https://dx.doi.org/10.4314/dujopas.v9i3a.6
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 9 No. 3a September 2023*

The Dynamical Equations of a Test Particle in the Restricted Three-Body Problem with a Triaxial Primary and Variable Masses

Oni Leke[1] and Shima Stephen[2]

[1, 2]Department of Mathematics,
College of Physical Science,
Joseph Sarwuan Tarka University,
P.M.B. 2373,
Makurdi, Benue-State
Nigeria

Email: shimastephen8562@gmail.com

Abstract
The restricted three-body problem (R3BP) defines the motion of an infinitesimal mass moving under the gravitational attractions of two main masses, called primaries. In this paper, we examine the dynamical equations of a test body in the frame of the circular R3BP. Both primaries are assumed to vary their masses in accordance with the Unified Mestschersky law (UML) and their motion determined by the Gylden-Mestschersky problem (GMP). Further, the first primary is assumed to be a triaxial variable mass body. The potential between the primaries is deduced and in furtherance, the non-autonomous equations of motion of the model are derived. The derived equations are time varying and are thus transformed to the autonomized forms with constant coefficients using the Mestschersky transformation (MT), the UML, the particular integral and solutions of the GMP. We also use a transformation we introduced, which helps in converting the time dependent triaxiality of the bigger primary to one that is constant. The derived systems of equations with variable and constant coefficients can be used to model the long-term motion of satellites and planets in binary systems.

Keywords: R3BP, Variable Masses, Triaxiality, Dynamical Equations, Test particle

INTRODUCTION
The restricted three-body problem (R3BP) defines the motion of an infinitesimal mass moving under the gravitational attractions of two main masses, called primaries, which move in circular orbits around their center of mass on account of their common attraction and the infinitesimal mass not inducing the motion of the primaries (Szebehely 1967). The approximate circular motion of the planets around the sun and the small masses of asteroids and the satellites of planets compared to the planet's masses, initially proposed the formulation of the restricted problem. There are so many examples of the restricted problem in space dynamics. One of them is the classical three-body problem viz; the Sun-Earth-Moon combination and describing the motion of the moon. The motion of a Trojan asteroid attracted by the Sun and Jupiter is another example, which has a similarity with the restricted problem. One of the foremost in space science is the creation of artificial bodies, which are required to

*Author for Correspondence

move in the neighborhood of two natural celestial bodies. This problem is also similar to the restricted problem.

The formulation of the classical R3BP considered all the bodies to be strictly spherical. But in actual sense, most celestial bodies are not perfect spheres. Some have the shape of an oblate spheroid while some are triaxial in nature (Sharma *et.al* 2001; Singh & Leke 2012; Saeed & Zotos 2021). For example, the Earth, Jupiter, Saturn and stars (Archerner, Antares and Altair) have the shape of an oblate spheroid while the Moon and several Post Asymptotic Giant Branch stars (Post AGB), Haumea, (a scalene dwarf planet, Figure **1**) are triaxial in shape. The rotation of a star produces an equatorial bulge due to centrifugal force; as a result, the stars are often non-spherical in shape.

Figure 1: Artist's conception of Haumea, a triaxial dwarf planet, with its two moons (Credit: NASA, ESA)

The lack of sphericity of the planets causes large perturbations from a two-body orbit. Interesting results of studies that have included the triaxiality of one, two or all the participating bodies in the R3BP have been produced up to date. Among them are, Khanna & Bhatnagar (1999), Sharma *et al.* (2001), Singh & Begha (2011), Singh (2013), Singh & Leke (2014), Singh & Umar (2014), Saeed & Zotos (2021), Alrebdi *et. al* (2022), Gyegwe *et. al* (2022), Vincent *et al* (2022) and Gahlot & Kishor (2023).

The classical R3BP adopts the masses of celestial bodies to be constant. However, the phenomenon of isotropic radiation or absorption in stars led scientists to formulate the restricted problem of three bodies with variable mass. The problem of the motion of astronomical objects with variable mass has many interesting applications in stellar, galactic, and planetary dynamics. As an example, we could mention motion of a satellite around a radiating star surrounded by a cloud and varying its mass due to particles of the cloud, and comets loosing part or all of their mass as a result of travelling around the Sun (or other stars) due to their interaction with the solar wind which blows off particles from their surfaces. Another interesting example of mass loss is the real physical scenario of those transiting exoplanets whose atmospheres are escaping because of the severe levels of energetic radiations, coming from their very close parent stars, hitting them. Due to the inclusion of mass variations, many researchers such as Gelf'gat (1973), Bekov (1988), Luk'yanov (1989), Singh & Leke (2010,2012,2013a, b, c) and Leke & Singh (2023), have carried out various studies to include mass variation of the primaries when the mass of the third body is kept constant while Singh & Ishwar (1984, 1985) and Zhang *et al.* (2012) have considered the formulation of the R3BP when the mass of the third body is assumed to vary with time and the masses of the primaries is kept constant . When the three masses vary with time, researchers such as El-Shaboury (1990), Bekov *et al*. (2005); Letelier & Da Silva (2011), and Singh & Leke (2013d), have investigated this formulation under different classifications.

