https://dx.doi.org/10.4314/dujopas.v9i4b.13
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 9 No. 4b December 2023*

A Study on an Economic Order Quantity Model Designated for Non-Instant Decaying Goods with Three-Stage Demand Rates, Linear Holding Cost and Linear Reciprocal Partial Backlogging Rate

Zulkifilu Muazu*[1], Babangida Bature[2], Yusuf Jibril[3], Mustapha Lawal Malumfashi[4]

[1,2,4]Department of Mathematics and Statistics,
Faculty of Natural and Applied Sciences,
Umaru Musa Yar'adua University Katsina,
Katsina State Nigeria
.
[3]Department of Mathematics and Statistics,
Federal Polytechnic Nasarawa,
Nasarawa State, Nigeria

Email: muazu.zulkifilu@umyu.edu.ng

Abstract

This study examines economic order quantity model for non-instant decaying goods with three-stage demand rates, linear holding cost and linear reciprocal partial backlogging amount. The average annual demand rates before goods start decaying, after goods start decaying and during stockouts are not the same and both taken as constant. Stockouts are permitted and incompletely backlogged, and the backlogging amount is flexible and varies on the waiting time for the subsequent top up. The model determined the best time with positive inventory, cycle length and order quantity that reduce entire flexible price. The essential and satisfactory situations for the occurrence and exclusivity of the best solutions are found. Arithmetical example was given to elucidate the theoretical outcomes of the model. Sensitivity scrutiny of some model constraints on best solutions is carried out and recommendations to reducing the entire variable cost of the inventory structure were similarly given.

Keywords: Non-instant decaying, three-stage demand rates, linear holding cost, linear reciprocal partial backlogging amount.

INTRODUCTION

In order to ensure that the right amount of inventory is demanded so that the trader do not have to make order too frequently and there is no excess of inventory sitting at hand, an economic order quantity (EOQ) model that established that right amount to demand such that the entire variable cost has a minimum value is developed by Harris (1913), and expected that the demand amount is constant, goods have infinite shelf lives and no stock out condition. Nevertheless, there is depletion of the inventory attributable to decaying in some cases. Henceforth, decaying plays significant function in several inventory structures and its consequences cannot be overlooked. An inventory model for stylish goods decaying at the end of the prescribed storing time was first studied by Whitin (1957). Ghare and Schrader (1963) presented a revised form of the EOQ model for exponentially decaying goods with a constant amount of decaying, where the consumption amount of decaying goods is expected to be

**Author for Correspondence*

closely interconnected to a negative exponential function of time. Covert and Philip (1974) extended Ghare and Schrader's (1963) model to develop an EOQ model for instantaneous decaying goods, where the amount of decaying follows two-constraint Weibull distribution. Philip (1974) extended Covert and Philip's (1974) model to develop an inventory model for instantaneous decaying goods, where decaying amount follows three-constraint Weibull distribution and stock out are not permitted to occur. Additionally, some interconnected studies on inventory models with the postulation that the decaying starts immediately the goods are received can be found in Tavakoli and Taleizadeh (2017), Chen (2018) and so on.

The postulation that the decaying starts from the instant of arrival of goods, for example, stylish goods, microelectronics, cars, rice, beans, yam, maize and so on, in store is unsuitable for these types of goods in developing inventory strategies. Thus, unaware of feature of these types of goods might cause businesses to overrate the entire relevant inventory cost, causing in unsuitable decision-making. Ouyang *et al.* (2006) studied an inventory model for Non-instant decaying goods with permissible delay in payment. Additionally, some interconnected studies on inventory models for Non-instant decaying goods under various postulations can be found in Chung (2009), Geetha and Udayakumar (2016), Babangida and Baraya (2018, 2019a, 2019b, 2020, 2021a, 2021b, 2022), Babangida *et al*. (2023) and so on.

A lot of the researches on inventory models studied holding cost to be constant. Nevertheless, genuinely, the holding cost of several goods might be in a dynamic state as the time value of money and price index changes. The cost of storing decaying and fresh goods as soon as additional storing capabilities and services are required might continuously be extreme. The holding cost for some goods detained in store is a linear function of time over which goods are stored. Frequently, the cost of holding goods, for example, fruits, vegetables, fish, meat, milk etc., in the store is sophisticated as soon as better-preserving capabilities are used to preserve the newness and avoid putrefaction, and subsequently this lesser decaying amount. Singhal and Singh (2018) studied an integrated top up model for decaying goods with multiple market demand rates under volume flexibility, where the decaying amount varies on the quality level and time and follows a two-constraint Weibull distribution. Holding cost is expected a linearly growing function of time. Additionally, some interconnected studies on inventory models with time-varying holding costs can be found in Mishra and Singh (2011), Tyagi *et al.* (2014), Tayal *et al.* (2015) and so on.

