over some existing estimators mentioned in the literature. The data was simulated using R Software by generating different samples of size 50, 100, and 200 were selected using simple random sampling without replacement from the population of size N = 1,000.

Table 1: Population Used for Simulation Study

Auxiliary variable	Study variable
$z_1 \sim Beta\left(1,3\right), z_2 \sim Gamma\left(5,7\right)$ $X=z_1\rho_{z_1z_2}+z_2\sqrt{1-\rho_{z_1z_2}^2}$	$Y=0.2X+0.5X^2+e$ $e\sim N(0,1)$

RESULTS AND DISCUSSION

Table 2: Biases, MSEs and PREs of Proposed and Existing Estimators for *n = 50*.

$n=50,\quad n_1=40,\quad n_2=10,\quad h_2=8$				
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	3.50727167	13.45363145	100.00000
	Ratio t_1	-0.39133991	0.15367127	8754.81266
	Exponential t_2	1.05007510	1.26445052	1063.99035
Proposed Estimators	T_1^{AM}	3.54202247	13.70891038	98.13786
	T_1^{GM}	0.05777009	0.02034883	66114.99766
	T_1^{HM}	-0.32448606	0.10725141	12544.01322
	T_2^{AM}	2.27867338	5.73673820	234.51709
	T_2^{GM}	1.98768842	4.38263133	306.97612
	T_2^{HM}	1.72762492	3.32710828	404.36410
	T_3^{AM}	0.32936759	0.15112749	8902.17383
	T_3^{GM}	-0.14054664	0.02630606	51142.71704

	T_3^{HM}	-0.32851267	0.10969460	12264.62462
	T_4^{AM}	1.38866895	2.17358888	618.95934
	T_4^{GM}	0.27811869	0.11059486	12164.78860
	T_4^{HM}	-0.26591462	0.07446513	18067.02186
$n = 50$, $n_1 = 30$, $n_2 = 20$, $h_2 = 15$				
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	2.63413328	7.89121270	100.00000
	Ratio t_1	-0.42552306	0.18123845	4354.04999
	Exponential t_2	0.70372046	0.62821997	1256.12256
Proposed Estimators	T_1^{AM}	2.65179251	7.99197448	98.73921
	T_1^{GM}	-0.13385340	0.02767950	28509.23013
	T_1^{HM}	-0.39107774	0.15358992	5137.84549
	T_2^{AM}	1.66892687	3.23463038	243.96026
	T_2^{GM}	1.43764371	2.42271419	325.71785
	T_2^{HM}	1.23148103	1.79845662	438.77693
	T_3^{AM}	0.13909870	0.05405839	14597.57274
	T_3^{GM}	-0.26016005	0.07137759	11055.58902
	T_3^{HM}	-0.39246389	0.15463715	5103.05102
	T_4^{AM}	0.97077689	1.14439053	689.55593
	T_4^{GM}	0.03743289	0.02344332	33660.82243
	T_4^{HM}	-0.35949510	0.13055761	6044.23822

Table 3: Biases, MSEs and PREs of Proposed and Existing Estimators for *n = 100*.

$n = 100$, $n_1 = 80$, $n_2 = 20$, $h_2 = 15$				
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	3.46775198	12.60875815	100.00000
	Ratio t_1	-0.39358503	0.15514205	8127.23464
	Exponential t_2	1.03388633	1.15094178	1095.51659
Proposed Estimators	T_1^{AM}	3.50138022	12.84859331	98.13337
	T_1^{GM}	0.05017056	0.01102441	114371.31066
	T_1^{HM}	-0.32870024	0.10892320	11575.82445
	T_2^{AM}	2.25081915	5.34186920	236.03644
	T_2^{GM}	1.96239991	4.06971955	309.81885
	T_2^{HM}	1.70466353	3.07939993	409.45504
	T_3^{AM}	0.32015065	0.12414197	10156.72502
	T_3^{GM}	-0.14551274	0.02443606	51598.98731
	T_3^{HM}	-0.33236173	0.11126992	11331.68646

