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On A Class of Four-Step Modified Backward Differentiation Formula (MBDF) For Solving Second-Order Ordinary Differential Equations

Kaze Atsi[1*], Pius Tumba[1] and Enoch Suleiman[1]
[1] Department of Mathematics,
Faculty of Science,
Federal University,
Gashua, Yobe state,
Nigeria

Email: kzeeat@yahoo.com

Abstract

This research was conducted to study a class of four-step modified backward differentiation formula (MBDF) for solving second-order ordinary differential equations, with one super-future point. A block method was constructed for the solution of problems of second-order ordinary differential equations. Adopting a step-number, $k = 4$, we obtained a number of discrete methods in block. The stability properties of the block method were investigated using Maple application. In order to ascertain its suitability, the method was tested on some initial valued problems of second-order ordinary differential equations. The numerical solutions of the problems were compared with the respective exact solutions and of other existing methods, and are presented in both tables and graphs.

Keywords: Interpolation and Collocation method, Modified Backward Differentiation Formula, Initial Value Problems, Second order Ordinary Differential Equations, Block method.

INTRODUCTION

Frequently, the general problem of second order ODEs originated in the field of engineering, dynamic system and sciences. The reduction of these problems to a system of first order ODEs are associated with lot of human efforts and many functions to be evaluated per iteration may jeopardize the accuracy of the method, Kuboye *et al.*, (2018). Collocation as an approach to develop block method is adopted in this work. The method has shown to bypass the disadvantages of computational time wastage associated with conventional methods, Adeyeye and Omar (2020). In this and many other researches, the solution for initial valued problem of second order ODEs of the form:

$$y'' = f(x, y(x), y'(x)),\ y(a) = y_0,\ y'(a) = y_1,\ x \in [a, b] \quad (1)$$

is examined.

**Author for Correspondence*

In order to find the most accurate method for the solution of problems of the form (1), a number of researches have been done. A block extended trapezoidal rule of the second kind (Etr2) for the direct solution of second order initial value problems of ODEs has been developed in Atsi and Kumleng (2021). Using the multi-step collocation approach, a single block method for step-number, $k=3$, has been derived. Adopting Taylor's series expansion in Adeyeye and Omar (2016), a 3-step second derivative numerical method to approximate (1) has been presented. In Anake *et al.,* (2012), a continuous one-step implicit hybrid block method for the direct solution of initial value problems (1), using power series were presented. A multistep collocation technique is used in Ehigie (2011) to develop 3-point explicit and implicit block methods, which has presented as suitable for generating solutions of the general second-order ordinary differential equations of the form (1). A four-point block method is proposed in Ukpebor (2019), with the use of Legendre polynomial as basis function in generating the continuous linear multistep for the solution of (1). A Continuous Two Step Trigonometrically-Fitted Second Order Method (TSTSOM) is used to solve an oscillatory second order problem of ordinary differential equations in Kayode *et al.,* (2021). Similarly, in (Abdelrahim and Omar, 2016; Awoyemi *et al.,* 2011; Obarhua 2019), numerical block methods have been adopted for solving problems of the form (1).

Interpolation and collocation technique is seen to be adopted in (Atsi and Kumleng, 2020; Kumleng *et al.*, 2013; Chollom *et al.,* 2007), for construction numerical methods in solving other problems.

As in the cases of the technique adopted in (Adeyeye and Omar, 2016; Anake *et al.,* (2012); Chollom *et al.,* 2007), a new four-step Modified Backward Differentiation Formula has been presented in this research for developing the method, used in block for the solution of (1).

METHODOLOGY

Derivation of the Methods

The new block method is constructed from a class of Modified Backward Differentiation Formula of step number four, $k = 4$, with one superfuture point, denoted by:

$$\sum_{j=0}^{2} \alpha_j(x) y_{n+j} = h[\beta_3(x) f_{n+3} + \beta_4(x) f_{n+4}] + h^2[\gamma_3(x) f_{n+3} + \gamma_4(x) f_{n+4} + \gamma_5(x) f_{n+5}] \quad (2)$$

where,

$\alpha_j(x)$ and $\beta_j(x)$ are called the continuous coefficients, and

h is the step size.