In the orthodox inventory model, stock out are not permitted. Nevertheless, occasionally clients' demands cannot be realised by the seller from the existing stores, this circumstance is known as stock out or shortage condition. Genuinely, stock out is inevitable attributable to countless uncertainties. Deb and Chaudhuri (1987) developed a heuristic method for top up of trended inventories considering stock out. According to Sharma (2003), allowing stock out to occur surges cycle length, spread the ordering cost over a long time and subsequently reducing the entire variable cost. Choudhury *et al.* (2013) established an EOQ model for non-instant decaying goods with stock-reliant demand amount and time-varying holding cost over a finite and infinite time horizon. Stock out are permitted and wholly backlogged.

Nevertheless, as soon as stock out occur, one cannot be sure that all clients are ready to hang on for a backorder attributable to clients' intolerant and dynamic nature of human beings. As soon as stock out occur, some clients whose necessities are not acute at that time might hang on for the back-orders to be realised, while others might opt to buy from other sellers. Subsequently, the opportunity cost attributable to lost sales should be studied. Bello and

Baraya (2018) developed an inventory model for Non-instant decaying item with two-phase demand rates and constant partial backlogging amount.

For the majority of goods, for example, stylish goods, microelectronics, cars and its spare parts, rice, beans, yam, maize, seasonal goods and so on, the extent of the waiting time for the subsequent top up would regulate whether or not the backlogging will be accepted. Consequently, the backlogging amount should be flexible and depend on the waiting time for the subsequent top up. That is, the longer the waiting time, the lesser the backlogging amount will be and vice versa. Additionally, some interconnected studies on inventory models with time-reliant incompletely backlogged stock out can be found in Geetha and Uthayakumar (2010), Babangida *et al.* (2023) and so on.

This research studied EOQ model for non-instant decaying goods with three-stage demand rates, linear holding cost and linear reciprocal partial backlogging amount. The average annual demand rates before goods start decaying, after goods start decaying and during stockouts are not the same and both studied as constant. Stock out are permitted and incompletely backlogged, and the backlogging amount is flexible and varies on the waiting time for the subsequent top up. The model established the best time with positive inventory, cycle length and order quantity that reduce entire variable cost. The essential and satisfactory circumstances for the occurrence and exclusivity of the best solutions are founded. Arithmetical example was given to elucidate the hypothetical outcomes of the model. Sensitivity scrutiny of some model constraints on best solutions is carried out and recommendations to reducing the entire variable cost of the inventory structure were given.

MATERIAL AND METHOD

Notation and assumptions

The inventory structure is settled based on the subsequent notation and postulations.

Notations

O	The ordering cost per order.
C_p	The buying price.
C_b	Stock out cost per unit of time.
C_π	The unit cost of lost sales per unit.
ω	The decaying rates function ($0 < \omega < 1$).
x_d	The length of time in which the goods exhibit no decaying.
x_1	Length of time in which the inventory has no stock out.
X	The length of the replenishment cycle time (time unit).
Q_m	The maximum inventory level.
B_m	The backorder level during the stock out time.
Q	The demand amount during the cycle length, i.e., $Q = (Q_m + B_m)$.

Assumptions

The model was developed under the subsequent postulations.

1. The top up amount is infinite.
2. The lead time is zero.
3. During the fixed time, x_d, there is no decaying and at the end of this time, the goods deteriorate at the amount ω.
4. There is no replacement or repair for decayed goods during the time under contemplation.
5. The average demand rates before decaying begins, after decaying sets in and during stock out are respectively given by α, β and γ.

6. Holding cost $C_1(x)$ per unit time is linear and is expected to be $C_1(x) = h_1 + h_2 x$; where $h_1 > 0$ and $h_2 > 0$.
7. Stockouts are permitted and incompletely backlogged during the stock out time, the backlogging amount is flexible and is reliant on the extent of the waiting time for subsequent top up i.e. the longer the waiting time is, the smaller the backlogging amount will be. The backlogging amount for negative inventory is given by $B(x) = \frac{1}{1+\sigma(X-x)}$, σ is backlogging constraint ($0 < \sigma < 1$)and $(X - x)$ is waiting time $(x_1 \leq x \leq X)$, $1 - B(x)$ is the remaining fraction lost

Formulation of the model

At the commencement of each top up cycle (i.e., at time $x = 0$), Q_m units of a single goods from the producer arrives. During the time interval $[0, x_d]$, the inventory level is depleting progressively attributable to market demand only and it is expected to be α. At time interval $[x_d, x_1]$, the inventory level is depleting attributable to combined consequences of demand from the clients and decaying and the demand amount at time is given by β. At time $x = x_1$, the inventory level depletes to zero. Stockouts occur at the time $x = x_1$ and are incompletely backlogged at the amount σ and the demand amount during time is given by γ. The nature of the inventory structure is explained in figure beneath

