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An Approximate Solution to Volterra Integral Equation of Second Kind with Quadrature Rule

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Abstract
In this paper, the Volterra integral equation of the second kind were solved. The functional under the integrals were approximated by the quadrature rule in which the scheme is obtained. The tested problems were solved and compared the results with the exact solutions, so that to illustrate the performance of the scheme derived.

Keywords: Volterra integral equation, quadrature rule, error bound, absolute error

Introduction
Several methods are available for solving Volterra integral equation but analytic and approximate solutions are the most common and easiest methods. Let us consider the linear Volterra integral equation of the second kind

$$u(x) = \lambda \int_0^x k(x,t)\, u(t)dt + g(x), \;\; x \in [0,T] \qquad (1)$$

where u(x) is an unknown function to be determine, K(x; t) is a smooth function and is regular value of the kernel. In particular, Rahbar and Hashemizabeh (2008) established an effective algorithm for solving Fredholm integral equation of second kind that implemented the quadrature and its modification. Kamyad and Mehrabizhad (2010) established a new algorithm by means of the calculus of variation and discretization scheme, so as to use for solving linear and nonlinear Volterra integral equations. Khan and Gondal (2011) propose new modification in standard Laplace decomposition algorithm to solve Abel's second kind integral equation. Prajapati *et al.* (2012) used friendly algorithm on the variation iteration method in solving singular Volterra integral equations with generalized Abels kernel. However, Yang and Hou (2013) established the numerical Scheme for solving Volterra integral equation with convolution kernel in which the equations were first converted to an algebraic. Also Khodabi and Maleknejad (2013) solved the stochastic Volterra integral

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equations numerically with triangular functions. Al-Jawary and Shehan (2015) derived an efficient method for solving singular Volterra integral equation of second kind analytically and make the comparisons between the results.
In this article we used some quadrature rule and derived a scheme for solving the second kind Volterra integral equations and the results obtained were compared with exact solutions.

Derivation of the Scheme
Consider the linear Volterra integral equation of second kind given in (1) as

$$u(x) = \lambda \int_0^x k(x,t)\, u(t)dt + g(x), \;\; x \in [0,T]$$

Substituting x by $x+\eta$ in the above equation for $\eta > 0$, we have

$$u(x+\eta) = \lambda \int_0^{x+\eta} k(x+\eta,t)\, u(t)dt + g(x+\eta), \;\; x \in [0,T] \quad (2)$$

by splitting of the interval we have

$$u(x+\eta) = \lambda \int_0^{\eta} k(x+\eta,t)\, u(t)dt + \lambda \int_{\eta}^{x+\eta} k(x+\eta,t)\, u(t)dt + g(x+\eta), \;\; x \in [0,T] \quad (3)$$

equivalently to

$$u(x+\eta) = I_\eta + \lambda \int_0^{x+\eta} k(x+\eta,t)\, u(t)dt + g(x+\eta), \;\; x \in [0,T] \quad (4)$$

where

$$I_\eta := \lambda \int_0^{\eta} k(x+\eta,t)\, u(t)dt$$

Since I_η is known, let us define a uniform grid X_h with step size $ih = T$

$$X_h \coloneqq \{x_i = ih, \;\; 0 \le i \le T\}$$

Setting $x = ih$ in (4) we have

$$u(x_i) = I_\eta + \lambda \int_0^{x_i} k(x_i,t)\, u(t)dt + g(x_i) \quad (5)$$

Now assume that I_η is approximated by the quadrature rule $Y(u) = \sum_{j=0}^{n-1} m_j u(t_j)$, $j = 0,1,2,3,\dots,n-1$, by such an approximation for $x_i \in [0,T]$, and (5) is reduced to the equation

$$u(x_i) = I_\eta + \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j) u(t_j) + g(x_i) \qquad i = 0,1,2,\dots,n-1 \quad (6)$$

