https://dx.doi.org/10.4314/dujopas.v9i4b.6
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490

Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 9 No. 4b December 2023*

Results on Prime and Semi-Prime Rings with Skew and Generalized Reverse Derivations

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Abstract

For this study, R represents a semiprime ring or a prime ring, as the case may be. The ring R is said to be semiprime if for any $a \in R$, $aRa = \{0\}$, implies $a = 0$. R is a prime ring if $aRb = \{0\}$, implies $a = 0$ or $b = 0$, $\forall\, a, b \in R$. By assuming that d is a skew derivation with an automorphism $\beta: R \to R$ associated with it, we prove some results on skew-derivations for semi-prime rings. In particular, we show that for a skew-derivation, if $d(a)d(b) \pm ab = 0, \forall\, a, b \in R$ then $d = 0$. Also, by introducing new differential identities, we establish that a prime ring with a generalized reverse derivation defined on it is commutative.

KEYWORDS: Semiprime ring, prime ring, reverse derivation, skew derivation, generalized reverse derivation.

INTRODUCTION

In this study, the symbol $[a, b]$ and (aob), represents the Lie product $ab - ba$ and the Jordan product $ab + ba$, respectively, where a, b are elements of a ring R. A ring R is said to be prime if $aRb = \{0\}$ for all $a, b \in R$, implies $a = 0$ or $b = 0$ and it is semiprime if $aRa = 0$ for all $a \in R$, implies that $a = 0$.

Posner (1957) introduced the following notion for a ring R: *A mapping $d: R \to R$ is said to be a derivation if $d(ab) = d(a)b + ad(b), \forall\, a, b \in R$.* The mapping d is said to be a derivation on R if it is an additive mapping. The concept of additive mapping was presented by Bresar & Vukman (1989) as follows: *A mapping f is called an additive mapping on R if $(a + b) = f(a) + f(b)\ \forall\, a, b \in R$.* Also, an additive mapping $F: R \to R$ is said to be a generalized derivation if there exists a derivation $d: R \to R$ such that $F(ab) = F(a)b + ad(b), \forall\, a, b \in R$. Derivations on rings and other algebraic structures abound in the literature (Mohammed *et al*., 2023; Balogun, 2014; Chaudhry and Ullah, 2011; Hvala, 1998; Bell and Kappe, 1989).

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Herstein (1957) initiated the concept of reverse derivations. According to Herstein, an additive mapping d on R is called a reverse derivation if $(ab) = d(b)a + bd(a)$, $\forall\, a, b \in R$. Aboubakr and Gonzalez (2015) defined a generalized reverse derivation as an additive mapping $F: R \to R$ such that there exists a mapping $d: R \to R$ and $F(ab) = F(b)a + bd(a)$, $\forall\, a, b \in R$. Sandhu (2018) proved the following result: *Let (μ_1, σ_1) and (μ_2, σ_2) be the skew-derivations of 2-torsion free prime ring R with $\mu_2\sigma_2 = \sigma_2\mu_2$. If the iterate $(\mu_1\mu_2, \sigma_1\sigma_2)$ is a skew derivation of R, we have either $(\mu_1, \sigma_1) = 0$ or $(\mu_2, \sigma_2) = 0$.*

Recently, Khan *et al*., (2020) proved that "if δ_1, δ_2 and δ_3 are skew-derivations associated with automorphisms β_1, β_2 and β_3 of 3!-torsion free prime rings R with $\delta_i\delta_j = \delta_j\delta_i$ for $i, j = 1,2,3$, $\delta_3^2 = \delta_3$ and the iterate $\delta_1\delta_2\delta_3$ is a skew-derivation of R, then at least one of $d_i = 0$, for $i = 1,2,3$.
Motivated by these works, we introduce new differential identities that make prime rings to be commutative. We also establish new results on skew-derivation for semi-prime rings.

THEORETICAL FRAMEWORK

We need the following definitions:

Definition 2.1
An additive mapping $\delta: R \to R$ of a prime ring R associated with an automorphism $\beta: R \to R$ is called a skew-derivation if $\delta(ab) = \delta(a)b + \beta(a)\delta(b)$, for all $a, b \in R$.

Definition 2.2
An additive mapping $F: R \to R$ is said to be a generalized reverse derivation if there exists a derivation $d: R \to R$ such that $F(ab) = F(b)a + bd(a), \forall\, a, b \in R$.
The following are results that will be extended in this work:

Theorem 2.3
Let R be a prime ring and d be a skew- derivative associated with an automorphism $\beta: R \to R$
If $d(x)d(y) \pm xy = 0$ for all $x, y \in R$ then $d = 0$.