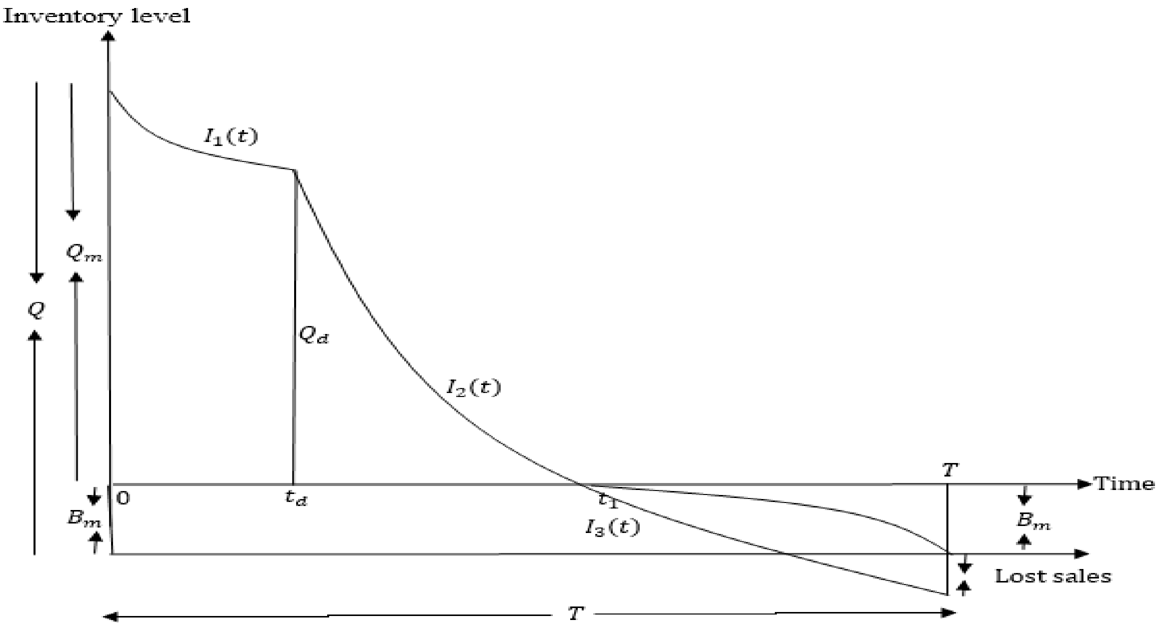

Figure 1 Graphical depiction of the Inventory

Based on the explanation in Figure 1, during the time interval $[0, X]$, the change of inventory at any time x is characterised by the subsequent differential equations

$$\frac{dI_1(x)}{dx} = -\alpha, \qquad 0 \leq x \leq x_d \qquad (1)$$

$$\frac{dI_2(x)}{dx} + \omega I_2(x) = -\beta, \qquad x_d \leq x \leq x_1 \qquad (2)$$

$$\frac{dI_3(x)}{dx} = -\frac{\gamma}{1 + \sigma(X - x)}, \qquad x_1 \leq x \leq X \qquad (3)$$

with boundary conditions $I_1(0) = Q_m$, $I_1(x_d) = I_2(x_d) = Q_d$ and $I_2(x_1) = I_3(x_1) = 0$.

The solution of equations (1), (2) and (3) are given by

$$I_1(x) = \frac{\beta}{\omega}\left(e^{\omega(x_1 - x_d)} - 1\right) + \alpha(x_d - x), \qquad 0 \leq x \leq x_d \qquad (4)$$

$$I_2(x) = \frac{\beta}{\omega}\left(e^{\omega(x_1-x)} - 1\right), \qquad x_d \le x \le x_1 \tag{5}$$

and

$$I_3(x) = -\frac{\gamma}{\sigma}\left[ln[1+\sigma(X-x_1)] - ln[1+\sigma(X-x)]\right] \qquad x_1 \le x \le X \tag{6}$$

From Figure1, Exploiting the condition $I_1(0) = Q_m$ in equation (4), the maximum inventory level is given by

$$Q_m = \frac{\beta}{\omega}\left(e^{\omega(x_1-x_d)} - 1\right) + \alpha x_d \tag{7}$$

Additionally, the value of Q_d can be derived at $x = x_d$, and then from equation (5), it follows that

$$Q_d = \frac{\beta}{\omega}\left(e^{\omega(x_1-x_d)} - 1\right) \tag{8}$$

The maximum backordered units B_m is gotten at $x = X$, and then from equation (6), it follows that

$$B_m = -I_3(X) = \frac{\gamma}{\sigma}\left[ln[1+\sigma(X-x_1)]\right] \tag{9}$$

Thus the demand size during entire time interval $[0,\ X]$ is

$$Q = Q_m + B_m = \frac{\beta}{\omega}\left(e^{\omega(x_1-x_d)} - 1\right) + \alpha x_d + \frac{\gamma}{\sigma}\left[ln[1+\sigma(X-x_1)]\right] \tag{10}$$

The cost of backorder during the interval $[x_1,\ X]$ is given by

$$SC = C_b\int_{x_1}^{X} -I_3(x)dx = C_b\int_{x_1}^{X}\frac{\gamma}{\delta}[\ln[1+\delta(X-x_1)] - \ln[1+\delta(X-x)]\,dx$$

$$= \frac{C_b\gamma}{\delta}\left((X-x_1) - \frac{ln\left(1+\delta(X-x_1)\right)}{\delta}\right). \tag{11}$$