The problem of the R3BP with variable masses in which the masses of the primaries vary in accordance with the unified Mestschersky law and their motion defined within the framework of the Gylden-Mestschersky problem has been studied extensively under different characterization.

In this present paper, our aim is to derive the dynamical equations of the R3BP with variable masses in which the bigger primary is a triaxial body while the second is a spherical one with the consideration that the mass variations and motion of the primaries are described by the unified Mestschersky law (1952) and the Gylden-Mestschersky problem (Gylden 1884, Mestscherskii, 1902). The paper is an extension of the dynamical equations of motion given by Gelf'gat (1973), Bekov (1988) and Luk'yanov (1989,1990) when the shape of the first primary departs from being a sphere to triaxial in nature.

METHODOLOGY

Gylden-Mestschersky Problem

By the problem of two bodies with variable masses, by analogy with the classical problem of two bodies with constant masses, one understands the problem of motion of two primary bodies, the masses m_1 and m_2 of which vary with time under certain laws and between which only the gravitational force acts. It is usually assumed that the separation of particles from (or their attachment to) the points take place in accordance with Mestschersky's hypothesis, i.e., a contact interaction occurs between the points of variable mass and the separating (or attaching) particles; it is assumed that the masses of the points vary continuously.

The absolute motion of the points is described by the Mestschersky equation for a point of variable mass,

$$\vec{F} = m\dot{\vec{v}} + \left(\vec{v} - \vec{u}\right)\dot{m} \tag{1}$$

where $\vec{F}$ is the sum of all the forces acting on the body and $\vec{v}$ is its velocity, both measured in an inertial coordinate system. Also, $\vec{u}$ is the velocity of the center of mass of the absorbed mass immediately before its union with the body (or of the ejected mass immediately after its ejection). The over dot denotes derivation with respect to the time variable.

Gylden represented the relative motion of mass m_2 about mass m_1 under the action of mutual gravitational force, as the sum of the masses of these points as varying with time by a certain law

$$m_1 + m_2 = \mu\left(t\right) \tag{2}$$

and wrote the differential equation of the problem in the form

$$\ddot{\vec{r}} + \frac{\mu\left(t\right)}{r^3}\vec{r} = 0 \tag{3}$$

Mestschersky showed that the Gylden problem (3) is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same.

There are two special cases of equation (1) to be considered. The first one is when the mass is ejected with the same velocity of the body at any moment $\left(\vec{v} = \vec{u}\right)$, that is, mass ejection does

not produce reactive forces. This case can be used to study the motion of a body ejecting mass isotropically (or radiating energy), since the total reactive momentum would be zero.
If $\vec{v} = \vec{u}$, then equation (1) reduces to the form

$$\vec{F} = m\dot{\vec{v}} \tag{4}$$

In this case the relative motion of the problem of two bodies with variable masses is described by the equation

$$\ddot{\vec{r}} = -G\frac{(m_1 + m_2)}{r^3}\vec{r} \tag{5}$$

Equation (5) is analogous to the equation of the classical problem of two bodies with constant masses, with the difference that now; the sum of the masses is a certain function of time. Equality of the velocities of ejected mass and the body at any moment means that isotropic variation of masses (in Mestschersky term) occurs. Equation (5) is justly called the *Gylden-Mestschersky problem* (GMP).

The Mestschersky Transformation and Unified Mestschersky Law
Mestschersky (1902) disclosed that the Gylden problem is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same, while the relative velocities of the particles separating from them (or attaching to them) equals zero everywhere. He found two laws of variation of the sum of the masses of the points

$$m_1 + m_2 = \mu(t) = \frac{\mu_0}{a + \alpha t} \tag{6}$$

$$m_1 + m_2 = \mu(t) = \frac{\mu_0}{\sqrt{\alpha + \beta t + \gamma t^2}} \tag{7}$$

where μ_0, a, α, β and γ are constants.

In the work of Mestschersky (1902), he reduced the GMP through the introduction of new variables and "time" to the equations of the classical problem of two bodies with constant masses by a transformation, which was thereafter known as the Mestschersky transformation (MT) and is given as

$$x = \xi R(t),\ y = \eta R(t),\ z = \varsigma R(t),\ \frac{dt}{d\tau} = R^2(t)$$
$$r_i = \rho_i R(t),\ (i = 1, 2), r = \rho_{12} R(t) \tag{8}$$

where $R(t) = \sqrt{\alpha t^2 + 2\beta t + \gamma}$; ξ, η, ζ, τ are the new variables and ρ_{12} is constant.
Later, Mestschersky (1952) came up with a law which considers the masses and their sum to vary in the same proportion in such a way that

$$\mu(t) = \frac{\mu_0}{R(t)},\ \mu_1(t) = \frac{\mu_{10}}{R(t)},\ \mu_2(t) = \frac{\mu_{20}}{R(t)} \tag{9}$$

where $\mu_1(t) = Gm_1(t), \mu_2(t) = Gm_2(t)$, $\mu(t) = \mu_1(t) + \mu_2(t)$, μ_{10} and μ_{20} are constants. The law (9) is called the unified Mestschersky law (UML) and it assures that the centre of the mass of the system moves inertially.

