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Axiomatization Multisets: A Comparative Analysis

[1]Balogun F., [2*]Wahab O. A. and [3]Isah A. I.
[1,] Department of Mathematics,
Federal University Dutsin-Ma

[2]Department of Mathematics,
Federal College of Education,
Katsina

[3]Department of Mathematical Sciences,
Kaduna State University

Email: ojowahab@gmail.com

Abstract
A multiset, unlike the classical set, allows for multiple instances of its elements. In this paper, we present a comparative analysis of theories on multisets. In particular, we examine, the first-order two-sorted multiset theory MST, and the single-sorted multiset theory MS that employs the same sort for multiplicities and the set they support. The logical strengths and significance of some axioms presented in these theories are investigated. The theory MST contains a copy of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) but is independent of ZFC. The single-sorted multiset theory describes a stronger theory that mirrors the Zermelo-Fraenkel set theory (ZF) and is equiconsistent with ZF and antifoundation. The two-sorted multisettheory MST is a conservative extension of the classical set theory, making it a suitable theory to assume when dealing with studies that involve multisets.

Keywords: Axiomatic system, Multisets, First-order logic, and Multiset theory.

INTRODUCTION
A multiset consists of elements, but the notion of a multiset is distinguished from that of a set by carrying information of how many times each element occurs in a given collection. In essence, multisets generalize the classical sets, and have practical implications in mathematics, physics, philosophy, logic, linguistics, and computer science (Balogun et al, 2020, 2021, 2022; Singh et al 2007; Blizard, 1988, 1991; Clement, 1988; Dershowitz & Manna, 1979). For instance, Dershowitz and Manna (1979) used multisets to prove that certain computer programs terminate. Relative to the classical sets, studies on axiomatization of multisets is yet to receive ample attention. Researchers such as Blizard (1988, 1991), Dang (2014), and Felisiak et al., (2020) have propounded some axioms in the context of multisets on the basis of the Zermelo-Fraenkel set theory *ZF*. Unlike the classical set, there is still no unified approach to axiomatizing multisets. In this work, a comparative analysis of some existing multiset theories is presented. The similarities and peculiarities of these multiset theories are discussed. Also, the implications of some of the modified axioms are investigated. We begin section 1.1 with notations and definitions of basic terminologies that will be used in the study. In section 2, the multiset theories to be investigated in this work are presented, in each case, the generalized ZFC axioms and axioms that are peculiar to the theory under consideration are highlighted. In section 3, a comparative analysis of the multiset theories from section 2 is presented. Concluding remarks and recommendations are proposed in section 4.

**Author for Correspondence*

NOTATIONS, DEFINITIONS, AND BASIC TERMINOLOGIES

The language of multiset theory is the first-order language having a single ternary relation, that relates two sets (in terms of membership) and the number of times one set belongs to another. The usual notation for universal and existential quantifiers ∀ and ∃, respectively, are used (∃! is used for *there exist a unique element/value*). Also, conventional logical symbols viz;~ (not), ∧ ($conjunction$), ∨ ($disjunction$), → (implication), ↔ ($biconditional$) are adopted. A three place predicate, e.g. $x \in_z y$, is usually employed for multisets, which reads x is in y exactly z times. As usual, *wff* will stand for a well-formed formula. A multiset M over a set S is a cardinal-valued function (also called generalized characteristic function) $M: S \to \aleph$ such that $x \in Dom(M)$ implies that $M(x)$ is a cardinal number. A relation R on a multiset is well-ordered if it is a total order and any nonempty multiset $M \subset Dom\,R$ has an R −minimal member. Distinct elements in a multiset are sometimes referred to as *objects*, while each occurrence of an object is called an element of the multiset. The number of occurrences of an object is called the multiplicity of the object.

As we proceed, more symbols and terms will be defined as they are introduced.

Multiset Theories

Multiset theories to be studied in this work will be presented in this section. Axioms that generalize the Zermelo-Fraenkel set theory (see Balogun & Wahab, 2022) will be highlighted and those peculiar to the theory under consideration will be discussed.