Adopting the interpolation and collocation technique, the D matrix for the method is obtained, as:

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_n+h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 & (x_n+h)^6 & (x_n+h)^7 \\ 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 & (x_n+2h)^6 & (x_n+2h)^7 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 & 6(x_n+3h)^5 & 7(x_n+3h)^6 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 & 6(x_n+4h)^5 & 7(x_n+4h)^6 \\ 0 & 0 & 2 & 6x_n+18h & 12(x_n+3h)^2 & 20(x_n+3h)^3 & 30(x_n+3h)^4 & 42(x_n+3h)^5 \\ 0 & 0 & 2 & 6x_n+24h & 12(x_n+4h)^2 & 20(x_n+4h)^3 & 30(x_n+4h)^4 & 42(x_n+4h)^5 \\ 0 & 0 & 2 & 6x_n+30h & 12(x_n+5h)^2 & 20(x_n+5h)^3 & 30(x_n+5h)^4 & 42(x_n+5h)^5 \end{bmatrix} \quad (3)$$

Some algebraic manipulations yield the proposed continuous scheme of the form (2). Hence, the continuous scheme is:

$$\begin{aligned} y(x) = \Bigg(& \frac{1}{30429960}\,\frac{1}{h^7}\big(30429960\,h^7 + 81488592\,h^6 x_n + 86048820\,h^5 x_n^2 + 47599663\,h^4 x_n^3 + 15132920\,h^3 \\ & x_n^4 + 2789406\,h^2 x_n^5 + 277492\,h\,x_n^6 + 11531\,x_n^7\big) - \frac{1}{30429960}\,\frac{1}{h^7}\big(x\,\big(81488592\,h^6 \\ & + 172097640\,h^5 x_n + 142798989\,h^4 x_n^2 + 60531680\,h^3 x_n^3 + 13947030\,h^2 x_n^4 + 1664952\,h\,x_n^5 + 80717\,x_n^6\big) \\ & \big) \\ & + \frac{1}{10143320}\,\frac{1}{h^7}\big(x^2\,\big(28682940\,h^5 + 47599663\,h^4 x_n + 30265840\,h^3 x_n^2 + 9298020\,h^2 x_n^3 \\ & + 1387460\,h\,x_n^4 + 80717\,x_n^5\big)\big) \\ & - \frac{1}{30429960}\,\frac{x^3\big(47599663\,h^4 + 60531680\,h^3 x_n + 27894060\,h^2 x_n^2 + 5549840\,h\,x_n^3 + 403585\,x_n^4\big)}{h^7} \\ & + \frac{1}{6085992}\,\frac{x^4\big(3026584\,h^3 + 2789406\,h^2 x_n + 832476\,h\,x_n^2 + 80717\,x_n^3\big)}{h^7} \\ & - \frac{1}{10143320}\,\frac{x^5\big(929802\,h^2 + 554984\,h\,x_n + 80717\,x_n^2\big)}{h^7} + \frac{1}{30429960}\,\frac{x^6\big(277492\,h + 80717\,x_n\big)}{h^7} \\ & - \frac{11531}{30429960}\,\frac{x^7}{h^7}\Bigg) y_n \end{aligned}$$

$$+\Big(-\frac{1}{3803745}\frac{1}{h^7}\big(x_n\big(28276128h^6+47135880h^5x_n+32122942h^4x_n^2+11527505h^3x_n^3+2299284h^2$$

$$x_n^4+241678hx_n^5+10454x_n^6\big)\big)+\frac{2}{3803745}\frac{1}{h^7}\big(x\big(14138064h^6+47135880h^5x_n$$

$$+48184413h^4x_n^2+23055010h^3x_n^3+5748210h^2x_n^4+725034hx_n^5+36589x_n^6\big)\big)$$