The opportunity cost per cycle attributable to lost sales is given by

$$LC = C_\pi\gamma\int_{x_1}^{X}\left(1 - \frac{\lambda}{1+\sigma(X-x)}\right)dx = C_\pi\gamma\left[(X-x_1) - \frac{ln\left(1+\sigma(X-x_1)\right)}{\sigma}\right]. \tag{12}$$

The entire variable cost per unit time is given by

$Z(x_1,X) = \frac{1}{X}${Ordering cost + Inventory holding cost + Decaying cost + Back-ordered cost + Lost sales cost}

$$= \frac{1}{X}\Bigg\{O + h_1\left[\frac{\beta x_d}{\omega}e^{\omega(x_1-x_d)} + \frac{\alpha}{2}x_d^2 + \frac{\beta}{\omega^2}e^{\omega(x_1-x_d)} - \frac{\beta}{\omega^2} - \frac{\beta x_1}{\omega}\right]$$

$$+ h_2\left[\frac{\beta x_d^2}{2\omega}e^{\omega(x_1-x_d)} + \frac{\alpha}{6}x_d^3 + \frac{\beta x_d}{\omega^2}e^{\omega(x_1-x_d)} - \frac{\beta x_1}{\omega^2} - \frac{\beta}{\omega^3} + \frac{\beta}{\omega^3}e^{\omega(x_1-x_d)} - \frac{\beta x_1^2}{2\omega}\right]$$

$$+ C_p\frac{\beta}{\omega}\left[e^{\omega(x_1-x_d)} - 1 - \omega(x_1-x_d)\right]$$

$$+ \left(C_\pi\gamma + \frac{\gamma C_b}{\sigma}\right)\left[(X-x_1) - \frac{ln\left(1+\sigma(X-x_1)\right)}{\sigma}\right]\Bigg\}. \tag{13}$$

Exploiting the famous estimates $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and $ln(1+x) = x - \frac{x^2}{2} + \cdots$, when $-1 < x \le 1$, in equation (13) yields

$$Z(x_1, X) = \frac{1}{X}\left\{\frac{1}{2}L_1x_1^2 - L_2x_1 + L_3 + \frac{\gamma(C_b + C_\pi\sigma)}{2}X^2 - \gamma(C_b + C_\pi\sigma)x_1X\right\}, \tag{14}$$

where

$L_1 = \beta\left[h_1(x_d\omega + 1) + h_2\left(\frac{x_d\omega}{2} + 1\right)x_d + C\omega\right] + \gamma[(C_b + C_\pi\sigma)], L_2 = \beta\left[h_1x_d^2\omega + \frac{h_2}{2}(1 + x_d\omega)x_d^2 + C_px_d\omega\right]$ and

$$L_3 = \left[O + h_1\left(\frac{\alpha}{2}x_d^2 - \frac{\beta x_d^2}{2} + \frac{\beta x_d^3\omega}{2}\right) + h_2\left(\frac{\alpha}{6}x_d^3 + \frac{\beta x_d^4\omega}{4}\right) + \frac{C_p\beta\omega x_d^2}{2}\right].$$

Optimal decision

This unit establishes the best ordering strategies that reduce the entire variable cost per unit time. The essential and satisfactory circumstances for the occurrence and exclusivity of best solutions are founded. The essential circumstances for the entire variable cost per unit time $Z(x_1, X)$ to be minimum are $\frac{\partial Z(x_1,X)}{\partial x_1} = 0$ and $\frac{\partial Z(x_1,X)}{\partial X} = 0$. The value of (x_1, X) gotten from $\frac{\partial Z(x_1,X)}{\partial x_1} = 0$ and $\frac{\partial Z(x_1,X)}{\partial X} = 0$ and for which the satisfactory condition $\left\{\left(\frac{\partial^2 Z(x_1,X)}{\partial x_1^2}\right)\left(\frac{\partial^2 Z(x_1,X)}{\partial X^2}\right) - \left(\frac{\partial^2 Z(x_1,X)}{\partial x_1\,\partial X}\right)^2\right\} > 0$ is satisfied gives a minimum value for the entire variable cost per unit time $Z(x_1, X)$.

The essential situations for the entire variable cost $Z_1(x_1, X)$ in equation (14) to be the minimum are $\frac{\partial Z(x_1,X)}{\partial x_1} = 0$ and $\frac{\partial Z(x_1,X)}{\partial X} = 0$, which give

$$\frac{\partial Z(x_1,X)}{\partial x_1} = \frac{\lambda}{X}\{L_1x_1 - L_2 - \gamma(C_b + C_\pi\sigma)X\}.$$

Setting $\frac{\partial Z(x_1,X)}{\partial x_1} = 0$ gives

$$L_1x_1 - L_2 - \gamma(C_b + C_\pi\sigma)X = 0 \qquad (15)$$

and

$$X = \frac{1}{\gamma(C_b + C_\pi\sigma)}(L_1x_1 - L_2). \qquad (16)$$

Note that $L_1x_1 - L_2 = \beta\left[h_1(x_d\omega(x_1 - x_d) + x_1) + \frac{h_2x_d\omega}{2}(x_1 - x_d)x_d + h_2\left(x_1 - \frac{x_d}{2}\right)x_d + C_p\omega(x_1 - x_d) + \gamma(C_b + C_\pi\sigma)x_1\right] > 0$, since $(x_1 - x_d) > 0$.