First-order Two-sorted Multiset Theory MST (Blizard, 1988)

Blizard (1988) describes a set as a multiset in which objects occur only once. The author initiated the first-order two-sorted theory MST, in which the naïve concept of a multiset is formalized based on the following assumptions:

a. A multiset is a collection M with repeated elements
b. Repeated elements are indistinguishable
c. Each occurrence of an element in the collection M contributes to the cardinality of M
d. The multiplicity of an object in M is a finite (positive) integral
e. Objects in M are not necessarily finite
f. The collection M is completely determined if its objects and their multiplicities are known.

A number of existing multiset theories are based on these assumptions with some differing only in the range of the characteristic function.

The language L of MST employs two sorts of variable symbols: multiset variable symbols $x, y, z, \ldots$ used to denote multisets and elements of multisets, and numerical variable symbols $k, l, m, n, \ldots$ used to denote multiplicities of objects in multisets. These two sorts of variables of L range over disjoint universes: M(multisets) and N(numbers). The theory MST implicitly assumes the axioms of Peano Arithmetic (PA) for its numeric variables. As noted by Blizard, the two-sorted language L is only used for convenience, as the same formulation can be achieved with a first-order one-sorted language.

Axioms of MST

One of the axioms peculiar to the multiset theory MST is the exact multiplicity axiom. Most researches on multisets adopt this axiom. An alternative approach (dynamic multisets) exists in the literature (William-West et al., 2021; Pagh et al.,2004), where the multiplicity of an object in a given multiset can vary with time – this has practical implications.

E*xact multiplicity* axiom

$$\forall x \forall y \forall n \forall m\ ((x \in^n y \wedge x \in^m y) \rightarrow n = m)$$

The axiom implies that the multiplicity with which an element belongs to a multiset is unique. The next axiom is a generalization of the ZF axiom of extensionality for multisets.

Axiom of extensionality

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\forall \boldsymbol{z} \forall \boldsymbol{n} (\boldsymbol{z} \in^{\boldsymbol{n}} \boldsymbol{x} \leftrightarrow \boldsymbol{z} \in^{\boldsymbol{n}} \boldsymbol{y}) \rightarrow \boldsymbol{x} = \boldsymbol{y})$$

If two multisets $x\ and\ y$ have exactly the same objects occurring with exactly the same multiplicities, then they are equal. The converse of this axiom follows by substitutivity of equality in the logic. Observe that with this axiom, unlike the Cantorian sets, two multisets with the same objects are not necessarily equal; rather, they are called *cognate* or *similar* multisets.
Another generalization of the ZF axioms is the empty set axiom for multisets.

E*mpty set* axiom

$$\exists \boldsymbol{y} \forall \boldsymbol{x} \forall \boldsymbol{n} (\sim \boldsymbol{x} \in^{\boldsymbol{n}} \boldsymbol{y})$$

The above axiom asserts that there is at least one multiset with no elements. The empty multiset constructed via this axiom is unique by the axiom of extensionality of MST and is denoted by $\emptyset$.Clearly$\forall x\ x \notin \emptyset$. The predicate $Set(u)$ is introduced in the theory MST, this defines

$$u = \emptyset \vee \forall x \forall n (x \in^n u \rightarrow n = 1)$$

Thus the empty multiset $\emptyset$ is reduced to the empty set since $Set(\emptyset)$ is true.
Blizard (1988) defines the inclusion relation for all multiset terms u and v via the formula: $\forall z \forall n (z \in^n v \rightarrow \exists m (n \leq m \wedge z \in^m v))$ which implies $u \subseteq v$ i.e. u is a *submultiset* of v (the term *msubset* is used for this notion in MST, and the multiset$v$ is called the parent multiset). The inclusion relation $\subseteq$ as defined in MST is reflexive and transitive. Through a combination of the exact multiplicity axiom and extensionality, it can be shown that $\subseteq$ is antisymmetric. Also, $\forall x \forall y ((x \subseteq y \wedge Set(y) \rightarrow Set(x))$ holds. Blizard's definition of the inclusion relation for multisets appears to be the most sustained in the literature compared to for instance, the definition proposed in Meyer and McRobbie (1982), where the inclusion relation is defined thus: If $\forall z (z \in x \rightarrow z \in y)$ then $x \subseteq y$. The relation $\subseteq$ as defined by Meyer & McRobbie is not antisymmetric. Their definition does not take into account the multiplicity of the term z and hence, does not preserve the principle that the cardinality of a submultiset is at most the cardinality of the parent multiset. *Whole* and *full* submultisets are introduced in MST via the following formulas:
$\forall z \forall n (z \in^n u \rightarrow z \in^n v)$ implies u is a whole submultiset of v, where
$u \subseteq v \wedge \forall z (z \in v \rightarrow z \in u)$ implies u is a full submultiset of v.
Notions of simple and regular multisets are also introduced in MST.