$$-\frac{2}{1267915}\frac{1}{h^7}\big(x^2\big(7855980h^5+16061471h^4x_n+11527505h^3x_n^2+3832140h^2x_n^3$$

$$+604195hx_n^4+36589x_n^5\big)\big)$$

$$+\frac{2}{3803745}\frac{x^3\left(16061471h^4+23055010h^3x_n+11496420h^2x_n^2+2416780hx_n^3+182945x_n^4\right)}{h^7}$$

$$-\frac{1}{760749}\frac{x^4\left(2305501h^3+2299284h^2x_n+725034hx_n^2+73178x_n^3\right)}{h^7}$$

$$+\frac{2}{1267915}\frac{x^5\left(383214h^2+241678hx_n+36589x_n^2\right)}{h^7}-\frac{2}{3803745}\frac{x^6\left(120839h+36589x_n\right)}{h^7}$$

$$+\frac{10454}{3803745}\frac{x^7}{h^7}\Big)y_{n+1}$$

$$+\Big(\frac{1}{30429960}\frac{1}{h^7}\big(x_n\big(144720432h^6+291038220h^5x_n+209383873h^4x_n^2+77087120h^3x_n^3$$

$$+15604866h^2x_n^4+1655932hx_n^5+72101x_n^6\big)\big)-\frac{1}{30429960}\frac{1}{h^7}\big(x\big(144720432h^6$$

$$+582076440h^5x_n+628151619h^4x_n^2+308348480h^3x_n^3+78024330h^2x_n^4+9935592hx_n^5+504707$$

$$x_n^6\big)\big)$$

$$+\frac{1}{10143320}\frac{1}{h^7}\big(x^2\big(97012740h^5+209383873h^4x_n+154174240h^3x_n^2+52016220h^2x_n^3$$

$$+8279660hx_n^4+504707x_n^5\big)\big)$$

$$-\frac{1}{30429960}\frac{1}{h^7}\big(x^3\big(209383873h^4+308348480h^3x_n+156048660h^2x_n^2+33118640hx_n^3$$

$$+2523535x_n^4\big)\big)+\frac{1}{6085992}\frac{x^4\left(15417424h^3+15604866h^2x_n+4967796hx_n^2+504707x_n^3\right)}{h^7}$$

$$-\frac{1}{10143320}\frac{x^5\left(5201622h^2+3311864hx_n+504707x_n^2\right)}{h^7}$$

$$+\frac{1}{30429960}\frac{x^6\left(1655932h+504707x_n\right)}{h^7}-\frac{72101}{30429960}\frac{x^7}{h^7}\Big)y_{n+2}$$

$$+\Big(\frac{1}{1267915}\frac{1}{h^6}\big(x_n\big(15496736h^6+38693160h^5x_n+36513894h^4x_n^2+16994835h^3x_n^3+4168743h^2x_n^4$$

$$+516906hx_n^5+25528x_n^6\big)\big)-\frac{1}{1267915}\frac{1}{h^6}\big(x\big(15496736h^6+77386320h^5x_n+109541682h^4$$

$$x_n^2+67979340h^3x_n^3+20843715h^2x_n^4+3101436hx_n^5+178696x_n^6\big)\big)$$

$$+\frac{6}{1267915}\frac{1}{h^6}\big(x^2\big(6448860h^5+18256947h^4x_n+16994835h^3x_n^2+6947905h^2x_n^3$$

$$+1292265hx_n^4+89348x_n^5\big)\big)$$

$$-\frac{2}{1267915}\frac{x^3\left(18256947h^4+33989670h^3x_n+20843715h^2x_n^2+5169060hx_n^3+446740x_n^4\right)}{h^6}$$

$$+\frac{1}{253583}\frac{x^4\left(3398967h^3+4168743h^2x_n+1550718hx_n^2+178696x_n^3\right)}{h^6}$$