Similarly,

$$\frac{\partial Z(x_1,X)}{\partial X} = -\frac{1}{X^2}\left\{\frac{1}{2}L_1x_1^2 - L_2x_1 + L_3 - \frac{X^2}{2}\gamma(C_b + C_\pi\sigma)\right\}.$$

Setting $\frac{\partial Z(x_1,X)}{\partial X} = 0$ gives

$$\frac{1}{2}L_1x_1^2 - L_2x_1 + L_3 - \frac{X^2}{2}\gamma(C_b + C_\pi\sigma) = 0. \qquad (17)$$

Substituting X from equation (16) into equation (17) yields

$$L_1\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_1^2 - 2L_2\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_1 - (2\gamma(C_b + C_\pi\sigma)L_3 - L_2^2) = 0. \qquad (18)$$

From equation (18), let

$$F(x_1) = L_1\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_1^2 - 2L_2\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_1 - (2\gamma(C_b + C_\pi\sigma)L_3 - L_2^2), \quad x_1 \in [x_d, \infty) \qquad (19)$$

and

$$\Delta = F(x_d) = L_1\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_d^2 - 2L_2\big(L_1 - \gamma(C_b + C_\pi\sigma)\big)x_d - (2\gamma(C_b + C_\pi\sigma)L_3 - L_2^2).$$

Then, the subsequent result is gotten.

Lemma 2.1.

(i) If $\Delta \le 0$, then the solution of $x_1 \in [x_d, \infty)$ (say x_1^*) which satisfies equation (18) not only occurs but similarly exclusive.

(ii) If $\Delta > 0$, then the solution of $x_1 \in [x_d, \infty)$ which satisfies equation (18) does not occur.

Proof of (i): Taking the first-order derivative of $F(x_1)$ with respect to $x_1 \in [x_d, \infty)$, it follows that

$\frac{F(x_1)}{dx_1} = 2(L_1x_1 - L_2)\big(L_1 - \gamma(C_b + C_\pi\sigma)\big) > 0$, since $(L_1x_1 - L_2) > 0$ and

$\big(L_1 - \gamma(C_b + C_\pi\sigma)\big) = \beta\left[h_1(x_d\omega + 1) + h_2\left(\frac{x_d\omega}{2} + 1\right)x_d + C_p\omega\right] > 0.$

Henceforth, $F(x_1)$ is a strictly growing function of x_1 in the interval $[x_d, \infty)$. Additionally, $\lim_{x_1 \to \infty} F(x_1) = \infty$ and $F(x_d) = \Delta \leq 0$. Consequently, by applying the intermediate value theorem, there occurs a exclusive x_1 say $x_1^* \in [x_d, \infty)$ such that $F(x_1^*) = 0$. Henceforth, x_1^* is the exclusive solution of equation (18). Thus, the value of x_1 (denoted by x_1^*) can be found from equation (18) and is given by

$$x_1^* = \frac{L_2}{L_1} + \frac{1}{L_1}\sqrt{\frac{(2L_1L_3 - L_2^2)\gamma(C_b + C_\pi\sigma)}{\left(L_1 - \gamma(C_b + C_\pi\sigma)\right)}}. \tag{20}$$

Once x_1^* is gotten, then the value of X (denoted by X^*) can be found from equation (16) and is given by

$$X^* = \frac{1}{\gamma(C_b + C_\pi\sigma)}(L_1x_1^* - L_2). \tag{21}$$

Equations (20) and (21) give the best values of x_1^* and X^* respectively for the cost function in equation (14) only if L_2 satisfies the inequality given in equation (22)

$$L_2^2 < 2L_1L_3. \tag{22}$$

Proof of (ii): If $\Delta > 0$, then from equation (19), $F_1(x_1) > 0$. Since $F_1(x_1)$ is a strictly growing function of $x_1 \in [x_d, \infty)$, $F_1(x_1) > 0$ for all $x_1 \in [x_d, \infty)$. Thus, a value of $x_1 \in [x_d, \infty)$ cannot be found such that $F_1(x_1) = 0$. This completes the proof.

Theorem 2.1

(i) If $\Delta \leq 0$, then the entire variable cost $Z_1(x_1, X)$ is convex and reaches its global minimum at the point (x_1^*, X^*), where (x_1^*, X^*) is the point which satisfies equations (18) and (15).

(ii) If $\Delta > 0$, then the entire variable cost $Z_1(x_1, X)$ has a minimum value at the point (x_1^*, X^*), where $x_1^* = x_d$ and $X^* = \frac{1}{\gamma(C_b + C_\pi\sigma)}(L_1x_d - L_2)$.