Singleton and Pair set axioms

Analogs of the singleton and pair set axioms are presented in MST based on the following:

i. $\forall x \forall n \exists y (x \in^n y \wedge \forall z (z \in y \leftrightarrow z = x))$
ii. $\forall x \forall y (x \neq y \rightarrow \forall n \forall m \exists z (x \in^n z \wedge y \in^m z \wedge \forall z' (z' \in z \leftrightarrow (z' = x \vee z' = y))))$

The first condition implies that given any multiset x, there is a multiset y that contains exactly n copies of x. The multiset y is unique by extensionality. Thus $[u]_1$ denotes the singleton $\{u\}$. By condition ii, the multisets x and y are required to be distinct, this makes the axiom consistent with the exact multiplicity axiom. In line with this, the pair $\{x, x\}$ does not exist in MST. For any two multiset terms u and v, the ordered pair set is given by

$$\langle u, v\rangle = \begin{cases} \{\{u\}, \{u, v\}\}, & if\ u \neq v \\ \{\{u\}, [u]_2\}, & if\ u = v \end{cases}$$

$set(\langle u, v\rangle)$ is true, and the multiset terms u,v are distinct elements. This differs from ZF since the Kuratowski pairs $\langle x, x\rangle$ which are singletons do not exist in MST.
The power set axiom described below generalizes the classical form of the axiom presented in the Zermelo-Fraenkel set theory with some limitations.

Power set axiom

$$\forall x \exists y (Set(y) \wedge \forall z (z \in y \leftrightarrow z \subseteq x))$$

For every multiset$x$ there is a set whose elements are exactly the submultisets of x. The set y is the power set of x and it is usually denoted by $\mathbb{P}(x)$. The power set axiom is unique by the following axiom which is a restriction of the extensionality axiom of MST to sets:

$$\forall x \forall y (Set(x) \wedge Set(y)) \rightarrow (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Example
Let$u = [x, y]_{3,1}$ then $\mathbb{P}([x, y]_{3,1}) = \{\emptyset, \{x\}, [x]_2, [x]_3, \{y\}, \{x, y\}, [x, y]_{2,1}, [x, y]_{3,1}\}$
This example shows that $\mathbb{P}(u)$ contains strictly less than 2^n elements, since by definition $Set(\mathbb{P}(u))$ holds. Hickman (1980) suggests that there is no good reason for admitting repeated elements in a power set. With this assumption, Cantor's power set theorem cannot be generalized to multisets. Another definition,$\breve{\mathbb{P}}(x)$, that preserves Cantor's theorem is proposed in MST, but not adopted (in order to avoid generating infinite multiplicities).
The next axiom is a formulation of the Zermelo-Fraenkel axiom of foundation for the theory MST.

Axiom of foundation

$$\forall y (y \neq \emptyset \rightarrow \exists x (x \in y \wedge \forall z (z \in x \rightarrow z \notin y)))$$

This axiom states that every nonempty multiset contains an element that is disjoint with it. The axiom ensures that there are no infinite descending $\in -$chains of the form $\ldots \in x_3 \in x_2 \in x_1$ within a multiset. Also, multisets such as $x = \{x\}$ and $\in -$loops of the form $x_1 \in x_2 \in \cdots \in x_n = x_1$ are avoided in MST.
Next, two forms of the axiom of union in MST are presented. The first form, just like the classical case, takes the maximum value of the multiplicities of an object in the multiset under consideration. While the additive union is introduced to deal with instances when the sum of the multiplicities of an object in the multisets under consideration is required.