$$-\frac{3}{1267915}\frac{x^5\left(1389581h^2+1033812hx_n+178696x_n^2\right)}{h^6}+\frac{2}{1267915}\frac{x^6\left(258453h+89348x_n\right)}{h^6}$$

$$-\frac{25528}{1267915}\frac{x^7}{h^6}\Big)f_{n+3}$$

$$+\left(\frac{1}{45644940}\,\frac{1}{h^5}\left(x_n\left(565381728\,h^6+1343045880\,h^5\,x_n+1189081042\,h^4\,x_n^2\right.\right.\right.$$

$$\left.\left.+\,517566665\,h^3\,x_n^3+119577339\,h^2\,x_n^4+14095183\,h\,x_n^5+667619\,x_n^6\right)\right)$$

$$-\frac{1}{45644940}\,\frac{1}{h^5}\left(x\left(565381728\,h^6+2686091760\,h^5\,x_n+3567243126\,h^4\,x_n^2\right.\right.$$

$$\left.\left.+\,2070266660\,h^3\,x_n^3+597886695\,h^2\,x_n^4+84571098\,h\,x_n^5+4673333\,x_n^6\right)\right)$$

$$+\frac{1}{15214980}\,\frac{1}{h^5}\left(x^2\left(447681960\,h^5+1189081042\,h^4\,x_n+1035133330\,h^3\,x_n^2\right.\right.$$

$$\left.\left.+\,398591130\,h^2\,x_n^3+70475915\,h\,x_n^4+4673333\,x_n^5\right)\right)$$

$$-\frac{1}{45644940}\,\frac{1}{h^5}\left(x^3\left(1189081042\,h^4+2070266660\,h^3\,x_n+1195773390\,h^2\,x_n^2\right.\right.$$

$$\left.\left.+\,281903660\,h\,x_n^3+23366665\,x_n^4\right)\right)$$

$$+\frac{1}{9128988}\,\frac{x^4\left(103513333\,h^3+119577339\,h^2\,x_n+42285549\,h\,x_n^2+4673333\,x_n^3\right)}{h^5}$$

$$-\frac{1}{15214980}\,\frac{x^5\left(39859113\,h^2+28190366\,h\,x_n+4673333\,x_n^2\right)}{h^5}$$

$$\left.+\frac{1}{45644940}\,\frac{x^6\left(14095183\,h+4673333\,x_n\right)}{h^5}-\frac{667619}{45644940}\,\frac{x^7}{h^5}\right)g_{n+3}$$

$$+\left(-\frac{1}{5071660}\,\frac{1}{h^6}\left(x_n\left(77597244\,h^6+188937540\,h^5\,x_n+173019611\,h^4\,x_n^2+78305040\,h^3\right.\right.\right.$$

$$\left.\left.x_n^3+18810882\,h^2\,x_n^4+2297364\,h\,x_n^5+112207\,x_n^6\right)\right)$$

$$+\frac{1}{5071660}\,\frac{1}{h^6}\left(x\left(77597244\,h^6+377875080\,h^5\,x_n+519058833\,h^4\,x_n^2\right.\right.$$

$$\left.\left.+\,313220160\,h^3\,x_n^3+94054410\,h^2\,x_n^4+13784184\,h\,x_n^5+785449\,x_n^6\right)\right)$$

$$-\frac{3}{5071660}\,\frac{1}{h^6}\left(x^2\left(62979180\,h^5+173019611\,h^4\,x_n+156610080\,h^3\,x_n^2+62702940\,h^2\right.\right.$$

$$\left.\left.x_n^3+11486820\,h\,x_n^4+785449\,x_n^5\right)\right)$$

$$+\frac{1}{5071660}\,\frac{1}{h^6}\left(x^3\left(173019611\,h^4+313220160\,h^3\,x_n+188108820\,h^2\,x_n^2\right.\right.$$