Proof of (i): When $\Delta \leq 0$, it is seen that x_1^* and X^* are the exclusive solutions of equations (18) and (15) respectively from Lemma 4.l(i). Taking the second derivative of $Z_1(x_1, X)$ with respect to x_1 and X and then finding the values of these functions at the point (x_1^*, X^*) yields

$$\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial x_1^2}\right|_{(x_1^*,\ X^*)} = \frac{1}{X^*}\left[\beta\left[h_1(x_d\omega + 1) + h_2\left(\frac{x_d\omega}{2} + 1\right)x_d + C_p\omega\right] + \gamma[(C_b + C_\pi\sigma)]\right] > 0,$$

$$\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial x_1 \partial X}\right|_{(x_1^*,\ X^*)} = -\frac{\gamma}{X^*}(C_b + C_\pi\sigma),$$

$$\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial X^2}\right|_{(x_1^*,\ X^*)} = \frac{\gamma}{X^*}(C_b + C_\pi\sigma) > 0$$

and

$$\left(\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial x_1^2}\right|_{(x_1^*,\ X^*)}\right)\left(\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial X^2}\right|_{(x_1^*,\ X^*)}\right) - \left(\left.\frac{\partial^2 Z_1(x_1,\ X)}{\partial x_1 \partial X}\right|_{(x_1^*,\ X^*)}\right)^2$$
$$= \frac{\gamma^2\beta(C_b + C_\pi\sigma)}{X^{*2}}\left[h_1(x_d\omega + 1) + h_2\left(\frac{x_d\omega}{2} + 1\right)x_d + C_p\omega\right] > 0. \tag{23}$$

It is Consequently determined from equation (23) and Lemma 4.1 that $Z(x_1^*,\ X^*)$ is convex and $(x_1^*,\ X^*)$ is the global minimum point of $Z(x_1,\ X)$. Henceforth, the values of x_1 and X in equations (20) and (21) respectively are best.

Proof of (ii): When $\Delta > 0$, $F(x_1) > 0$ for all $x_1 \in [x_d,\ \infty)$. Thus, $\frac{\partial Z(x_1,\ X)}{\partial X} = \frac{F(x_1)}{X^2} > 0$ for all $x_1 \in [x_d, \infty)$ which implies $Z(x_1, X)$ is a strictly growing function of X. Thus, $Z(x_1,\ X)$ has a minimum value when X is minimum. Consequently, $Z(x_1,\ X)$ has a minimum value at the point $(x_1^*,\ X^*)$, where $x_1^* = x_d$ and $X^* = \frac{1}{\gamma(C_b + C_\pi\sigma)}(L_1x_d - L_2)$. This completes the proof.

Thus, after obtaining the best values of x_1^* and X^*, the best Economic Order Quantity (denoted by EOQ^*) can be computed as follows:

EOQ^* =Entire demand before decaying sets in+entire demand after decaying sets in+entire number of decayed goods + the entire number of goods back-ordered

$$= \int_0^{x_d} \alpha dx + \int_{x_d}^{x_1^*} \beta dx + \left[\frac{\beta}{\omega}\left(e^{\omega(x_1^*-x_d)} - 1\right) - \beta(x_1^* - x_d)\right] + \frac{\gamma}{\sigma}\left[ln[1 + \sigma(X^* - x_1^*)]\right]$$

$$= \alpha x_d + \frac{\beta}{\omega}\left(e^{\omega(x_1^*-x_d)} - 1\right) + \frac{\gamma}{\sigma}\left[ln[1 + \sigma(X^* - x_1^*)]\right]. \quad (24)$$

Arithmetical examples

This unit provides some arithmetical examples to elucidate the hypothetical outcomes of the model developed.

Example 2.1 (Case 1).

Study an inventory structure with the subsequent input constraints: $O =$ ₦$350/\text{order}$, $C_p =$ ₦$45/\text{units/year}$, $h_1 =$ ₦$15/\text{units/year}$, $h_2 =$ ₦$5/\text{units/year}$, $C_b =$ ₦$20/\text{units/year}$, $C_\pi =$ ₦$5/\text{units/year}$, $\omega = 0.05$ units/year, $\alpha = 980$ units, $\beta = 180$ units, $\gamma = 15$ units, $\lambda = 450$ units, $x_d = 0.2136$ year (78 days) and $\sigma = 0.8$. It is seen that $\Delta = -16.5278 < 0$, $L_2^2 = 3.78255$, $2L_1L_3 = 102.8074$ and Henceforth $L_2^2 < 2L_1L_3$. Substituting the above values in equations (20), (21), (14) and (24), the values of best time with positive inventory, cycle length, entire variable cost and economic order quantity are respectively gotten as follows: $\boldsymbol{t_1^* = 0.2625}$ year (96 days), $\boldsymbol{T^* = 0.5186}$ year (189 days), $\boldsymbol{Z_1}(\boldsymbol{t_1^*}, X^*) =$ ₦$\mathbf{1837.8012}$ per year and $\boldsymbol{EOQ^*} = \mathbf{327.6931}$ units per year.

Sensitivity analysis

The sensitivity scrutiny of some constraints is carried out by varying each of these constraints from -20% to 20% taking one constraint at a time and keeping the remaining constraints unchanged. The consequences of changes of these constraints on decision variables for Example 2.1 is summarised in Tables 2.1.

Table 2.1: Effect of changes of some constraints on decision variables for Example 5.1.