Axiom of union

$$\forall x \exists z' \forall z \forall n (z \in^n z' \leftrightarrow \exists y (z \in^n y \wedge y \in x) \wedge (\forall y \forall m ((z \in^m y \wedge y \in x) \rightarrow m \leq n) \vee$$
$$(\forall m \exists y \exists k (z \in^k y \wedge y \in x \wedge m \leq k) \rightarrow \forall y \forall m ((z \in^m y \wedge y \in x) \rightarrow n \leq m))))$$

For any multiset x there is a multiset $\boldsymbol{z}'$which consists of all elements of elements of x. The multiplicity n of z in $\boldsymbol{z}'$is the maximum multiplicity of z as an element of element of x, if such a maximum exists. This maximum exists for finite multisets, in the infinite case, the minimum multiplicity is taken – this always exists.
The binary multiset union operation $\cup$ is given by:
$u \cup v = \cup\{u, v\}$ for distinct multisets terms u, v, otherwise,

$$u \cup u = \cup\{u\}$$

This multiset union in MST has the following property for any two multisets:

$$\forall x \forall y \forall z \forall n (z \in^n x \cup y$$
$$\leftrightarrow [(z \in^n x \wedge z \notin y) \vee (z \notin x \wedge z \in^n y) \vee \exists k \exists l (z \in^k x \wedge z \in^l y \wedge n = \max(k, l))])$$

Also, for any multiset x in MST, $x \cup x = x$ by extensionality.

If x is finite, then the sum of products of multiplicities of elements of elements of x is always finite.

Additive union axiom

The multiset $x \uplus y$ has the following property:

$$\forall x \forall y \forall z \forall n(z \in^n x \uplus y \leftrightarrow [(z \in^n x \wedge z \notin y) \vee (z \notin x \wedge z \in^n y)]) \vee$$
$$\exists k \exists l(z \in^k x \wedge z \in^l y \wedge n = k + l)])$$

The next two axioms of MST to be discussed are axiom schemata formed via a combination of other axioms.

Axiom schema of separation

This axiom states that for every $wff \varphi(x,n)$ of L with free variables including x and n but excluding y and n' the universal closure

$$\forall x \forall n \forall n'((\varphi(x,n) \wedge \varphi(x,n')) \rightarrow n = n' \rightarrow$$
$$\forall z \exists y \forall x \forall n(x \in^n y \leftrightarrow [x]_n \subseteq z \wedge \varphi(x,n))$$

is an axiom of MST. The formula $\varphi(x,n)$ must be "functional", this ensures that the multiset y is well-defined. Each formula produces an axiom and for each axiom, the multiset y is contained in z since $\forall x \forall n(x \in^n y \rightarrow [x]_n \subseteq z)$. Multisets that are produced by the separation schema are submultisets of multisets that exist by already established axioms.

In MST, the separation schema is used to prove the existence of root sets, intersection multiset and relative complement multisets. For example, if $\varphi(x,n)$ is given by $x = x \wedge n = 1$, by the separation schema we have $\forall z \exists y \forall x \forall n(x \in^n y \leftrightarrow (\{x\} \subseteq z \wedge n = 1))$. In the separation schema, the multiset y is the root set z^* of z such that $\forall z Set(z^*)$.

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or any multisets term v the intersection of v is denoted by $\cap v$. If follows from the separation schema that $\cap x \subseteq \cup x$, this produces the desired relation $\cap x \subseteq \cup x \subseteq \uplus x$.

In general, $\cap x$ is the unique set that contains all elements of elements of x that belong to every element of x. The multiplicity of an element in $\cap x$ is the minimum multiplicity of that element as an element of every element of x. For multiset terms u and v (resp. numeric terms s, t), the binary intersection operation $\cap$ (resp. binary function *min*) is defined as follows:

$$u \cap v = \begin{cases} \cap \{u,v\} & if\ u \neq v \\ \cap \{u\} & if\ u = v \end{cases} \text{ (resp. } \min(s,t) = \begin{cases} s & if\ s \leq t \\ t & otherwise \end{cases} \text{)}$$

The terms u and v are disjoint if $u \cap v = \emptyset$.

The multiset intersection as defined in MST has the usual properties of multiset intersection as follows:

$\forall x \forall y \forall z \forall n(z \in^n x \cap y \leftrightarrow \exists k \exists l(z \in^k x \wedge z \in^l y \wedge n = \min(k,l)))$. Also, by the law of trichotomy in PA, the function *min* is well-defined.

