Bi-Commutative Digital Contraction Mapping and Fixed Point Theorem On Digital Image and Metric Spaces

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Abstract

The contraction mapping theorem is a powerful and most useful method in constructing the solution of linear and nonlinear systems. The principle of Banach fixed point guarantees that a self-contraction mapping of a complete metric space has a unique fixed point which can be obtain using the limit of iteration technique defined by repeated images under mapping of an arbitrary starting point in space. In this paper, we propose a Bi-commutative (II-commuting) digital maps for Kannan and Reich contraction mapping theorem and analyze the existence and uniqueness of the fixed point in the framework of digital metric spaces. We applied the contraction and commutative maps methods in proving our results. The results obtained satisfies the existence and uniqueness condition of the Banach contraction principle for a digital contraction mapping in digital metric space.

Keywords: Bi-Commutative map, Digital Contraction Mapping, Metric Space, Fixed Point.

INTRODUCTION

The contraction mapping principle plays a very vital role in the study of dynamical system, linear and non-linear differential and integral equations. The principle was first stated and proved by Banach, and also around the same period, the notion of abstract metric space was established which gave rise to the general background for the principle of contraction mappings in a complete metric space (Nawab & Iram, 2019). A contraction map is self-continuous map defined in a metric-space by a pair (M, d) satisfying the condition that, there exist some positive real number $L \in \mathbb{R}$ such that $0 < L < 1$ and for $a, b \in M$, then we define a metric function $d(T(a), T(b)) \leq L\, d(a, b)$. Where L is a Lipschitz constant of the map T. Hence, this implies that every contraction map is a Lipschitz continuous function and it is also uniformly continuous. The Banach contraction principle states that for every self-continuous map defined on metric function that is complete, there exist a unique point, and that for every element in a metric space, the iterated functions sequence of such element converges to a fixed point (Asha *et al.*, 2016). The notion of contraction is very important and practicable for

iteration of functions where mappings are frequently used in a complete metric space.

The theory of fixed points in topological space involves rigorous study of spaces that contains the necessary conditions for its existence and uniqueness. Digital topology is a topology that studies the properties and attributes of two or three dimensional digital images that conform to a topological feature (Choonkil, 2019). Over the years, researchers in the field of pure mathematics have shown the uniqueness and existence of a fixed point using different types of continuous mapping and methods. Ozgur *et al.* (2020) proposed commutative and compatible type mapping in the framework of digital metric spaces and established some fixed point results using the property of commutative mappings. They also proved some common fixed point results for compatible mappings and their variants in the area of digital metric spaces. Deepak (2018), proposed intimate mappings in metric space and in digital metric space. The author further established a common fixed point result for two corresponding intimate mappings in digital metric spaces. Sang-Eon (2016), proposed a Banach contraction mapping result for digital images. The author also extends the concept of contraction map and Banach fixed point theorem to image processing and limited to only single type contraction mapping. Roa *et al.* (2019), obtained some results for bi-mapping and quadruple self-mappings by modifying distances of sub-compatible functions in five variables and deficit functions with generalization of only contractive type mapping condition. Karim *et al. (2021)*, introduced some results and graphical representation for Chatterjea principle type contraction map in a generalized metric-space but limited to single type mapping.

Definition 1.1 (Digital image) (Kalaiarasia and Jain, 2022, Sridevi *et al., 2016)*
An order pair (X, μ) with μ-adjacency and X as a finite subset of $\mathbb{Z}^n$ for some positive integers is called a digital image.

Definition 1.2 (Digital metric space) (Kalaiarasia and Jain, 2022)
Let $(X, \mu) \subset \mathbb{Z}^n$ and ρ be a metric function on $\mathbb{Z}^n$ such that $\rho: X \times X \to Z^n$ by

$$\rho(r, s) = (\sum_1^n (r_i - s_i)^2)^{\frac{1}{2}} \qquad (1.1)$$

Then, the following properties holds for $x, y, z \in X$

I. $\rho(x, y) \geq 0, \rho(x, y) = 0$ iff $x = y$.

II. $\rho(x, y) = \rho(y, x)$

III. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

The pair (X, μ) with the metric function ρ define as (X, ρ, μ) is called digital metric space.
Definition 1.3 (Fixed point) A self-mapping $T: X \to X$ on a set X with a point $x_0 \in X$ is a fixed point on the map T, if $Tx_0 = x_0$.