Constraints	% Change in in constraint	% Change in $\boldsymbol{t_1^*}$	% Change in $\boldsymbol{T^*}$	% Change in $\boldsymbol{EOQ^*}$	% Change in $\boldsymbol{Z(t_1^*, T^*)}$
θ	−20	0.6799	0.2240	0.1639	−0.0802
	−10	0.3377	0.1113	0.0814	−0.0398
	+10	−0.3334	−0.10982	−0.0804	0.0393
	+20	−0.6624	−0.2182	−0.1597	0.0782
C_p	−20	0.6553	0.2145	0.1613	−0.0796
	−10	0.3256	0.1066	0.0801	−0.0396
	+10	−0.3215	−0.1052	−0.0791	0.0391
	+20	−0.6390	−0.2091	−0.1572	0.0778
σ	−20	−0.8282	0.8951	1.6657	−0.8704
	−10	−0.4109	0.4421	0.8148	−0.4318
	+10	0.4045	−0.4315	−0.7809	0.4251
	+20	0.8027	−0.8528	−1.5299	0.8436
C_b	−20	−5.4027	6.1367	4.7939	−5.6781
	−10	−2.5667	2.8261	2.2268	−2.6975
	+10	2.3366	−2.4433	−1.9532	2.4557
	+20	4.4746	−4.5786	−3.6828	4.7026
C_π	−20	−0.8282	0.8951	0.7090	−0.8704
	−10	−0.4109	0.4421	0.3506	−0.4318
	+10	0.4045	−0.4315	−0.3430	0.4251

	+20	0.8027	−0.8528	−0.6787	0.8436

DISCUSSION OF THE RESULTS

Based on the computed outcomes shown in Table 6.1, the subsequent decision-making intuitions are gotten.

(a) As soon as the amount of decaying, (θ), surges, the best time with positive inventory (x_1^*), cycle length (X^*) and Economic Order Quantity (EOQ^*) decrease, while entire variable cost $\left(Z(x_1^*, X^*)\right)$ surges and vice versa. As soon as the number of decayed goods surges, then the entire variable cost will be extreme. Henceforth, the trader will demand fewer amount to avoid goods being decaying as soon as the decaying amount surges. This decreases the inventory holding cost and subsequently reducing the entire variable cost. The amount of decaying can similarly be reduced by refining the equipment in the silo.

(b) As soon as the unit buying cost, $\left(C_p\right)$, surges, the best time with positive inventory (x_1^*), cycle length (X^*) and Economic Order Quantity (EOQ^*) decrease, while the entire variable cost $\left(Z(x_1^*, X^*)\right)$ surges and vice versa. In a real market circumstances, the higher the cost of an item, the higher the entire variable cost and vice versa. The trader will demand fewer amount as soon as unit buying cost surges.

(c) As soon as the backlogging constraint, (σ), surges, the best time with positive inventory (x_1^*) and entire variable cost $\left(Z(x_1^*, X^*)\right)$ increase, while the cycle length (X^*) and Economic Order Quantity (EOQ^*) decrease and vice versa. This means that if few clients are ready to hang on for the backorder, the trader should demand large to avoid stock out and curtail the cycle length.

(d) As soon as the stock out cost, (C_b), surges, the best time with positive inventory (x_1^*) and entire variable cost $\left(Z(x_1^*, X^*)\right)$ decrease, while the cycle length (X^*) and Economic Order Quantity (EOQ^*) increase and vice versa. This means that as soon as the Stockouts cost increase, entire variable cost surges and the number of back-ordered goods reduce drastically which in turn decreases the entire variable cost.

(e) As soon as the cost of lost sales, (C_π), surges, the best time with positive inventory (x_1^*) and entire variable cost $\left(Z(x_1^*, X^*)\right)$ decrease, while the cycle length (X^*) and Economic Order Quantity (EOQ^*) increase and vice versa.

CONCLUSION

In this unit, EOQ model for non-instant decaying goods with three-stage demand rates, linear time-reliant holding cost and Stockouts is developed. Stockouts are permitted and incompletely backlogged. The extent of the Waiting time would regulate whether backlogging will be accepted or not. Henceforth, the backlogging amount is flexible and varies on the waiting time for the subsequent top up. The best time with positive inventory, cycle length and order quantity that reduce the entire variable cost are determined. Arithmetical example was given to elucidate the hypothetical outcomes of the model. Sensitivity scrutiny of some model constraints is carried out to see the effect of changes of these constraints on decision variables. The outcomes show that the trader reduces the entire variable cost by ordering fewer goods to curtail the best time with positive inventory and cycle length as soon as the amount of decaying, unit buying cost, stock out cost, backlogging constraint and cost of lost sales increase respectively.

REFERENCES

Babangida, B. and Baraya, Y. M. (2018). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand under trade credit policy. *Journal of the Nigerian Association of Mathematical Physics*, **47**(4), 93−110.

Babangida, B. and Baraya, Y. M. (2019a). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand and linear holding cost under trade credit policy. ABACUS: *Journal of the Mathematical Association of Nigeria*, **46**(1), 191−217.