In MST, the relative complement is defined via the separation schema as follows:

For the numeric terms, we know that $PA \vdash \forall m \forall k(k < m \rightarrow \exists! n(m = k + n))$, the unique n is denoted by $m - k$. The relative complement is given by the following wff:

$\varphi(x,n) = (x \in^n z \wedge x \notin y') \vee \exists m \exists k(x \in^m z \wedge x \in^k y' \wedge k < m \wedge n = m - k)$, where $\varphi(x,n)$ is functional. Given multiset terms u and v with $v \subseteq u$, the relative complement of v in u is contained in u.Contrary to what holds in ZF, in MST an element can occur in a submultiset and in its relative complement, since MST admits repetition of elements (Hickman, 1980 describes this as rather counter intuitive).

Axiom schema of replacement

This axiom schema states that if the $wff \varphi(x,y)$ of L is a functional then for any multiset z there is a unique multiset z' that is the image of z under φ. The least multiplicity is taken whenever we have more than one x in z that is mapped to y by φ.This condition (taking the least multiplicity) is imposed to ensure that the axiom schema of replacementis consistent with the exact multiplicity axiom of MST. The universal closure of

$$\forall x \forall y \forall y'((\varphi(x,y) \wedge \varphi(x,y')) \rightarrow y = y') \rightarrow$$

$\forall z \exists z' \forall y \forall n(y \in^n z' \leftrightarrow [\exists x(x \in^n z \wedge \varphi(x,y)) \wedge \forall x \forall m((x \in^m z \wedge \varphi(x,y)) \rightarrow n \leq m)])$ is an axiom of MST, where x and y are free variables in $\varphi(x,y)$ but y' and z' are not. Unlike in ZF, the separation schema cannot be deduced from the replacement schema in MST due to the restrictions on the replacement schema in MST.

Infinite multiset axiom

Just as in ZF, the infinite multiset axiom is stated thus:

$$\exists y(\emptyset \in y \wedge \forall x(x \in y \rightarrow x \cup \{x\} \in y))$$

Let $x = \hat{n} \wedge y = [z]_n$ be the $wff \varphi(x,y)$ in the replacement schema, this produces the infinite set $\{\{z\}, [z]_2, [z]_3, \dots\}$. This can be used to illustrate the operations of union, intersection and additive union in MST. Let x be the infinite set then, $\cup x = \uplus x = \cap x = \{z\}$, since the only multiplicity of z as an element of elements of x that exists is its minimum multiplicity.

Choice multiset axiom

For a multiset$y$, any multisety' that satisfies the following axiom is a choice multiset for y:

$$\forall y[[y \neq \emptyset \wedge \forall x(x \in y \rightarrow x \neq \emptyset) \wedge$$
$$\forall x \forall z((x \in y \wedge z \in y \wedge x \neq z) \rightarrow x \cap z = \emptyset)] \rightarrow$$
$$\exists y'(\forall x \forall n(x \in^n y \rightarrow \exists x'(x' \in^n y' \wedge x' \in x \ \wedge$$
$$\forall x''((x'' \in x \wedge x'' \in y') \rightarrow x'' = x'))) \wedge$$
$$\forall x' \forall n(x' \in^n y' \rightarrow \exists x(x \in^n y \wedge x' \in x)))].$$

Corresponding to elements$x$ in $y$are elements x' in y' having the same multiplicity. A choice multiset is unique if every element x of y is a simple multiset. If $Set(y)$ then y' is a choice set for y since $Set(y')$. If the condition $HSet(y)$ (i.e. hereditary set y) is included in the choice multiset axiom, then $HSet(y')$ follows,and the wff obtained is equivalent to the classical axiom of choice of ZFC.

Single-sortedmultiset theory MS (Dang, 2014)

The single-sortedmultiset theory is a one-sorted account of multisets with multiplicities and sets coming from the same domain and following the same axioms. This theory assumes that multiplicities are no longer predefined cardinal numbers, rather they are treated as multisets with their own internal structures. In the single-sorted theory, a number x is intuitively regarded as being less than y (multiplicities) if $x \subset y$. The axioms proposed equally mirror the ZF theory with some exceptions.