METHODOLOGY

Lipschitz condition (Christiana, 2023)
Let $f(x, t)$ be a vector value function. It is said to satisfy a Lipschitz condition in a region $\mathbb{R}$ in space (x, t), if, for some constant say L such that

$$\left|\left|f(x, t) - f(y, t)\right|\right| \leq L \,|x - y| \qquad (2.1)$$

Where L is known as Lipschitz constant and whenever $(x, t) \in \mathbb{R}, (y, t) \in \mathbb{R}^1$

Digital contraction mapping (Asha *et al.,* 2016)
Let (X, ρ, μ) be any digital metric space and that $T: (X, \rho, \mu) \to (X, \rho, \mu)$ be a self-digital map. If there exist $L \in (0,1)$ such that $\forall x_1, x_2 \in X$,

$$d\big(T(x_1), T(x_2)\big) \le L \, d(x_1, x_2) \tag{2.2}$$

Then T is known as digital contraction map, and T has a unique point fixed point $\alpha \in X$ for each $c \in X, \lim\limits_{n \to \infty} T^n c = \alpha$. This implies that, for every digital contraction map on a digital metric space has a unique fixed point.

Lemma 2.1 (Asha *et al.,* 2016)
Suppose $M \subseteq \mathbb{Z}^n$ and let (M, ρ, μ) be a digital metric space. Then there exist a sequence $\{r_m\}$ sequence in the set M such that $\rho(r_{m+1}, r_m) < \rho(r_m, r_{m-1}), m = 1, 2 \dots$.
Remark: $\rho(r_1, r_0) = \sqrt{n}$, where n is a nonnegative integer in as much as ρ is a metric in $M \subseteq \mathbb{Z}^n$.

Theorem 2.1 (Kannan contraction) (Adeyemi, 2021)
Let X be a metric space and let $T: X \to X$ be a mapping, then there exist $0 < L < \frac{1}{2}, \forall \, x, y \in X,$ such that

$$d(Tx, Ty) \le L \, [d(x, Tx) + d(y, Ty)] \tag{2.3}$$

Then T has a unique fixed point $\alpha \in X$ and for every $x \in X$, the sequences $\{x_n\}$ iterates $\{T^n x\}$ by
$x_{n+1} = Tx^n$.

Contraction conditions theorems for digital metric space by Choonkil *et al.,* 2019
The authors Choonkil *et al.,* 2019, worked on fixed point theorems of different type contraction conditions for digital metric space and proposed the following

Corollary 2.1 (Choonkil *et al.,* 2019)
Let T be a contraction mapping on a complete metric space X and L be a contraction constant with a fixed point a_0. Then, for every $x_0 \in X$, with T-iterates $\{x_n\}$, the following estimate holds:

$$\rho(x_n, a_0) \le \frac{L^n}{1 - L} \rho\big(x_0, T(x_0)\big),$$

$$\rho(x_n, a_0) \le L \, \rho(x_{n-1}, a_0),$$

And

$$\rho(x_n, a_0) \le \frac{L}{1 - L} \rho(x_{n-1}, x_n),$$

Theorem 2.2 (Reich fixed point) (Choonkil *et al.,* 2019)

Let (M, ρ, μ) be a digital metric space with a mapping H into itself satisfying the following

$$\rho\big(H(u), H(v)\big) \le l\rho(u, Hu) + q\rho(v, Hv) + r\rho(u, v), \ \forall \, u, v \in H \tag{2.4}$$

And all the positive real numbers l, q, r satisfies $l + q + r < 1$, then H has a unique fixed point in H.

Commutative Mapping (Asha and Kumari, 2016)

Let (X, d, ρ) be a complete digital metric space and $T_1, T_2 : X \to X$ be two maps defined on X. Then T_1, T_2 are said to be commutative if
$$T_1 \circ T_2(x) = T_2 \circ T_1(x), \forall x \in X \tag{2.5}$$

Corollary 2.2 (Asha and Kumari, 2016)

Let H and G be two mappings of a complete digital-metric space (M, ρ, μ) into itself. Suppose that H is continuous and $G(u) \subseteq H(u)$. If there exists $L \in (o, 1)$ and a positive integer r such that

$$\rho\big(G_r(u), G_r(v)\big) \leq L \, \rho\big(H(u), H(v)\big), \forall \, u, v \in M$$

Then H and G contains only one point that is common.