Particular Solutions of the Gylden-Mestschersky Problem.
We let $\mu(t) = G(m_1 + m_2)$ in equation (5) to get

$$\ddot{\vec{r}} + \frac{\mu}{r^2}\frac{\vec{r}}{r} = 0 \tag{10}$$

Equation (10) takes the form

$$\ddot{r} - \frac{C^2}{r^3} + \frac{\mu}{r^2} = 0, \tag{11}$$

where $\quad C = r^2\dot{\theta} \;$ or $\; r^2\omega = C$

r is the distance between the bodies, θ is the angle between the straight line passing through m_1 and m_2, and a certain fixed straight line in the plane of motion. $\dot{\theta} = \omega(t)$ is angular velocity of revolution of the bodies and C is the constant of area integral.

Now, equation (11) has the particular solutions

$$\ddot{r} = \rho_{12}\frac{(\alpha\gamma - \beta^2)}{R^3(t)} \tag{12}$$

$$\omega(t) = \frac{\omega_0}{R^2(t)} \tag{13}$$

$$C = \rho_{12}^2\omega_0 \tag{14}$$

and

$$r\mu = \kappa C^2 \tag{15}$$

where

$$\kappa = \frac{\beta^2 - \alpha\gamma + \omega_0^2}{\omega_0^2}$$

Equation (15) is a particular integral of the Gylden-Mestschersky problem and κ is a constant. When $\beta^2 - \alpha\gamma = 0$, we get $\kappa = 1$ and this corresponds to the case when the masses are constant. When $\beta^2 - \alpha\gamma > 0$, this means that $\kappa > 1$ and when $\beta^2 - \alpha\gamma < 0$, this implies that $\kappa < 1$. Since kappa cannot be zero, the range is such that $0 < \kappa < \infty$.

Equations of Motion

The investigations of the motion and libration points of the infinitesimal mass in the restricted problem, under the condition that the motion of the variable-mass main bodies are determined by the Gylden-Mestschersky problem with isotropic mass variation of the primaries varying in proportion to each other in accordance with the unified Mestschersky law was studied by Bekov (1988). We consider a rotating frame of reference $O\,x\,y\,z$, where O is the origin and suppose that m_1 and m_2 are the masses of the primary bodies and m_3 is the mass of the third body. Let the radius vector from m_3 to m_1 be r_1, m_3 to m_2 be r_2 and the distance between the two primaries be r and let ω be the angular velocity. We consider same formulation by Bekov (1988) with further assumptions that the bigger primary is a triaxial body. In this premise, the kinetic energy in the rotating frame of reference $0xyz$ is given by

$$T = \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m_3\omega(x\dot{y} - y\dot{x}) + \frac{1}{2}m_3(x^2 + y^2)\omega^2 \tag{16}$$

while the potential energy has the form (Sharma *et. al* 2001)

$$V = -Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_1}{2m_1r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)\right] \tag{17}$$

where G is the gravitational constant.

$$I_1 = \frac{m_1}{5}\left(b^2 + c^2\right),\ I_2 = \frac{m_1}{5}\left(a^2 + c^2\right),\ I_3 = \frac{m_1}{5}\left(a^2 + b^2\right), \tag{18}$$

$I_i\left(i=1,2,3\right)$ are the moments of inertia about the principal axes of m_1 while I is the moment of inertia about the line joining the center of m_1 and m_3, and are defined by

$$I = I_1 l_1^2 + I_2 l_2^2 + I_3 l_3^2 \tag{19}$$

where $l_i\left(i=1,2,3\right)$ be the direction cosine with respect to the principal axes of m_1 and are such that

$$l_1 = \frac{x}{r_1},\ l_2 = \frac{y}{r_1} \text{ and } l_3 = \frac{z}{r_1}$$

Therefore, the potential (17) is cast to the forms

$$V = -Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_1}{2m_1 r_1^3}\left\{I_1 + I_2 + I_3 - \frac{3}{r_1^2}\left(I_1 x^2 + I_2 y^2 + I_3 z^2\right)\right\}\right]$$

If the body is displaced to the points $\left(x_1, 0, 0\right)$ the potential becomes

$$V = -Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_1}{2m_1 r_1^3}\left\{I_1 + I_2 + I_3 - \frac{3}{r_1^2}\left(I_1\left(x - x_1\right)^2 + I_2 y^2 + I_3 z^2\right)\right\}\right]$$

where $r_1^2 = \left(x - x_1\right)^2 + y^2 + z^2$ and $r_2^2 = \left(x - x_2\right)^2 + y^2 + z^2$

Hence, the potential can be rewritten as

$$V = -Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_1}{2m_1 r_1^3}\left\{-2I_1 + I_2 + I_3 - \frac{3}{r_1^2}\left(\left(I_2 - I_1\right)y^2 - \frac{3}{r_1^2}\left(I_3 - I_1\right)z^2\right)\right\}\right]$$

Substituting equations (18) at once gives

$$V = -Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_1}{10 r_1^3}\left\{2a^2 - b^2 - c^2 - \frac{3}{r_1^2}\left(a^2 - b^2\right)y^2 - \frac{3}{r_1^2}\left(a^2 - c^2\right)z^2\right\}\right]$$

Next, we assume that the triaxial shape of the first primary also changes with time as the mass changes, so that the potential V is cast into the form

$$V = -Gm_3\left[\frac{m_1(t)}{r_1} + \frac{m_2(t)}{r_2} + \frac{m_1(t)\left(2\sigma_1 - \sigma_2\right)(t)}{2r_1^3} - \frac{3m_1(t)y^2\left(\sigma_1 - \sigma_2\right)(t)}{2r_1^5} - \frac{3m_1(t)z^2\sigma_1(t)}{2r_1^5}\right] \tag{20}$$

where

$\sigma_1(t) = \dfrac{a^2 - b^2}{5r^2}$, $\sigma_2(t) = \dfrac{b^2 - c^2}{5r^2}$, is the triaxiality of the first primary.