	T_4^{AM}	1.36935109	1.99935854	630.64017
	T_4^{GM}	0.26915385	0.08933189	14114.51005
	T_4^{HM}	-0.27135662	0.07535302	16732.91741
$n = 100,$ $n_1 = 60,$ $n_2 = 40,$ $h_2 = 30$				
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	2.64721033	7.49206269	100.00000
	Ratio t_1	-0.42647650	0.18194787	4117.69728
Proposed Estimators	Exponential t_2	0.70783008	0.56851435	1317.83176
	T_1^{AM}	2.66439283	7.58686862	98.75039
	T_1^{GM}	-0.13555344	0.02289350	32725.71382
	T_1^{HM}	-0.39289218	0.15461838	4845.51871
	T_2^{AM}	1.67752020	3.04241719	246.25363
	T_2^{GM}	1.44501134	2.26884534	330.21478
	T_2^{HM}	1.23778529	1.67525249	447.21991
	T_3^{AM}	0.14067679	0.03735126	20058.39398
	T_3^{GM}	-0.26133382	0.06999776	10703.28951
	T_3^{HM}	-0.39414438	0.15559088	4815.23269
	T_4^{AM}	0.97618797	1.05551769	709.79982
	T_4^{GM}	0.03689994	0.01200851	62389.59509
	T_4^{HM}	-0.36189641	0.13149468	5697.61652

Table 4: Biases, MSEs and PREs of Proposed and Existing Estimators for *n* = *200*.

$n = 200,$ $n_1 = 150,$ $n_2 = 50,$ $h_2 = 38$				
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	3.336806616	11.409953567	100.00000
	Ratio t_1	-0.405038318	0.164132168	6951.68638
	Exponential t_2	0.977289739	0.993651203	1148.28559
Proposed Estimators	T_1^{AM}	3.364708214	11.599183607	98.36859
	T_1^{GM}	-0.001755572	0.003392118	336366.66081
	T_1^{HM}	-0.350887023	0.123412032	9245.41421
	T_2^{AM}	2.157048178	4.782941934	238.55513
	T_2^{GM}	1.876141835	3.622988091	314.93213
	T_2^{HM}	1.625379427	2.723544561	418.93765
	T_3^{AM}	0.286125710	0.091963773	12407.00897
	T_3^{GM}	-0.178285760	0.033076029	34496.14059
	T_3^{HM}	-0.353486392	0.125222204	9111.76552
	T_4^{AM}	1.303019346	1.756373181	649.63151

	T_4^{GM}	0.210530177	0.051465279	22170.19673
	T_4^{HM}	-0.302099003	0.091843817	12423.21360
$n = 200,$	$n_1 = 130,$	$n_2 = 70,$	$h_2 = 53$	
	Estimators	**Bias**	**MSEs**	**PREs**
Exiting Estimators	Sample t_0	2.92623586	8.80993643	100.00000
	Ratio t_1	-0.42062531	0.17696421	4978.37197
Proposed Estimators	Exponential t_2	0.81478259	0.69833004	1261.57202
	T_1^{AM}	2.94634396	8.92987944	98.65684
	T_1^{GM}	-0.09259045	0.01102648	79898.00402
	T_1^{HM}	-0.38136972	0.14559183	6051.12025
	T_2^{AM}	1.87050923	3.61531695	243.68365
	T_2^{GM}	1.61777027	2.70944587	325.15639
	T_2^{HM}	1.39243138	2.01193173	437.88446
	T_3^{AM}	0.19707864	0.04781838	18423.74547
	T_3^{GM}	-0.23482628	0.05606907	15712.64915
	T_3^{HM}	-0.38290744	0.14675909	6002.99214
	T_4^{AM}	1.10679771	1.27735638	689.70074
	T_4^{GM}	0.09609941	0.01488222	59197.73827
	T_4^{HM}	-0.34529224	0.11953367	7370.25506

The outcomes presented in the tables demonstrate that the suggested estimators exhibit lower Mean Squared Errors (MSEs) and higher Percentage Relative Efficiencies (PREs), surpassing the performance of existing estimators. This superiority is particularly evident when dealing with data affected by both measurement error and nonresponse concurrently. Estimators with minimal MSEs and maximal PREs are deemed more potent and efficient compared to their counterparts.

CONCLUSION

In this paper, the problem of estimation of finite population variance in the presence of measurement error and nonresponse was considered simultaneously. The estimators proposed were a combination of the mean of sample variance estimator, ratio and exponential variance estimators arithmetically, geometrically and harmonically by following Tarq et al. (2021) under SRS design. The efficiency conditions are also obtained in which the proposed estimators may perform better than the existing estimators. The results obtained for MSEs and PREs of the simulation study show that the proposed estimators performed better than other estimators available in the literature when the population is characterized with measurement error and nonresponse simultaneously. The PREs of the suggested estimators increase and the ARBs decrease by increases in the sample sizes. Based on the numerical findings, we recommend the proposed estimators for the estimation of population variance.

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