Theorem 2.3 (Asha and Kumari, 2016)

Suppose H is continuous and is a self-map in complete digital metric space (M, ρ, μ), then the map H contains a unique point that belong to M if and only if there exist an $L \in (0,1)$ such that a mapping $G: M \to M$ commutes with H and $G(u) \subseteq H(u)$ satisfies

$$\rho(G(u), G(v)) \leq L \, \rho(H(u), H(v)) \qquad (2.6)$$

For every $u, v \in M$. It is true that H and G have a common fixed point that is unique if equation (2.6) holds.

RESULTS AND DISCUSSION

Results

Theorem 3.1 (Kannan contraction with two-commuting maps) Let (M, μ) be a digital image and μ an adjacency relation between the objects of M, where $M \subset Z^n$. Let (M, ρ, u), be a digital metric space. Suppose Ψ and φ be two mappings each maps to itself on M satisfying the following

$$\rho(\Psi \,(u), \Psi \,(v) \leq g\{\rho\big(u, \Psi \,(u)\big) + \rho\big(v, \Psi \,(v)\big)\}$$

$$\rho(\varphi \,(u), \varphi \,(v) \leq g\{\rho\big(u, \varphi \,(u)\big) + \rho\big(v, \varphi \,(v)\big)\} \,, \text{ and}$$

$$\Psi(u) \subset \varphi(u)$$

For every $u, v \in M$ and $g \in (0, \frac{1}{2})$, that Ψ and φ have a common fixed point in M and its unique, provided Ψ and φ commute on M.

Proof: Suppose $\rho(\Psi \,(u), \Psi \,(v) \leq g\{\rho\big(u, \Psi \,(u)\big) + \rho\big(v, \Psi \,(v)\big)\}$ and

$\rho(\varphi \,(u), \varphi \,(v) \leq g\{\rho\big(u, \varphi \,(u)\big) + \rho\big(v, \varphi \,(v)\big)\}$. Since $\Psi(u) \subset \varphi(u)$ and

$\rho(\Psi \,(u), \Psi \,(v) \leq \rho(\varphi \,(u), \varphi \,(v))$. Then

$$\rho(\Psi \,(u), \Psi \,(v) \leq g\{\rho\big(u, \Psi \,(u)\big) + \rho\big(v, \Psi \,(v)\big)\}$$

$$\leq \rho(\varphi \,(u), \varphi \,(v) \leq g\{\rho\big(u, \varphi \,(u)\big) + \rho\big(v, \varphi \,(v)\big)\}$$

It implies that $\rho(\Psi \,(u), \Psi \,(v) \leq \rho(\varphi \,(u), \varphi \,(v))$

Let $u_o \in M$ and consider the iterate of sequence $\Psi(u_{n+1}) = \varphi(u_o)$, for $n = 0,1,2,3, \dots$

And $\Psi(\Psi(u_n)) = \Psi(u_{n+1})$ such that

$$\Psi(u_1) = \varphi(u_0)$$

$$\Psi(u_1) = \varphi(u_0)$$

$$\vdots$$

$$\Psi(u_{n+1}) = \varphi(u_n)$$

By theorem 2.2, we obtained

$$\rho(\Psi(u_1), \Psi(u_2)) \leq g\rho(\Psi(u_0), \Psi(u_1)) \leq g\{\rho\left(\Psi(u_0), \Psi(\Psi(u_0))\right) + \rho(\Psi(u_1), \Psi(\Psi(u_1)))\}$$

$$\leq g\{\rho\left(\Psi(u_0), \Psi(u_1)\right) + \rho(\Psi(u_1), \Psi(u_0))\}$$

$$\leq g\rho\left(\Psi(u_0), \Psi(u_1)\right) + g\rho(\Psi(u_1), \Psi(u_2))$$

$$\rho\left(\Psi(u_1), \Psi(u_2)\right) - g\rho(\Psi(u_1), \Psi(u_2)) \leq g\rho(\Psi(u_0), \Psi(u_1))$$