$$\left.\left.+\,45947280\,h\,x_n^3+3927245\,x_n^4\right)\right)$$

$$-\frac{1}{1014332}\,\frac{x^4\left(15661008\,h^3+18810882\,h^2\,x_n+6892092\,h\,x_n^2+785449\,x_n^3\right)}{h^6}$$

$$+\frac{3}{5071660}\,\frac{x^5\left(6270294\,h^2+4594728\,h\,x_n+785449\,x_n^2\right)}{h^6}$$

$$\left.-\frac{1}{5071660}\,\frac{x^6\left(2297364\,h+785449\,x_n\right)}{h^6}+\frac{112207}{5071660}\,\frac{x^7}{h^6}\right)f_{n+4}$$

$$+\Bigg(\frac{1}{45644940}\,\frac{1}{h^5}\big(x_n\,(302338656\,h^6+743568210\,h^5x_n+690763489\,h^4x_n^2$$

$$+318400940\,h^3x_n^3+78152418\,h^2x_n^4+9774346\,h\,x_n^5+488933\,x_n^6)\big)$$

$$-\frac{1}{45644940}\,\frac{1}{h^5}\big(x\,(302338656\,h^6+1487136420\,h^5x_n+2072290467\,h^4x_n^2$$

$$+1273603760\,h^3x_n^3+390762090\,h^2x_n^4+58646076\,h\,x_n^5+3422531\,x_n^6)\big)$$

$$+\frac{1}{15214980}\,\frac{1}{h^5}\big(x^2\,(247856070\,h^5+690763489\,h^4x_n+636801880\,h^3x_n^2$$

$$+260508060\,h^2x_n^3+48871730\,h\,x_n^4+3422531\,x_n^5)\big)$$

$$-\frac{1}{45644940}\,\frac{1}{h^5}\big(x^3\,(690763489\,h^4+1273603760\,h^3x_n+781524180\,h^2x_n^2$$

$$+195486920\,h\,x_n^3+17112655\,x_n^4)\big)$$

$$+\frac{1}{9128988}\,\frac{x^4\,(63680188\,h^3+78152418\,h^2x_n+29323038\,h\,x_n^2+3422531\,x_n^3)}{h^5}$$

$$-\frac{1}{15214980}\,\frac{x^5\,(26050806\,h^2+19548692\,h\,x_n+3422531\,x_n^2)}{h^5}$$

$$+\frac{1}{45644940}\,\frac{x^6\,(9774346\,h+3422531\,x_n)}{h^5}-\frac{488933}{45644940}\,\frac{x^7}{h^5}\Bigg)g_{n+4}$$

$$+\Bigg(-\frac{1}{45644940}\,\frac{1}{h^5}\big(x_n\,(12371616\,h^6+31200840\,h^5x_n+30053054\,h^4x_n^2+14524675\,h^3x_n^3$$

$$+3781143\,h^2x_n^4+508001\,h\,x_n^5+27703\,x_n^6)\big)+\frac{1}{45644940}\,\frac{1}{h^5}\big(x\,(12371616\,h^6$$

$$+62401680\,h^5x_n+90159162\,h^4x_n^2+58098700\,h^3x_n^3+18905715\,h^2x_n^4+3048006\,h\,x_n^5$$

$$+193921\,x_n^6)\big)-\frac{1}{15214980}\,\frac{1}{h^5}\big(x^2\,(10400280\,h^5+30053054\,h^4x_n$$

$$+29049350\,h^3x_n^2+12603810\,h^2x_n^3+2540005\,h\,x_n^4+193921\,x_n^5)\big)$$

$$+\frac{1}{45644940}\,\frac{1}{h^5}\big(x^3\,(30053054\,h^4+58098700\,h^3x_n+37811430\,h^2x_n^2$$

$$+10160020\,h\,x_n^3+969605\,x_n^4)\big)$$

$$-\frac{1}{9128988}\,\frac{x^4\,(2904935\,h^3+3781143\,h^2x_n+1524003\,h\,x_n^2+193921\,x_n^3)}{h^5}$$

$$+\frac{1}{15214980}\,\frac{x^5\,(1260381\,h^2+1016002\,h\,x_n+193921\,x_n^2)}{h^5}$$

$$-\frac{13}{45644940}\,\frac{x^6\,(39077\,h+14917\,x_n)}{h^5}+\frac{27703}{45644940}\,\frac{x^7}{h^5}\Bigg)g_{n+5} \qquad (4)$$