$r_1^2 = (x - x_1)^2 + y^2 + z^2$, $\quad r_2^2 = (x - x_2)^2 + y^2 + z^2$.

Now, let p_x, p_y and p_z be the generalized components of momentum, then

$$\rho_x = m_3\left(\dot{x} - \omega y\right),\quad \rho_y = m_3\left(\dot{y} + \omega x\right),\quad \rho_z = m_3 \dot{z} \tag{21}$$

Substituting system (21) in (16), and simplifying yields

$$T = \frac{1}{2m_3}\left(\rho_x^2 + \rho_y^2 + \rho_z^2\right)$$

Now using equations (21) in the Hamiltonian H and simplifying, gives

$$H = \frac{1}{2m_3}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega\left(y p_x - x p_y\right) + V \tag{22}$$

Therefore, we have

$$\dot{p}_x = -\left(-\omega p_y\right) - \frac{\partial V}{\partial x}, \quad \dot{p}_y = -\omega p_x - \frac{\partial V}{\partial y}, \quad \dot{p}_z = -\frac{\partial V}{\partial z} \tag{23}$$

Now since the primaries move within the framework of the GMP and their masses vary with time in accordance with the UML, then p_x, p_y, p_z and the angular velocity ω will all be time dependent. Thus, differentiating system p_x, p_y and p_z with respect to time, respectively, we get

$$\dot{p}_x = m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega}y\right), \quad \dot{p}_y = m_3\left(\ddot{y} + \omega\dot{x} + \dot{\omega}x\right), \quad \dot{p}_z = m_3\ddot{z} \tag{24}$$

From systems (23) and (24), we have

$$\begin{aligned} m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega}y\right) &= \omega p_y - \frac{\partial V}{\partial x} \\ m_3\left(\ddot{y} + \omega\dot{x} + \dot{\omega}x\right) &= -\omega p_x - \frac{\partial V}{\partial y} \\ m_3\ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned} \tag{25}$$

Substituting the equations of system (21) in those of system (25), gives

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega}y - \frac{1}{m_3}\frac{\partial V}{\partial x} \\ \ddot{y} + 2\omega\dot{x} &= \omega^2 y - \dot{\omega}x - \frac{1}{m_3}\frac{\partial V}{\partial y} \\ m_3\ddot{z} &= -\frac{1}{m_3}\frac{\partial V}{\partial z} \end{aligned} \tag{26}$$

Now, from equation (20), we have

$$\begin{aligned} \frac{\partial V}{\partial x} &= Gm_3\left[\frac{m_1(x-x_1)}{r_1^3} + \frac{3}{2}\frac{m_1(2\sigma_1-\sigma_2)(t)(x-x_1)}{r_1^5} + \frac{m_2(x-x_2)}{r_2^3} - \frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)(x-x_1)y^2}{r_1^7} \right. \\ &\quad \left. - \frac{15}{2}\frac{Gm_1\sigma_1(t)(x-x_1)z^2}{r_1^7}\right] \\ \frac{\partial V}{\partial y} &= Gm_3\left[\frac{m_1 y}{r_1^3} + \frac{3}{2}\frac{m_1(2\sigma_1-\sigma_2)(t)y}{r_1^5} + \frac{3m_1(\sigma_1-\sigma_2)(t)y}{r_1^5} + \frac{m_2 y}{r_2^3} - \frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)y^3}{r_1^7} \right. \\ &\quad \left. - \frac{15}{2}\frac{Gm_1\sigma_1(t)yz^2}{r_1^7}\right] \\ \frac{\partial V}{\partial z} &= Gm_3\left[\frac{m_1 z}{r_1^3} + \frac{3}{2}\frac{m_1(2\sigma_1-\sigma_2)(t)z}{r_1^5} + \frac{m_2 z}{r_2^3} - \frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)y^2 z}{r_1^7} - \frac{15}{2}\frac{Gm_1\sigma_1(t)z^3}{r_1^7}\right] \end{aligned} \tag{27}$$

On substituting equations (27) in the respective equations of system (26), we get