In this multiset theory, the language $\mathcal{L}_\mathcal{M}$ has one sort of variable and, as usual, two predicate symbols viz, the identity/equality =, and the ternary symbol ∈. Dang (2014) uses over-lined symbols for the membership and subset relations to differentiate them from their corresponding symbols in classical set theory.
The axioms and axiom schemata of the single-sorted multiset theory MS are as follows: unique multiplicity, extensionality, empty set, comprehension, pairing, subset, union, replacement, power set, and infinity.

Axiom of unique multiplicity

The single-sorted theory assumes the exact multiplicity axiom of MST. Again we have, $(\forall x, y, a, b)(x \in_a y \wedge x \in_b y \rightarrow a = b)$.

Axiom of extensionality
Similar to the theory MST, the axiom of extensionality is stated thus:
$(\forall x, y)(x = y \leftrightarrow (\forall a, b)(a \in_b x \leftrightarrow a \in_b y))$.
The difference being that the variables x, y, a, b are from the same domain.

Axiom of empty set
The empty set axiom is given by:

$$(\exists x)(\forall y)\ y \overline{\notin}\ x$$

As usual, the empty set is unique by extensionality and it is denoted by $\overline{\emptyset}$.

Axiom schema of comprehension
For all formulas φ with two free variables and possibly parameters, the axiom schema of comprehension is as follows:

$$(\forall x)(\exists y)(\forall z, b)(z \in_b y \leftrightarrow z \in_b x \wedge \varphi(z, b))$$

The set obtained via this axiom schema, for each formula, inherits the multiplicities of the initial set.

Remark
In the single-sorted theory a multiset with the property $(\forall x, y)(x \in_y a \leftrightarrow \varphi(x, y))$ is denoted by $\{x \otimes y: \varphi(x, y)\}$, where $\{x_1 \otimes y_1, \ldots, x_n \otimes y_n\}$ represents $(\forall x, y)(x \in_y a \leftrightarrow (x = x_1 \wedge y = y_1) \vee \ldots \vee (x = x_n \wedge y = y_n))$.
These multisets if they exist are unique by extensionality.

Axiom of pairing
The pairset axiom for multisets in MS is stated as follows:

$$(\forall x, y)(\exists a) a = \{x \otimes \overline{\emptyset}, y \otimes \overline{\emptyset}\}$$

Where the ordered pair consisting of the elements x and y is given by,

$$\langle x, y \rangle := \{\{x \otimes \overline{\emptyset}\} \otimes \overline{\emptyset}, \{x \otimes \overline{\emptyset}, y \otimes \overline{\emptyset}\} \otimes \overline{\emptyset}\}$$

Axiom of union
In the single-sorted theory, the axiom of union is stated as follows:

$$(\forall x)(\exists b)(\forall a)(b \sqsubset a \leftrightarrow (\forall y \,\overline{\in}\, x) y \sqsubset a)$$

As in classical set theory, the union of x is denoted by $\cup x$, and the binary union$\cup \{x \otimes \overline{\emptyset}, y \otimes \overline{\emptyset}\}$ is denoted by $x \cup y$.

Axiom schema of replacement
For all formulas φ with two free variables and possibly with parameters, the axiom schema of replacement is given by,
$(\forall x)((\forall a \,\overline{\in}\, x)(\exists! b)\varphi(a, b) \rightarrow (\exists y)(\forall b, d)(b \in_d y \leftrightarrow (\exists a \,\overline{\in}\, x)\varphi(a, b) \wedge (\forall e)(d \sqsubset e \leftrightarrow (\forall a \,\overline{\in}\, x)(\varphi(a, b) \rightarrow \frac{x}{a} \sqsubset e))))$.
The multiset obtained via this axiom schema is unique for each x and for each functional φ. Recall that multisets and their multiplicities come from the same domain and follow the same axioms, the author proved via the schema of multiplicity replacement that the multiplicity of $a \,\overline{\in} \bigcup x$ is the union of multiplicities of a in all $b \,\overline{\in}\, x$.