We now interpolate (4) at $x_{n+3}, x_{n+4}, x_{n+5}$, and collocate the first and second derivative of (4) at $x_n, x_{n+1}, x_{n+2}, x_{n+5}$ and x_n, x_{n+1}, x_{n+2}, to obtain the following equations:

$$
\left.
\begin{aligned}
y_{n+5} &= \frac{3200}{253583}y_n - \frac{38449}{253583}y_{n+1} + \frac{288832}{253583}y_{n+2} + \frac{467964}{253583}hf_{n+3} + \frac{260736}{253583}hf_{n+4} \\
&\quad - \frac{106844}{760749}h^2 g_{n+3} + \frac{543712}{760749}h^2 g_{n+4} + \frac{56668}{760749}h^2 g_{n+5} \\
y_{n+4} &= \frac{9891}{1267915}y_n - \frac{136832}{1267915}y_{n+1} + \frac{1394856}{1267915}y_{n+2} + \frac{1095552}{1267915}hf_{n+3} + \frac{1323228}{1267915}hf_{n+4} \\
&\quad - \frac{791904}{1267915}h^2 g_{n+3} - \frac{375528}{1267915}h^2 g_{n+4} + \frac{9888}{1267915}h^2 g_{n+5} \\
y_{n+3} &= \frac{9454}{1267915}y_n - \frac{132453}{1267915}y_{n+1} + \frac{1390914}{1267915}y_{n+2} + \frac{444918}{1267915}hf_{n+3} + \frac{709452}{1267915}hf_{n+4} \\
&\quad - \frac{916896}{1267915}h^2 g_{n+3} - \frac{274332}{1267915}h^2 g_{n+4} + \frac{8682}{1267915}h^2 g_{n+5} \\
y_{n+2} &= -\frac{580}{44217}h^2 g_{n+5} + \frac{3836}{14739}h^2 g_{n+4} - \frac{1228}{14739}h^2 g_{n+3} - \frac{46106}{44217}g_{n+2}h^2 - \frac{2494}{4913}hf_{n+4} \\
&\quad + \frac{7184}{4913}hf_{n+3} + \frac{5136}{4913}y_{n+1} - \frac{223}{4913}y_n \\
y_{n+1} &= \frac{28705}{45984}h^2 g_{n+5} - \frac{29395}{1916}h^2 g_{n+4} - \frac{650905}{22992}h^2 g_{n+3} - \frac{253583}{45984}g_{n+1}h^2 + \frac{67899}{1916}hf_{n+4} \\
&\quad - \frac{13956}{479}hf_{n+3} - \frac{10159}{3832}y_{n+2} + \frac{13991}{3832}y_n \\
y_n &= \frac{31516}{130377}h^2 g_{n+5} - \frac{751079}{130377}h^2 g_{n+4} - \frac{452204}{43459}h^2 g_{n+3} + \frac{23053}{130377}g_n h^2 + \frac{572538}{43459}hf_{n+4} \\
&\quad - \frac{469008}{43459}hf_{n+3} - \frac{146989}{43459}y_{n+2} + \frac{190448}{43459}y_{n+1} \\
11411235 f_{n+5}h &= 13000043h^2 g_{n+3} + 23046536h^2 g_{n+4} + 3268379h^2 g_{n+5} + 25658451hf_{n+3} \\
&\quad - 15199776hf_{n+4} + 136128y_n - 1224816y_{n+1} + 1088688y_{n+2} \\
22822470 f_{n+2}h &= -29521364h^2 g_{n+3} - 11479298h^2 g_{n+4} + 390748h^2 g_{n+5} - 13430448hf_{n+3} \\
&\quad + 28649448hf_{n+4} + 543201y_n - 8689872y_{n+1} + 8146671y_{n+2} \\
1267915 f_{n+1}h &= 2937351h^2 g_{n+3} + 1404792h^2 g_{n+4} - 53277h^2 g_{n+5} + 2406267hf_{n+3} \\
&\quad - 3345872hf_{n+4} - 167544y_n - 1872432y_{n+1} + 2039976y_{n+2} \\
1267915 f_n h &= -15705048h^2 g_{n+3} - 8398296h^2 g_{n+4} + 343656h^2 g_{n+5} - 15496736hf_{n+3} \\
&\quad + 19399311hf_{n+4} - 3395358y_n + 9425376y_{n+1} - 6030018y_{n+2}
\end{aligned}
\right\} \quad (5)
$$