Theorem 2.4
Let R be prime ring and d be a skew- derivative associated with an automorphism $\beta: R \to R$
If $d(y)d(x) + yx = 0$ for all $x, y \in R$, then $d = 0$.

Theorem 2.5
Let R be a semiprime ring and I a non-zero ideal of R. Suppose F is a multiplicative (generalized)-reverse derivation associated with the mapping d on R. If $F(xy) - xoy = 0$ for all $x, y \in I$, then R is commutative.

DISCUSSION AND RESULTS

In this section, we extend the results presented in section 2.

RESULTS ON SEMIPRIME RINGS

We have the following results for semiprime rings:

Theorem 1
Suppose R is a semiprime ring and d is a skew- derivation associated with an automorphism β from R to itself.
If $d(a)d(b) + ab = 0$ or $d(a)d(b) - ab = 0\ \forall\, a, b \in R$ then the skew-derivation is zero.

Proof:
First we consider the case

$$d(a)d(b) + ab = 0 \quad (1)$$

$\forall\ a, b, \in R$. Putting bc in place of b in equation (1), we have

$$d(a)d(bc) + abc = 0$$

By definition 1.1 we have

$$d(a)\big(d(b)c + \beta(b)d(c)\big) + abc = 0\ \ \forall\ a, b, c\ \in R$$

$$d(a)d(b)c + d(a)\beta(b)d(c) + abc = 0$$

But $d(a)d(b) = -ab$

$$-abc + d(a)\beta(b)d(c) + abc = 0$$

$$d(a)\beta(b)d(c) + abc - abc = 0$$

$$d(a)\beta(b)d(c) = 0 \quad (2)$$

Replacing a with ac in equation (2), we obtain

$$d(ac)\big(\beta(b)d(c)\big) = 0$$

Again by definition 1.1, we have

$$(d(a)c\big(\beta(a)d(c)\big)\beta(b)d(c) = 0$$

$$d(a)c\beta(b)d(c) + \beta(a)d(c)\beta(b)d(c) = 0 \quad (3)$$

Replacing $\beta(b)$ with $c\beta(b)$ in equation (2), we get

$$d(a)c\beta(b)d(c) = 0\ \ \forall\ a, b, c, \in R \quad (4)$$

Subtracting equation (4) from equation (3), we obtain

$$\beta(a)d(c)\beta(b)d(c) = 0\ \ \forall\ a. b, c, \in R \quad (5)$$

Replacing $\beta(a)d(c)$ with $d(c)$ in equation (5), we get

$$d(c)\beta(y)d(c) = 0\ \ \forall\ a, b, c \in R$$

Now, d is a skew derivation and β is an automorphism associated with it
Thus,

$$d(c)R\ d(c)\ , \forall\ \ z \in R$$

By Semiprimeness of the ring R, this implies
$d(c) = 0$. Hence, we have the required result.
Similarly the second case follows.

Theorem 2
Let R be a semiprime ring and d be a skew- derivative associated with an automorphism $\beta: R \rightarrow R$.
If $d(b)d(a) + ba = 0$ or $d(b)d(a) - ba = 0$ for all $a, b \in R$ then the skew-derivation is zero.
Proof: First we consider the case

$$d(b)d(a) + ba = 0 \quad (1)$$

for all $a, b, \in R$. Replacing a with ac in equation (1), we have

$$d(b)d(ac) + bac = 0$$

By definition 1.1, we have

$$d(b)\big(d(a)c + \beta(a)d(c)\big) + bac = 0\ \ \forall\ a, b, c\ \in R$$

$$d(b)d(a)c + d(b)\beta(a)d(c) + bac = 0$$

But $d(b)d(a) = -bac$

$$-bac + d(b)\beta(a)d(c) + bac = 0$$

$$d(b)\beta(a)d(c) + bac - bac = 0$$

$$d(b)\beta(a)d(c) = 0 \quad (2)$$

Replacing b with bc in equation (2), we obtain

$$d(bc)\big(\beta(a)d(c)\big) = 0$$

By definition 1.1, we have

$$(d(b)c + \beta(b)d(c))\big(\beta(a)d(c)\big) = 0$$

$$d(b)c\beta(a)d(c) + \beta(b)d(c)\beta(a)d(c) = 0 \quad (3)$$

Replacing $\beta(a)$ with $c\beta(a)$ in equation (2), we get

$$d(b)c\beta(a)d(c) = 0 \ \forall \ a.b,,c \in R \quad (4)$$

Subtracting equation (4) from equation (3), we obtain

$$\beta(b)d(c)\beta(a)d(c) = 0 \ \forall \ a.b,c \in R \quad (5)$$

Substituting $d(c)$ for $\beta(b)d(c)$ in equation (5), we get

$$d(c)\beta(a)d(c) = 0 \ \forall \ a,c \in R$$

Now, d is a skew derivation and β is an automorphism associated with it
Thus,

$$d(c)R\, d(c) = 0, \ \forall \ c \in R$$

By the semiprimeness of the ring R, we have
$d(c) = 0$. Hence the result.
The second case follows.