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega}y - \frac{Gm_1(x-x_1)}{r_1^3} - \frac{Gm_2(x-x_2)}{r_2^3} - \frac{3}{2}\frac{Gm_1(2\sigma_1-\sigma_2)(t)(x-x_1)}{r_1^5} \\ &\quad + \frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)(x-x_1)y^2}{r_1^7} + \frac{15}{2}\frac{Gm_1\sigma_1(t)(x-x_1)z^2}{r_1^7} \end{aligned}$$

$$\ddot{y}+2\omega\dot{x}=\omega^2 y-\dot{\omega}x-\frac{Gm_1 y}{r_1^3}-\frac{Gm_2 y}{r_2^3}-\frac{3}{2}\frac{Gm_1(2\sigma_1-\sigma_2)(t)y}{r_1^5}-\frac{3Gm_1(\sigma_1-\sigma_2)(t)y}{r_1^5}$$
$$+\frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)y^3}{r_1^7}+\frac{15}{2}\frac{Gm_1\sigma_1(t)yz^2}{r_1^7} \qquad (28)$$
$$\ddot{z}=-\frac{Gm_1 z}{r_1^3}-\frac{Gm_2 z}{r_2^3}-\frac{3}{2}\frac{Gm_1(2\sigma_1-\sigma_2)(t)z}{r_1^5}+\frac{15}{2}\frac{Gm_1(\sigma_1-\sigma_2)(t)y^2 z}{r_1^7}+\frac{15}{2}\frac{Gm_1\sigma_1(t)z^3}{r_1^7}$$

Now, we let

$$Gm_1(t)=\mu_1(t),\ Gm_2(t)=\mu_2(t) \qquad (29)$$

So that equations (28) take the form:

$$\ddot{x}-2\omega\dot{y}=\omega^2 x+\dot{\omega}y-\frac{\mu_1(x-x_1)}{r_1^3}-\frac{\mu_2(x-x_2)}{r_2^3}-\frac{3}{2}\frac{\mu_1(2\sigma_1-\sigma_2)(t)(x-x_1)}{r_1^5}$$
$$+\frac{15}{2}\frac{\mu_1(\sigma_1-\sigma_2)(t)(x-x_1)y^2}{r_1^7}+\frac{15}{2}\frac{\mu_1\sigma_1(t)(x-x_1)z^2}{r_1^7}$$
$$\ddot{y}+2\omega\dot{x}=\omega^2 y-\dot{\omega}x-\frac{\mu_1 y}{r_1^3}-\frac{\mu_2 y}{r_2^3}-\frac{3}{2}\frac{\mu_1(4\sigma_1-3\sigma_2)(t)y}{r_1^5}+\frac{15}{2}\frac{\mu_1(\sigma_1-\sigma_2)(t)y^3}{r_1^7}$$
$$+\frac{15}{2}\frac{\mu_1\sigma_1(t)yz^2}{r_1^7} \qquad (30)$$
$$\ddot{z}=-\frac{\mu_1 z}{r_1^3}-\frac{\mu_2 z}{r_2^3}-\frac{3}{2}\frac{\mu_1(2\sigma_1-\sigma_2)(t)z}{r_1^5}-\frac{3\mu_1\sigma_1(t)z}{r_1^5}+\frac{15}{2}\frac{\mu_1(\sigma_1-\sigma_2)(t)y^2 z}{r_1^7}$$
$$+\frac{15}{2}\frac{\mu_1\sigma_1(t)z^3}{r_1^7}$$

where $r_1^2=(x-x_1)^2+y^2+z^2$, $r_2^2=(x-x_2)^2+y^2+z^2$

$$\omega^2(t)=\frac{\mu(t)}{\kappa r^3}\left[1+\frac{3(2\sigma_1-\sigma_2)(t)}{2r^2}\right]$$

κ is the constant of the integral (15) of the GMP; $\mu(t)$ is defined in equations (9)

These equations describe the motion of the third body having infinitesimal mass in the gravitational field of the triaxial bigger primary and the spherical smaller primary in the Barycentric coordinate system $O\ x\ y\ z$, rotating with an angular velocity $\omega(t)$ about the $z-$axis perpendicular to the plane of motion of the primaries, while the $x-$axis always passes through these points with the consideration that both primaries experience mass variations with time.

Now from the property of the center of mass μ_1 at $(x_1,0,0)$ and μ_2 at $(x_2,0,0)$ with the consideration that the origin is taken as the center of mass, we have

$$\mu_1 x_1+\mu_2 x_2=0 \qquad (31)$$

From $r^2=(x_2-x_1)^2$, we have $x_2=r+x_1$.

Substituting for x_2 in equation (31) and solving, we get

$$x_1 = -\frac{\mu_2 r}{\mu_1 + \mu_2}\ , \quad x_2 = \frac{\mu_1 r}{\mu_1 + \mu_2} \tag{32}$$

The expressions (32) connect the Barycentric coordinates x_1 and x_2 with the mutual distance r.

Autonomization of the equations with variable coefficients

The equations of motion (30) are non-integrable differential equations with variable coefficients. The solutions even for particular steady-state solutions –the EPs are difficult to seek directly from equation (30), because these equations contain unknown functions of time. In order to transform system (30), we use the MT (8); the UML (9); the particular integral (15) and solutions of the GMP.