$$(1 - g)\rho(\Psi(u_1), \Psi(u_2)) \leq g\rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_1), \Psi(u_2)) \leq \frac{g}{1 - g}\rho(\Psi(u_0), \Psi(u_1))$$

$$\rho\left(\Psi(u_2), \Psi(u_3)\right) \leq \left(\frac{g}{1 - g}\right)^2 \rho\left(\Psi(u_0), \Psi(u_1)\right)$$

$$\vdots$$

$$\rho(\Psi(u_n), \Psi(u_{n+1})) \leq \left(\frac{g}{1 - g}\right)^n \rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_{n+1}), \Psi(u_{n+2})) \leq \left(\frac{g}{1 - g}\right)^{n+1} \rho(\Psi(u_0), \Psi(u_1))$$

Let $v = \frac{g}{1-g}$, then from the above inequality we obtained

$$\rho(\Psi(u_n), \Psi(u_{n+1})) \leq v^n \rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_{n+1}), \Psi(u_{n+2})) \leq v^{n+1} \rho(\Psi(u_0), \Psi(u_1))$$

The last inequality above and the following holds by corollary 2.1,

$$\rho\left(\Psi(u_n), \Psi(u_{n+r})\right) \leq \rho\left(\Psi(u_n), \Psi(u_{n+1})\right) + \rho\left(\Psi(u_{n+1}), \Psi(u_{n+2})\right) + \cdots + \rho(\Psi(u_{n+r-1}), \Psi(u_{n+r}))$$

$$\leq (v^n + v^{n+1} + \cdots + v^{n+r-1})\rho(\Psi(u_0), \Psi(u_1))$$

$$\leq v^n(1 + v + v^2 + \cdots)\rho(\Psi(u_0), \Psi(u_1))$$

Note that $(1 + v + v^2 + \cdots)$ is an infinite geometric series i.e. $S_\infty = \frac{u_1}{1-r}$ where $u_1 = 1$ and $r = v$, then $\frac{u_1}{1-r} = \frac{1}{1-v}$. Therefore,

$$\rho(\Psi(u_n), \Psi(u_{n+r})) \leq v^n\left(\frac{1}{1-v}\right)\rho(\Psi(u_0), \Psi(u_1)) \leq \frac{v^n}{1-v}\rho(\Psi(u_0), \Psi(u_1))$$

Since $0 \leq v < 1$, $\frac{v^n}{1-v}\rho(\Psi(u_0), \Psi(u_1)) \to 0$ as $n \to \infty$ it implies that $\{\Psi(u_n)\}$ is known as Cauchy sequence in (M, ρ, u), because of the completeness of the digital metric space (M, ρ, u), the sequence $\{\Psi(u_n)\}$ converges to a point l in (M, ρ, u) and $\varphi(u) = \Psi(u_{n+1})$ also converges to a point l in (M, ρ, u). Since $(u, u) - continuity$ of Ψ implies $(u, u) - continuity$ of φ then

$\{\varphi(\Psi(u_n))\}$ converges to $\varphi(l)$. Nevertheless, since Ψ and φ commute on M and $\varphi(\Psi(u_n)) = \Psi(\varphi(u_n))$ then $\{\Psi(\varphi(u_n))\}$ converges to $\Psi(l)$. This implies that $\Psi(l) = \varphi(l)$. Since the limit of both sequence are unique, it implies that

$\Psi(\Psi(l)) = \Psi(\varphi(l))$. By corollary 2.2 and theorem 2.2, we obtained

$$\rho(\varphi(l), \varphi(\varphi(l))) \le v\rho(\Psi(l), \Psi(\Psi(l))) \le \rho(\varphi(l), \varphi(\varphi(l)))$$