Stability Analysis of Block Scheme

The schemes (5) are represented in block form as:

$$AY_m = ay_m + hBF_m + hbf_m + h^2CG_m + h^2cg_m \quad (6)$$

where, $Y_m = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}]^T$, $y_m = [y_n]^T$, $f_m = [f_n]^T$, $F_m = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}]^T$, $g_m = [g_n]^T$ and $G_m = [g_{n+1}, g_{n+2}, g_{n+3}, g_{n+4}, g_{n+5}]^T$.

A, B, C and a, b, c are 10 by 5 and 10 by 1 matrices respectively.

Order and Error Constant

As it has been employed in Atsi and Kumleng (2021), the order and error constant of the new block method are presented thus:

the order is,

$$p = [7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7]^T$$

with error constant,

$C_{p+1} =$

$$\left[-\frac{1240237}{479271870}, -\frac{143197}{133131075}, -\frac{151607}{152149800}, \frac{16591}{5306040}, -\frac{8037893}{57939840}, -\frac{19760569}{328550040}, -\frac{111152819}{2520}, -\frac{157303543}{2520}, \frac{1241837}{120}, -\frac{1602141}{20}\right]$$

Consistency

The new block scheme is consistent, since $p \geq 1$, see Atsi and Kumleng (2020).

Zero Stability

The root of the first characteristic polynomial satisfies $|r| \leq 1$, are, $|r| = [0, 0, 0, 0, 0, 0, 0, 0, 1]$ or $[1]$. Hence, the new block method is zero-stable, see Chollom *et al.,* (2007).

Convergence

As exploited in Atsi and Kumleng (2020), the new block scheme is convergent, since it is both consistent and zero stable.

Numerical Implementation

The proposed method was tested in the following examples:

Example 1: see Atsi and Kumleng (2021)

$$y'' = -1001\, y' - 1000\, y, \qquad y(0) = 1, y'(0) = -1, h = 0.1$$

Exact Solution: $y(x) = e^{-x}$

The numerical results of this problem and comparisons are presented in Table 1.

Example 2: see Ukpebor (2019)

$$y'' = y', y(0) = 0, y'(0) = -1, h = 0.1$$

with the exact solution $y(x) = 1 - \exp(x)$

RESULTS AND DISCUSSION

Table I Comparisons of errors in the solutions of example 1

x	Absolute Error in Atsi and Kumleng (2021)	Absolute Error in New MBDF Method
0.1	$8.0e-10$	$6.9867e-12$
0.2	$4.1e-9$	$1.0029e-12$
0.3	$9.3e-9$	$7.8587e-12$
0.4	$3.2e-9$	$1.0472e-11$
0.5	$7.0e-10$	$6.3223e-11$
0.6	$2.3e-9$	$1.0047e-11$
0.7	$6.0e-10$	$9.3665e-12$
0.8	$3.5e-9$	$2.6451e-12$
0.9	$6.6e-9$	$1.0685e-11$
1.0	$3.3e-9$	$2.3268e-11$