From the MT (8), we have

$$x = \xi R(t) \ \Rightarrow \ x = \xi(\alpha t^2 + 2\beta t + \gamma)^{\frac{1}{2}}$$

Differentiating with respect to t and denoting differentiation with respect to τ by dashes, we get

$$\dot{x} = \frac{\xi(\alpha t + \beta)}{R(t)} + \frac{\xi'}{R(t)}\ , \dot{y} = \frac{\eta(\alpha t + \beta)}{R(t)} + \frac{\eta'}{R(t)}\ , \ \dot{z} = \frac{\varsigma(\alpha t + \beta)}{R(t)} + \frac{\varsigma'}{R(t)} \tag{33}$$

$$\ddot{x} = \frac{\xi(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\xi''}{R^3(t)}\ , \ \ddot{y} = \frac{\eta(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\eta''}{R^3(t)}, \quad \ddot{z} = \frac{\zeta(\alpha\gamma - \beta^2)}{R^2(t)} + \frac{\zeta''}{R^3(t)}$$

Also, from a particular solution (13) of the GMP, we get

$$\dot{\omega} = -\frac{2\omega_0(\alpha t + \beta)}{R^4(t)}$$

Further, in view of the variable triaxiality of the first primary, we introduce a transformation

$$\sigma_i(t) = \delta_i R^2(t) \tag{34}$$

where $\quad \delta_1 = \dfrac{a_{11} - b_{11}}{5\rho^2} \quad \delta_2 = \dfrac{b_{11} - c_{11}}{5\rho^2}$

Here, $\delta_i << 1(i = 1,2)$ is the triaxiality of the first primary while a_{11}, b_{11} and c_{11} are the lengths of its semi-axes of the triaxial body for the autonomized system, and ρ is the distance between the primaries.

Substituting all the above in system (30) and simplifying, yields

$$\xi'' - 2\omega_0\eta' = \omega_0^2\xi - \xi(\alpha\gamma - \beta^2) - \frac{\mu_{10}(\xi - \xi_1)}{\rho_1^3} - \frac{\mu_{20}(\xi - \xi_2)}{\rho_2^3} + \frac{3\mu_{10}(\xi - \xi_1)(2\delta_1 - \delta_2)}{2\rho_1^5} + \frac{15\mu_{10}(\xi - \xi_1)\delta_1\zeta^2}{2\rho_1^7}$$

$$\eta'' + 2\omega_0\xi' = \omega_0^2\eta - \eta(\alpha\gamma - \beta^2) - \frac{\mu_{10}\eta}{\rho_1^3} - \frac{\mu_{20}\eta}{\rho_2^3} + \frac{3\mu_{10}\eta(2\delta_1 - \delta_2)}{2\rho_1^5} + \frac{15\mu_{10}\eta\delta_1\zeta^2}{2\rho_1^7} \tag{35}$$

$$\zeta'' = (\alpha\gamma - \beta^2)\zeta - \frac{\mu_{10}\zeta}{\rho_1^3} - \frac{\mu_{20}\zeta}{\rho_2^3} - \frac{3\mu_{10}(2\delta_1 - \delta_2)\zeta}{2\rho_1^5} - \frac{3\mu_{10}\delta_1\zeta}{\rho_1^5} + \frac{15\mu_{10}\delta_1\zeta^3}{2\rho_1^7}$$

Performing same substitution on r_1^2 and r_2^2, and simplifying gives

$$\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2$$

$$\rho_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2$$

Also, we have from equations (32), that

$$x_1 = -\frac{\mu_{20}\rho_{12}R(t)}{\mu_0}, \; x_2 = \frac{\mu_{10}\rho_{12}R(t)}{\mu_0}$$

And can be expressed such that

$$x_1 = \xi_1 R(t), \qquad x_2 = \xi_2 R(t) \tag{36}$$

where $\xi_1 = -\frac{\mu_{20}}{\mu_0}\rho_{12}$, $\xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}$ are constants.

Hence, the system (30) of equations of motion with variable coefficients is now reduced to the autonomous form such that

$$\xi'' - 2\omega_0\eta' = \frac{\partial\Omega}{\partial\xi}, \qquad \eta'' + 2\omega_0\xi' = \frac{\partial\Omega}{\partial\eta}, \qquad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{37}$$

where

$$\Omega = \frac{\omega_0^2\left(\xi^2+\eta^2\right)}{2} - \frac{\left(\alpha\gamma-\beta^2\right)\left(\xi^2+\eta^2+\zeta^2\right)}{2} + \frac{\mu_{10}}{\rho_1} + \frac{\mu_{20}}{\rho_2} - \frac{\mu_{10}\left(2\delta_1-\delta_2\right)}{2\rho_1^3} - \frac{3\mu_{10}\left(\delta_1-\delta_2\right)\eta^2}{2\rho_1^5} - \frac{3\mu_{10}\delta_1\zeta^2}{2\rho_1^5}$$

$$\rho_1^2 = (\xi-\xi_1)^2+\eta^2+\zeta^2, \quad \rho_2^2 = (\xi-\xi_2)^2+\eta^2+\zeta^2 \tag{38}$$

$$\xi_1 = \frac{-\mu_{20}}{\mu_0}\rho_{12}, \quad \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}$$

Next, we make choice for units of measurements such that at initial time t_0, we choose

$$\mu_0 = G \tag{39}$$

For the unit of time and length, we choose them, respectively, such that

$$\rho_{12} = 1 \tag{40}$$

We now introduce the mass parameter υ, expressed as

$$\frac{\mu_{10}}{\mu_0} = 1-\upsilon, \; \frac{\mu_{20}}{\mu_0} = \upsilon \quad , \quad \text{where } 0 < \upsilon \le \frac{1}{2} \tag{41}$$

Substituting (39) in (41), we have

$$\mu_{10} = G\left(1-\upsilon\right) \quad , \quad \mu_{20} = G\upsilon \tag{42}$$

Now from the particular integral (15) and (39), we simplify to get

$$G = \kappa \tag{43}$$

If these measurements are substituted in equation (15), we get

$$\left(\kappa-1\right)\omega_0^2 = \beta^2 - \alpha\gamma \tag{44}$$

where κ is the constant of integration of the GMP.