We can write $\varphi(l) = \Psi(\Psi(l))$, and that

$$\varphi(l) = \varphi(\varphi(l)) = \Psi(\varphi(l))$$

This shows that $\varphi(l)$ is a fixed point between maps Ψ and φ. To show the uniqueness, we shall assume two points of the maps Ψ and φ. Let the points be r and s respectively and $r \ne s$, for all $r, s \in M$. Then $\Psi(r) = r = \varphi(r)$ and $\varphi(s) = s = \Psi(s)$ such that

$$\rho(r, s) \le \rho(\varphi(r), \varphi(s)) \le v\rho(\Psi(r), \Psi(s)) \le v\rho(r, s)$$

$$\rho(r, s) - v\rho(r, s) \le 0$$

$$(1 - v)\rho(r, s) \le 0$$

$$1 - v \le 0$$

$$1 \le v$$

Which implies that $v = 1$ and $v > 1$ which is a contradiction because $0 \le 1$. Suppose $r = s$

$$\rho(r, s) \le \rho(\varphi(r), \varphi(s)) \le v\rho(\Psi(r), \Psi(s)) \le v\rho(r, s) = 0$$

Therefore $\rho(r, s) = 0$. Hence $r = s$ which shows the fixed point is unique.

Theorem 3.2 (Reich contraction theorem with two-commuting maps)

Suppose Ψ be a map in a digital metric space (M, ρ, u) onto itself, then the following holds

$$\rho(\Psi(u), \Psi(v)) \le f\rho\left(\Psi(u), \Psi(\Psi(u))\right) + g\rho\left(\Psi(v), \Psi(\Psi(v))\right) + h\rho(\Psi(u), \Psi(v))$$

For all $u, v \in M$. Let φ be another self map satisfying the following

$$\rho(\varphi(u), \varphi(v)) \le f\rho\left(\varphi(u), \varphi(\varphi(v))\right) + g\rho\left(\varphi(u), \varphi(\varphi(v))\right) + h\rho(\varphi(u), \varphi(v))$$

For all non-negative real numbers f, g, h with $f + g + h < 1$. Then Ψ and φ have a fixed point in M and its unique, provided Ψ and φ satisfied commutative property in M

Proof: Let $\Psi(u_0) \in M$. Suppose the sequence $\Psi(u_{n+1}) = \Psi(\Psi(u_n))$ and $\Psi(u_n) = \Psi(u_{n-1})$, since $\Psi(u) \subset \varphi(u)$ by theorem 2.3 we have that

$\rho(\Psi(u), \Psi(v)) \le \rho(\varphi(u), \varphi(v))$. Hence,

$$\rho(\Psi(u), \Psi(v)) \le f\rho\left(\Psi(u), \Psi(\Psi(u))\right) + g\rho\left(\Psi(v), \Psi(\Psi(v))\right) + h\rho(\Psi(u), \Psi(v))$$

$$\le \rho(\varphi(u), \varphi(v)) \le f\rho\left(\varphi(u), \varphi(\varphi(u))\right) + g\rho\left(\varphi(v), \varphi(\varphi(v))\right) + h\rho(\varphi(u), \varphi(v))$$

Since $\Psi(u) \subset \varphi(u)$ it implies that $\Psi(u) \leq \varphi(u)$ by corollary 2.2. Therefore, $\rho(\Psi(u), \Psi(v)) \leq \rho(\varphi(u), \varphi(v)) \leq f\rho\left(\Psi(u), \Psi(\Psi(u))\right) + g\rho\left(\Psi(v), \Psi(\Psi(v))\right) + h\rho(\Psi(u), \Psi(v))$. Let $u = u_1$ and $v = u_2$

$$\rho(\Psi(u_1), \Psi(u_2)) \leq f\rho\left(\Psi(u_0), \Psi(\Psi(u_0))\right) + g\rho\left(\Psi(u_1), \Psi(\Psi(u_1))\right) + h\rho(\Psi(u_0), \Psi(u_1))$$

$$\leq f\rho(\Psi(u_0), \Psi(u_1)) + g\rho(\Psi(u_1), \Psi(u_2)) + h\rho(\Psi(u_0), \Psi(u_1))$$

Subtract $g\rho(\Psi(u_1), \Psi(u_2))$ from both sides of the later, yields

$$\rho(\Psi(u_1), \Psi(u_2)) - g\rho(\Psi(u_1), \Psi(u_2)) \leq f\rho(\Psi(u_0), \Psi(u_1)) + h\rho(\Psi(u_0), \Psi(u_1))$$

$$(1 - g)\rho(\Psi(u_1), \Psi(u_2)) \leq (f + h)\rho(\Psi(u_0), \Psi(u_1))$$

We have $\rho(\Psi(u_1), \Psi(u_2)) \leq \frac{f+h}{1-g}\rho(\Psi(u_0), \Psi(u_1))$