Table II Comparisons of errors in the solutions of example 2

x	Absolute Error in Ukpebor (2019)	Absolute Error in New MBDF method
0.1	$3.354884e-009$	$1.0950e-10$
0.2	$3.304564e-008$	$2.8252e-10$
0.3	$1.233789e-007$	$4.6576e-10$
0.4	$3.155858e-007$	$6.6236e-10$
0.5	$6.588997e-007$	$9.2662e-10$
0.6	$1.211782e-006$	$1.5125e-09$
0.7	$2.043321e-006$	$2.2458e-09$
0.8	$3.234818e-006$	$3.0430e-09$
0.9	$4.881592e-006$	$3.9143e-09$
1.0	$7.095032e-006$	$4.9548e-09$

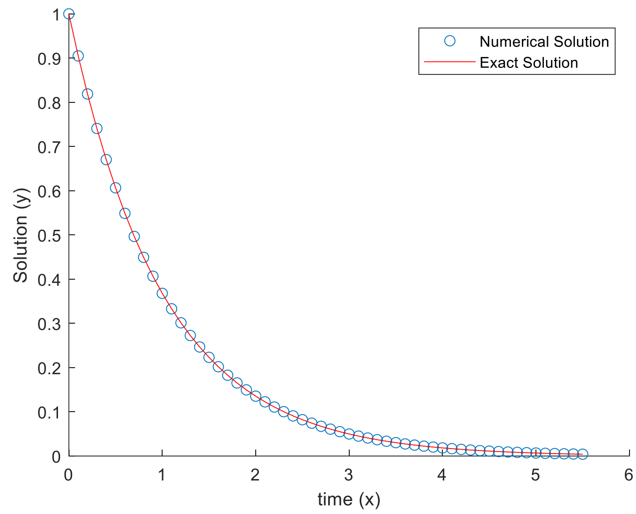
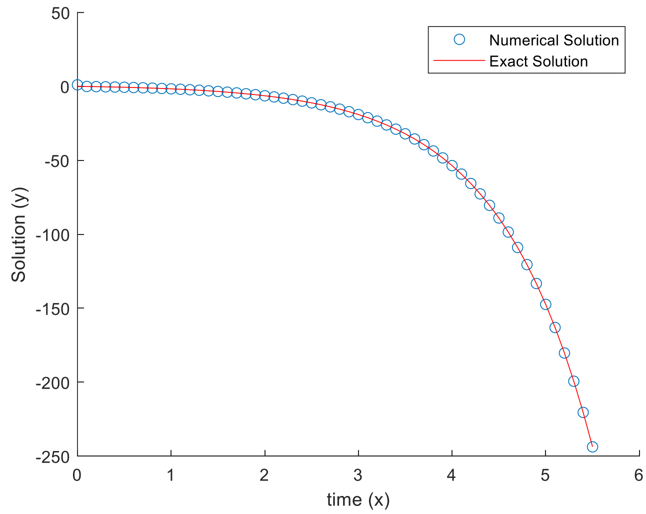


Figure I showing the exact solution and numerical solution for example 1

Figure II showing the exact solution and numerical solution for example 2



DISCUSSION OF FINDINGS

Two problems of IVP of ODEs were solved with step size, $h = 0.1$, in each example. In the tabulated error (exact/numerical) results, it can be seen that the new four-step modified backward differentiation formula (MBDF) method presents minimum errors in comparison with the errors in Atsi and Kumleng (2021) and Ukpebor (2019) for example 1 and 2, respectively. The new MBDF method demonstrated a good accuracy with the exact solution as seen in figure 1 and 2 for example 1 and 2, respectively. Hence, the new MBDF method may be suitable for solving initial valued problems of ODEs.

CONCLUSION

In this research Step number $k = 4$, was considered, resulting to ten discrete methods used in block for solution of two different second-order ODEs. The construction, analysis and implementation of the method were done in Maple and Matlab mathematical applications. In comparison of results, the new method competes better than other existing methods. The authors in the next work will work on the construction methods using higher step numbers and establishing the A-stability of the method.

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