RESULT ON PRIME RINGS

Next, we present the following result for a prime ring R:

Theorem 3

Let R be a prime ring and F be a generalized reverse derivation associated with the reverse derivation d on R. If $F(ab) \pm aob = 0, \forall \ a,b \in R,$ then the ring R is commutative.
Proof:

$$F(ab) \pm aob = 0 \ , \forall \ a,b \in R \quad (1)$$

Put $a = za$

$$0 = F(zab) - zaob \ , \forall \ a,b,z \in R \quad (2)$$

By definition 1.2

$$0 = F(ab)z + abd(z) - (aob)z + (aob)z - zab - baz, \ \forall \ a,b,z \in R$$

$$0 = \big(F(ab) - (aob)\big)z + abd(z) + abz - baz - zab - ba, \ \forall \ a,b,z \in R \quad (3)$$

$$0 = \big(F(ab) - (aob)\big)z + abd(z) + [ab,z] + b[a,z], \forall \ a,b,z \in R \quad (4)$$

Since $F(ab) - aob = 0$, then

$$0 = abd(z) + [ab,z] + b[a,z], \forall \ a,b,z \in R \quad (5)$$

Put $y = xy$

$$0 = a^2bd(z) + [a^2b,z] + ab[a,z], \forall \ a,b,z \in R$$

$$0 = a^2bd(z) + a[ab,z] + [a,z]ab = ab[a,z], \forall \ a,b,z \in R \quad (6)$$

From equation (5) right multiplying by a

$$0 = a^2bd(z) + a[ab,z] + ab[a,z] \ , \forall \ a,b,z \in R \quad (7)$$

(6) minus (7) we have,

$$0 = [a,z]ab, \forall \ a,b,z \in R \quad (8)$$

Put $z = zu,\ u \in R,$ we have

$$0 = [a,zu]ab, \forall \ a,b,z,u \in R \quad (9)$$

This implies that,

$$(z[a,u] + [a,z]u)ab = 0, \forall \ a,b,z,u \in R$$

$$z[a,u]ab + [a,z]uab = 0, \forall \ a,b,z,u \in R \quad (10)$$

But $u = zu,$we have

$$z[a,zu]ab + [a,z]zuab = 0, \forall \ a,b,z,u \in R \quad (11)$$

But $[a,zu]ab = 0$
Thus we have,

$$[a,z]zuab = 0, \forall \ a,b,z,u \in R \quad (12)$$

But , $u = zu$

$$[a,z]uab = 0, \forall \ a,b,z,u \in R \quad (13)$$

Replacing zu instead of u, $(u = zu)$ in equation (13)

$$[a,z]uzab = 0, \forall \ a,b,z,u \in R \quad (14)$$

Put $b = zb$ in (13)

$$[a,z]uazb = 0, \forall \ a,b,z,u \in R \quad (15)$$

(14) minus (15), we have
$[a,z]uzab - [a,b]uazb = 0, \forall\, a,b,z,u \in R$ (16)
$[a,z](zuab - uazb) = 0, \forall\, a,b,z,u \in R$
$[a,z](u(zab - azb)) = 0, \forall\, a,b,z,u \in R$ (17)
$[a,z](u(za - az)b) = 0, \forall\, a,b,z,u \in R$
$[a,z]u[z,a]b = 0, \forall\, a,b,z,u \in R$ (18)
Put $b = 1$
$[a,z]u[z,a] = 0, \forall\, a,z,u \in R$
$[a,z]R[z,a] = 0, \forall\, a,z \in R$ (19)
Since R is a prime ring we have
$[a,z] = 0$ or $[z,a] = 0, \forall\, a,z \in R$
$az - za = 0$ or $-az = 0$, $\forall\, a,z \in R$
Hence,
$az = za$ or $za = az, \forall\, a,z \in R$.
Therefore the ring R is commutative, hence our result.

CONCLUSION

In this work, we established some new results on skew-derivation and generalized reverse derivation for prime and semi-prime rings by extending some existing results. In particular, we proved the following for semi-prime rings with skew-derivations: *if* $d(a)d(b) \pm ab = 0$ *for all* $a,b \in R$ *then* $d = 0$. Furthermore, by introducing new differential identities, we proved that a prime ring R with a generalized reverse derivation F, which is associated with a reverse derivation d on R, is commutative if $F(ab) \pm aob = 0, \forall\, a,b \in R$.

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