Similarly $\rho(\Psi(u_2), \Psi(u_3)) \leq \left(\frac{f+h}{1-g}\right)^2 \rho(\Psi(u_0), \Psi(u_1))$ By corollary 2.1,

$$\vdots$$

$$\rho(\Psi(u_n), \Psi(u_{n+1})) \leq \left(\frac{f+h}{1-g}\right)^n \rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_{n+1}), \Psi(u_{n+2})) \leq \left(\frac{f+h}{1-g}\right)^{n+1} \rho(\Psi(u_0), \Psi(u_1))$$

Let $v = \frac{f+h}{1-g}$ and $v < 1$. Then

$$\rho(\Psi(u_n), \Psi(u_{n+1})) \leq v^n \rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_{n+1}), \Psi(u_{n+2})) \leq v^{n+1} \rho(\Psi(u_0), \Psi(u_1))$$

By theorem 2.2, the following holds

$$\rho(\Psi(u_n), \Psi(u_{n+r}))$$
$$\leq \rho(\Psi(u_n), \Psi(u_{n+1})) + \rho(\Psi(u_{n+1}), \Psi(u_{n+2})) + \cdots + \rho(\Psi(u_{n+r-1}), \Psi(u_{n+r}))$$

$$\leq (v^n + v^{n+1} + \cdots + v^{n+r-1})\rho(\Psi(u_0), \Psi(u_1))$$

$$\leq v^n(1 + v + v^2 + \cdots)\rho(\Psi(u_0), \Psi(u_1))$$

Where $(1 + v + v^2 + \cdots)$ is an infinite geometric series, and series i.e. $S_\infty = \frac{u_1}{1-r}$ where $u_1 = 1$ and $r = v$, implies $\frac{1}{1-v}$. Therefore, the following holds

$$\rho(\Psi(u_n), \Psi(u_{n+r})) \leq v^n \left(\frac{1}{1-v}\right)\rho(\Psi(u_0), \Psi(u_1))$$

$$\rho(\Psi(u_n), \Psi(u_{n+r})) \leq \frac{v^n}{1-v}\rho(\Psi(u_0), \Psi(u_1))$$

Since $0 \leq v < 1$, then $\frac{v^n}{1-v}\rho(\Psi(u_0), \Psi(u_1)) \to 0$ as $n \to \infty$ which shows that $\{\Psi(u_n)\}$ is a Cauchy sequence in (M, ρ, u). Due to the completeness (M, ρ, u), the sequenced $\{\Psi(u_n)\}$

converges to a point c in (M, ρ, u) and $\varphi(u) = \Psi(u_{n+1})$ also converges to a point c in (M, ρ, u). Since $(u, u) - continuity$ of Ψ implies $(u, u) - continuity$ of φ then $\{\varphi(\Psi(u_n))\}$ converges to $\varphi(c)$. Since Ψ and φ commute on M and $\varphi(\Psi(u_n)) = \Psi(\varphi(u_n))$ then $\{\Psi(\varphi(u_n))\}$ converges to $\Psi(c)$. This implies that $\Psi(c) = \varphi(c)$.

Since the limit of both sequence are unique, it implies that $\Psi(\varphi(c)) = \varphi(\Psi(c))$

By theorem 2.3, we obtained $\rho(\varphi(c), \varphi(\varphi(c))) \leq v\rho(\Psi(c), \Psi(\Psi(c))) \leq \rho(\varphi(c), \varphi(\varphi(c)))$

With the fact that $\varphi(c) = \Psi(\Psi(c))$ and $\varphi(c) = \varphi(\varphi(c)) = \Psi(\varphi(c))$

This means that $\varphi(c)$ is a common fixed point of maps Ψ and φ.