Babangida, B. and Baraya, Y. M. (2019b). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand and complete backlogging under trade credit policy. ABACUS: *Journal of the Mathematical Association of Nigeria*, **46**(1), 488−505.

Babangida, B. and Baraya, Y. M. (2020). An inventory model for non-instantaneous deteriorating items with time dependent quadratic demand, two storage facilities and shortages under trade credit policy. *International Journal of Modelling in Operations Management*, **8**(1), 1−44. **DOI:** 10.1504/IJMOM.2020.10029879.

Babangida, B. and Baraya, Y. M. (2021a). An EOQ model for non-instantaneous deteriorating items with two-phase demand rates and two level pricing strategies under trade credit policy. *Transaction of the Nigerian Association of Mathematical Physics*, **17**(4), 117−130.

Babangida, B. and Baraya, Y. M. (2021b). An EOQ model for non-instantaneous deteriorating items with time dependent quadratic rate, linear holding cost and partial backlogging rate under trade credit policy. *Transaction of the Nigerian Association of Mathematical Physics*, **17**(4), 131−144.

Babangida, B. and Baraya, Y. M. (2022). An EOQ model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost and time dependent partial backlogging rate under trade credit policy. ABACUS: *Journal of the Mathematical Association of Nigeria*, **49**(2), 91−125.

Babangida, B., Baraya, Y. M., Ohanuba, O. F., Hali, A. I. and Malumfashi, M. L. (2023). An Order Inventory Model for Delayed Deteriorating Items with Two-Storage Facilities, Time-Varying Demand and Partial Backlogging Rates Under Trade-Credit Policy. *UMYU Scientifica*, **2**(1), 336-341.

Bello, Y. and Baraya, Y. M. (2018). An inventory model for non-instantaneous deteriorating item with two-phase demand rate and partial backlogging. *Journal of the Nigerian Association of Mathematical Physics*, **47**(4), 77–86.

Chen, Z. (2018). Optimal inventory replenishment and pricing for a single-manufacturer and multi-retailer system of deteriorating items. *International Journal of Operational Research*, **31**(1), 112–139.

Choudhury, K. D., Karmakar, B., Das, M. and Datta, T. K. (2013). An inventory model for deteriorating items with stock dependent demand, time-varying holding cost and shortages. *Journal of the Operational Research Society*, **23**(1), 137–142.

Chung, K. J. (2009). A complete proof on the solution procedure for non-instantaneous deteriorating items with permissible delay in payment. *Computers and Industrial Engineering*, **56**(1), 267–273.

Covert, R. B. and Philip, G. C. (1974). An EOQ model with Weibull distribution deterioration. *IIE Transactions*, **5**(4), 323–326.

Deb, M. and Chaudhuri, K. S. (1987). A note on the heuristic for replenishment of trended inventories considering shortages. *Journal of the Operational Research Society*, **38**(5), 459–463.

Geetha, K. V. and Udayakumar, R. (2016). Optimal lot sizing policy for non-instantaneous deteriorating items with price and advertisement dependent demand under partial backlogging. *International Journal of Applied and Computational Mathematics*, **2**(1), 171–193.

Geetha, K. V. and Uthayakumar, R. (2010). Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments. *Journal of Computational and Applied Mathematics*, **233**(10), 2492–2505.

Ghare, P. M. and Schrader, G. F. (1963). A model for an exponentially decaying inventory. *Journal of Industrial Engineering*, **14**(5), 238–243.

Harris, F. (1913). How many parts to make at once, factory. *The Magazine of Management*, **10**(2), 135–136.

Mishra, V. K. and Singh, L. S. (2011). Deteriorating inventory model for time dependent demand and holding cost with partial backlogging. *International Journal of management science and engineering management*, **6**(4), 267–271.

Ouyang, L. Y., Wu, K. S. and Yang, C. T. (2006). A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments. *Computers and Industrial Engineering*, **51**(4), 637–651.

Philip, G. C. (1974). A generalized EOQ model for items with Weibull distribution. *AIIE Transactions*, **6**(2), 159–162.

Sharma, J. K. (2003). Operations research theory and application. *Beri Macmillan Indian Limited*, 584–585.

Singhal, S. and Singh, S. R. (2018). Supply chain system for time and quality dependent decaying items with multiple market demand and volume flexibility. *International Journal of Operational Research*, **31**(2), 245–261.

Tavakoli, S. and Taleizadeh, A. A. (2017). An EOQ model for decaying item with full advanced payment and conditional discount. *Annals of Operations Research*, **259**(3), 1–22.

Tayal, S., Singh, S. R., Sharma, R. and Singh, A. P. (2015). An EPQ model for non-instantaneous deteriorating item with time dependent holding cost and exponential demand rate. *International Journal of Operational Research*, **23**(2), 145–162.

Tyagi, A. P., Pandey, R. K. and Singh, S. R. (2014). Optimal replenishment policy for non-instantaneous deteriorating items with stock dependent demand and variable holding cost. *International Journal of Operational Research*, **21** (4), 466–488.

Whitin, T. M. (1957). Theory of inventory management. *Princeton University Press, Princeton.*