Axiom of power set
The power set axiom of the single-sorted multiset theory is presented as follows:

$$(\forall x)(\exists y) y = \mathcal{P}x$$

Where the canonical power set of x is given by

$$\mathcal{P}x := \{y \otimes \overline{\emptyset} : y \sqsubset x\}$$

Axiom of infinity
In order to present the axiom of infinity in the MS theory, an analogue of the von Neumann $\omega$is first defined.
$\alpha \in ON$ represents the formula given by the following:
$\forall x \,\overline{\in}\, \alpha(\frac{\alpha}{x} = \overline{\emptyset} \wedge (\forall y \,\overline{\in}\, x)\left(y \,\overline{\in}\, \alpha \wedge \frac{x}{y} = \overline{\emptyset}\right))$where $\frac{y}{x}$ represents the unique multiplicity of x in y,and the relation $\overline{\in}$ restricted to α is well-ordered.
Also, $\alpha^+ = \alpha \cup \{\alpha \otimes \overline{\emptyset}\}$ and
$\alpha < \beta$ implies $\alpha \,\overline{\in}\, \beta$ and α, β are both ordinals
The axiom of infinity is thus stated:

$$(\exists x \in ON)(\overline{\emptyset} \in_{\overline{\emptyset}} x \wedge (\forall y \,\overline{\in}\, x)\left(\left(y = \overline{\emptyset} \vee (\exists z \,\overline{\in}\, x) y = z^+\right) \wedge y^+ \in_{\overline{\emptyset}} x\right))$$

The multiset produced by the axiom of infinity is denoted $\overline{\omega}$ and $n + 1$ stands for n^+.
Remark
Dang (2014) extended the multiset theory MS by proposing an additional axiom – *axiom of finite multiplicities* - to handle multiplicities.

$$(\forall x, y, a)(x \in_a y \rightarrow a \,\overline{\in}\, \overline{\omega}) \wedge (\forall x)(\forall a \,\overline{\in}\, \overline{\omega})(\exists y) y = \{x \otimes a\}$$

The new theory, denoted $MS_{\overline{\omega}}$, consists of all the axioms of MS with the finite multiplicities axiom. The theory $MS_{\overline{\omega}}$ is equivalent to the theory MST, however the multiplicity n in $MS_{\overline{\omega}}$ corresponds to the multiplicity $n + 1$ in MST.

DISCUSSION

The two multiset theories discussed in this work are formulated on the basis of first-order logic. However, the multiset theory MST uses two sorts of variable symbols with different sets of axioms for each sort, while the theory MS is a single-sorted theory having same variables and axioms for multisets and multiplicities. The following axioms are common to both multiset theories: exact multiplicity, extensionality, empty set, pairing, union, power set and infinity. Also, both theories have the following axiom schemata in common: replacement, comprehension (or separation as the case may be). The inclusion relation employed in MST is, as usual, antisymmetric, while in MS, a definition that is antifoundational is introduced. However, Dang (2014) describes additional axiom that makes this relation antisymmetric. Also, in the theory MS new axioms are included to deal with multiplicities separately. For instance, the axiom of finite multiplicities is introduced; this axiom extends MS and makes it equivalent to MST.Dang also showed that the consistency strength of the multiset theory MS is equivalent to that of ZF by proving that, consistency of MS implies that of ZF. The multiset theory MST extends and contains a copy of the ZFC theory. Also, when considering only hereditary sets in MST, the algebra of multisets in MST is identical to that of sets in ZFC.

CONCLUSIONS

This paper presents a concise account of some multiset theories, in particular, our discussions are centered around the first-order two-sorted theory MST, and the single-sorted theory MS. The theory MST appears to be the most sustained multiset theory in the literature. A number of existing multiset theories assume the axioms of MST with some modifications.
The single-sorted multiset theory MS presents axioms that are consistent with the axioms of MST. The MS theory comes across as a stronger multiset theory. It differs from most of the existing theories as it assumes a single-sort for its multiset and numeric variables, and has the same axioms for these variables.

Another multiset theory that could be of interest is the generalized multiset theory (Felisiak et al. 2020). The generalized multiset theory makes a clear distinction between true multisets (which exist by axiomatic theories), and artificial multisets (which are set-based multisets). This theory allows the multiplicity of an object to be an arbitrary real number - this includes a negative number. The similarities and peculiarities of the generalized multiset theory relative to multiset theories discussed in this work could also be analyzed. In Singh and Isah (2016), multiset of equivalence relation approach is discussed. The authors presented a multiset as a set with an equivalence relation; the concept of sort is applied to formulate the mathematics of multisets. This approach could be of interest as it uses a modified and extended version of the extant theory.

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