Finally we are to show that the fixed point is unique. Suppose q and w are the fixed points of the two commuting maps Ψ and φ, respectively. Let $q \neq w$, for all $q, w \in M$. Then

$\Psi(q) = q = \varphi(q)$ and $\varphi(w) = w = \Psi(w)$. This follows that

$$\rho(q, w) \leq \rho(\varphi(q), \varphi(w)) \leq v\rho(\Psi(q), \Psi(w)) \leq v\rho(q, w)$$

$$\rho(q, w) - v\rho(q, w) \leq 0$$

$$(1 - v)\rho(q, w) \leq 0$$

$$1 - v \leq 0$$

$$1 \leq v$$

With $v = 1$ or $v > 1$ which is a contradiction because $0 \leq 1$. Then, suppose $q = w$

$$\rho(q, w) \leq \rho(\varphi(q), \varphi(w)) \leq v\rho(\Psi(q), \Psi(w)) \leq v\rho(q, w) = 0$$

Therefore, $\rho(q, w) = 0$. Hence $q = w$ which shows the uniqueness of the fixed points.

Discussion

The contraction conditions for digital metric spaces was proposed by Choonkil *et al.,* 2019. In this study, we have proposed a bi-commutative maps for Kannan and Reich contraction fixed point theorem for digital image and metric spaces. We have shown the commutativity of the pair maps in the digital contraction mappings and established a unique and common fixed point of the mappings. The unique and common fixed point is established by assuming two different points say r, q and s, w for both maps Ψ and φ in theorem 3.1, and 3.2. If these two points are not equal, the result will contradict the Lipschitz constant, but we have shown that the two assumed points are equal $(r, q = s, w)$. Hence, $q = w$ is a unique and common fixed point.

CONCLUSION

In this paper, we study bi-commutative maps for various contraction conditions in digital image and metric spaces. We proposed a Dual Commutative maps for Kannan contraction theorem and Reich contraction theorem for digital image and metric spaces and established the existence and uniqueness of the fixed point. We applied the contraction and commutative maps methods in proving our results as shown in section 3.1. The results obtained satisfies the existence and uniqueness condition of the Banach contraction principle for a digital contraction mapping in digital metric space.

REFERENCES

Adeyemi T. A. (2021). A review of f-contraction mapping on metric spaces. Bachelor Degree Project Submitted to Department of Computer Science and Mathematics, Mountain Top University, Nigeria. pp 1-54. http://ir.mtu.edu.ng/jspui/handle/123456789/274

Asha, R., Kumari, J. & Asha R. (2016). Common fixed point theorem in digital metric spaces. *International Journal of Scientific and Engineering Research,* 7(12).

Choonkil, P., Ozgur, E., Sanjay, K., Deepak, J., & Jung, R. (2019). Fixed point theorem for various contraction conditions in digital metric spaces. *J Computational Analysis and Applications*, 26(8).

Christiana R. I. (2023). Fixed-Point Theorems Involving Lipschitz in the Small. *Abstract and Applied Analysis*, Volume 2023, Article ID 5236150, https://doi.org/10.1155/2023/5236150.

Deepak, J. (2018). Common fixed point theorem for intimate mappings in digital metric spaces. *International Journal of Mathematics Trends and Technology*, 56(2).

Karim, C., Mustapha, K., & Abdessamed, K. (2021). Fixed point result for contraction mappings in generalized metric spaces with graph. *Journal of Function Spaces*, https://doi.org/10.1155/2021/ss40347.

Nawab, H. & Iram, I. (2019). Contraction mappings and application. Recent Advances in Integral Equations, 10.5772/intechopen.81571.

Ozgur, E., Deepak, J., Sanjay, K., Choonkil, P., & Dong, Y. (2020). Commuting and compatible mappings in digital metric spaces. *Journal of Fixed Point Theory Application*, 22(5).

Rao, C. S., Murphy, K. S., & Murphy, M. P. R. (2019). Common fixed point theorems on complete metric space for two self-maps using generalize altering distances functions in five variables and deficit functions. *Global Journal of Pure and Applied Mathematics*, 15(3), 241–250.

Kalaiarasia R. and Jain R. (2022). Fixed point theory in digital topology. *International Journal Nonlinear Analysis Application,* 13 (Special issue for ICDACT-2021), 157–163. *http://dx.doi.org/10.22075/ijnaa.2022.6375*

Sang-Eon, H. (2016). Banach fixed point theorem from the viewpoint of digital topology. *Journal Nonlinear Science Application*, 9, 895–905.

Sridevi, K., Kameswari, M. V. R., & Kiran, D. M. (2017). Fixed point theorem for digital contraction type mappings in digital metric spaces. *International Journal of Mathematics Trends and Technology, 48(3).*