Also, from equation (14), we get $C = 1$.

Therefore, substituting the units of measurement in equations of system (37) and (38), we get the equations of motion of the autonomized system with constant coefficients in the forms

$$\xi'' - 2\omega_o\eta' = \frac{\partial\Omega}{\partial\xi}, \quad \eta'' + 2\omega_o\xi' = \frac{\partial\Omega}{\partial\eta}, \quad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{45}$$

where

$$\Omega = \frac{\kappa\omega_0^2\left(\xi^2+\eta^2\right)}{2} + \frac{\omega_0^2(\kappa-1)\zeta^2}{2} + \frac{\kappa(1-\upsilon)}{\rho_1} + \frac{\kappa\upsilon}{\rho_2} + \frac{\kappa(1-\upsilon)(2\delta_1-\delta_2)}{2\rho_1^3} - \frac{3\kappa(1-\upsilon)(\delta_1-\delta_2)\eta^2}{2\rho_1^5}$$
$$-\frac{3\kappa(1-\upsilon)\delta_1\zeta^2}{2\rho_1^5}$$
$$\omega_0^2 = 1 + \frac{3(2\delta_1-\delta_2)}{2}, \quad \rho_1 = \sqrt{(\xi+\upsilon)^2+\eta^2+\zeta^2}, \quad \rho_2 = \sqrt{(\xi+\upsilon-1)^2+\eta^2+\zeta^2}$$

DISCUSSION

The paper investigates derivations of the dynamical equations of a test particle in the frame of the R3BP when the primaries undergo mass variations in accordance with the UML and their motion governed by the GMP with further assumption that the bigger primary is a triaxial body. The equations of motion of the non-autonomous equations have been derived in equations (30) and thereafter transformed to the autonomized forms with constant coefficients (45) using the MT, the particular solutions of the GMP, the UML and we introduced a transformation that helps to convert the variable triaxiality to one which is constant. These two systems of equations are different from those of Bekov (1988, 2005), Luk'yanov (1989), Sharma *et. al*. (2001), Singh & Leke (2010, 2012, 2013a), Singh & Begha (2011), Taura & Leke (2022), Alrebdi. *et.al* (2022) and, Leke &Singh (2023). The study of the R3BP is of great theoretical, practical, historical and educational relevance, and in its many variant, has had important implications in several scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics. The R3BP is still an interesting and active research field that has been receiving considerable attention from scientists and astronomers because of its applications in dynamics of the solar and stellar systems, lunar theory, and artificial satellites.

CONCLUSION

The paper modeled the R3BP to include mass variations of the primaries when the bigger primary is a triaxial body. The primaries undergo mass variation with time in accordance with unified Mestschersky Law (1952) and their motion described by the Gylden-Mestschersky problem. The equations of motion of the time dependent dynamical system were derived and transformed to the system of equations with constant coefficients using the Mestschersky transformation, the unified Mestschersky law, the particular solutions of the Gylden-Mestschersky problem and our own introduced transformation that helped in converting the time dependent triaxiality of the bigger primary to one with constant triaxiality. These models may be used to study the dynamics of a particle under the gravitational influence of a binary system, such as in the **Kruger 60** or **Achird** system.

REFERENCES

Alrebdi. H.I, Fredy, L.D and Zotos, E.E.(2022) on the equilibria of the restricted three-body problem with a triaxial rigid body II: Prolate primary. *Results in Physics*, *38*, 105623

Bekov, A. A.(1988). Libration points of the restricted problem of Three Bodies with variable Mass. *Soviet Astronomy Journal, 33*, 92-95

Bekov, A.A., Beysekov, A.N. and Aldibaeva, L.T. (2005). On the dynamics of non-stationary binary stellar systems with non-isotropic mass flow. *Astronomical and Astrophysical Transactions*, *24*, 311–316

El-Shaboury, S.M. (1990). Existence of particular solutions in the generalized restricted problem of three bodies with variable masses. *Astrophysics and Space Science 174*, 151-154.

Gahlot, P and Kishor, R.(2023) Artificial equilibrium points and their linear stability analysis in the solar sail problem with triaxial second primary. *Advances in Space Research* 71, 3262-3280

Gelf'gat, B.E. (1973). *Current Problems of Celestial Mechanics and Astrodynamics*, Nauka, Moscow

Gyegwe, J. M, Vincent, A.E and Perdiou, A. E. (2022) On the stability of triangular equilibrium points in the photogravitational R3BP with an oblate infinitesimal and triaxial primaries for the binary Lalande 21258 system, *Approximation and Computation in Science and Engineering*, 180, 397-41

Gylden, H. (1884). Die Bahnbewegungen in Einem Systeme von zwei Körpern in dem Falle, dass die Massen Ver Nderun- Gen Unterworfen Sind, *Astronomische Nachrichten*. *109*, 1-6.

Khanna, M. and Bhatnagar, K.B. (1999). Existence and stability of libration points in the restricted three-body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid". *International Journal of Pure and Applied. Mathematics 30*, 721–733.

Leke, O and Singh, J. (2023). "Out-of-plane equilibrium points of extra-solar planets in the central binaries PSR B1620-26 and Kepler-16 with cluster of material points and variable masses. *New Astronomy, 99*, 101958 Elsevier

Letelier, P. S. and Da Silva, T. A. (2011). Solutions to the restricted three-body problem with variable mass. *Astrophysics and Space Science*., *332*, 325–329

Luk'yanov, L. G. (1989). Particular solutions in the restricted problem of three bodies with variable masses. *Astronomical Journal of Academy of Sciences of USSR, 66, 180-187*

Mestschersky, I.V. (1902). Ueber die Integration der Bewegungs- gleichungen im Probleme zweier Körper von ver nderli- cher Masse, Astronomische Nachrichten. *159*, 229-242.

Mestschersky, I.V. (1952) *Works on the mechanics of bodies of variable mass*, GITTL, Moscow, p. 205.

Saeed, T.and Zotos, E.E. (2021) On the equilibria of the restricted three-body with a triaxial rigid body 1: oblate primary, *Results in Physics* 23, 103990

Sharma, R.K., Taqvi, Z.A. and Bhatnagar, K. B. (2001). Existence and stability of libration points in the restricted three-body problem when the primaries are triaxial rigid bodies and sources of radiation *Celestial Mechanics and Dynamical Astronomy*, *79*, 119-126.

Singh, J. and Ishwar, B. (1984). Effects of small perturbations in the Coriolis and centrifugal forces on the locations of equilibrium points in the restricted problem of three bodies with variable mass, *Celestial Mech*anics, *32*, 297-305.

Singh, J. and Ishwar, B. (1985). Effects of small perturbations in the Coriolis and centrifugal forces on the stability of triangular points in the restricted problem of three bodies with variable mass, *Celestial Mech*anics, *35*, 201-207.

Singh, J. and Leke, O. (2010). Stability of the photogravitational restricted three-body problem with variable masses. *Astrophysics and Space Science, 326*, 305- 314.

Singh, J. and Begha, J.M. (2011). Stability of equilibrium points in the generalized perturbed restricted three-body problem. *Astrophysics and Space Science, 331*,511-517.

Singh, J. and Leke, O. (2012) "Equilibrium points and stability in the restricted three- body problem with oblateness and variable masses" *Astrophysics and Space Science, Vol. 340: 27-41. Springer*

Singh, J. (2013). The equilibrium points in the perturbed R3BP with triaxial and luminous primaries. *Astrophysics and Space Science, 346*, 41-50

Singh, J. and Leke, O. (2013a) “Effects of oblateness, perturbations, radiation and varying masses on the stability of equilibrium points in the restricted three-body problem”. *Astrophysics and Space Science 344:* 51-61. *Springer*

Singh, J., and Leke, O. (2013b) “Existence and stability of equilibrium points in the Robe’s restricted three-body problem with variable masses,” *International Journal of Astronomy and Astrophysics, 3:* 113–122, *Scientific Research USA*

Singh, J. and Leke, O. (2013c) “Robe’s restricted three-body problem with variable masses and perturbing forces” *ISRN Astronomy* and *Astrophysics, 2013, Article ID 910354 Hindawi*

Singh, J. and Leke, O. (2013d) “On Robe’s Circular Restricted Problem of Three Variable Mass Bodies” *Journal of Astrophysics 2013, Article ID 898794, Hindawi*

Singh, J., and Leke, O. (2014) “Analytic and numerical treatment of motion of dust grain particle around triangular equilibrium points with post-AGB binary star and disc” *Advances in Space Research 54,* 1659–1677. *Elsevier*

Singh, J. and Umar, A. (2014). The collinear libration points in the elliptic R3BP with a triaxial primary and an oblate secondary. *International Journal of Astronomy and Astrophysics*, *4*, 391- 398

Szebehely, V.G. (1967). *Theory of Orbits*. Academic Press, New York

Taura, J.J and Leke, O. (2022). “Derivation of the dynamical equations of motion of the R3BP with variable masses and disk” *FUDMA Journal of Sciences*, Vol 6, 125-133

Vincent, A. E., Perdiou A.E, and E.A Perdios. (2022). Existence and stability of equilibrium points in the R3BP with triaxial-radiating primaries and an oblate massless body under the effect of the circumbinary disc *Frontiers in Astronomy and Space Sciences*, *9*, 872459

Zhang, M, J, Chang-Yin, Z and Yong-Qing, X. (2012). On the triangular libration points in photogravitational restricted three-body problem with variable mass. *Astrophysics and Space Science, 337*, 107–113