Lemma: Special Gronwall lemma: Let (e_n) and (e_j) be non-negative sequences and C a nonnegative constant if

$$u_n \le C + \sum_{k=0}^{n-1} g_k u_k \qquad for \quad n \ge 0$$

then

$$u_n \le C e^{\sum_{j=0}^{n-1} g_j} \qquad for \quad n \ge 0$$

Error Bound of the Scheme in Midpoint Rule Approach
In this section we present the error bound for the convergence of the scheme.
Theorem 3.1 Consider (1) and assume the integral I_η is known for a chosen particular solution. Then the approximate solution obtained by the Quadrature method converges with order 2 to the exact solution.
Proof
The solution u of the exact solution satisfies

$$u(x_i) = I_\eta + \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j) u(t_j) + g(x_i) + \psi(\lambda, x_i), \qquad i \ge 0 \quad (7)$$

where $\psi(\lambda, x_i)$ is the consistency error given by

$$\psi(\lambda, x_i) = \lambda \int_0^x k(x,t) u(t)dt - \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j) u(t_j) \quad (8)$$

But the exact solution is

$$u(x) = \lambda \int_0^x k(x,t)u(t)dt + g(x) \qquad (9)$$

Setting $e_i = u(x) - u(x_i)$ for $i \geq 0$ and by utilizing (9) and (7) this gives

$$e_i = \lambda \sum_{j=0}^{n-1} m_j k(x,t_j)u(t_j) - \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j)u(t_j) + \psi(\lambda, x_i)$$

$$= \lambda \sum_{j=0}^{n-1} m_j (k(x,t_j)u(t_j) - k(x_i,t_j)u(t_j)) + \psi(\lambda, x_i)$$

$$= \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j)[u(x) - u(t_j)] + \psi(\lambda, x_i), \qquad i \geq 0$$

Since $x_i \in [0,T]$ for $i \geq 0$ after the variable interchanging this yield

$$e_i = \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j)e_j + \psi(\lambda, x_i), \qquad i \geq 0 \qquad (10)$$

Taking the modulus in (10) we have

$$|e_i| \leq \lambda \textstyle\sum_{j=0}^{n-1} |m_j k(x_i,t_j)||e_j| + |\psi(\lambda, x_i)|, \quad i \geq 0 \qquad (11)$$

On the other hand from (8) we have

$$|\psi(\lambda, x_i)| = |\lambda \int_0^x k(x,t)u(t)dt - \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j)u(t_j)|$$

$$= |\lambda \sum_{j=0}^{n-1} m_j k(x,t_j)u(t_j) - \lambda \sum_{j=0}^{n-1} m_j k(x_i,t_j)u(t_j)|$$

$$\leq \lambda \sum_{j=0}^{n-1} m_j |(k(t_j,x)u(x) - k(x_i,t_j)u(t_j))|$$

But for a regular kernel after the variables interchanging we have

$$|\psi(\lambda, x_i)| \leq \lambda \sum_{j=0}^{n-1} m_j \, |u(x) - u(t_j)|$$

This implies

$$|\psi(\lambda, x_i)| \leq \lambda \textstyle\sum_{j=0}^{n-1} m_j \, |e_j|, \quad i \geq 0 \qquad (12)$$

Therefore, by substituting (12) into (11) we have

$$|e_i| \leq \lambda \sum_{j=0}^{n-1} |m_j k(x_i,t_j)||e_j| + \lambda \sum_{j=0}^{n-1} m_j \, |e_j|, \quad i \geq 0$$

$$\leq \lambda \textstyle\sum_{j=0}^{n-1} m_j |e_j||k(x_i,t_j)| + \lambda + \lambda \sum_{j=0}^{n-1} m_j \, |e_j| - \lambda, \qquad i \geq 0 \qquad (13)$$

By applying the special Gronwall lemma for the discrete in (13) we have

$$|e_i| \leq \lambda^2 e^{\left(\sum_{j=0}^{n-1} m_j |e_j|\right)|k(x_i,t_j)|} + \lambda^2 e^{\left(\sum_{j=0}^{n-1} m_j |e_j|\right)}, \quad i \geq 0 \qquad (14)$$

We obtained the error bound as

$$|e_i| \leq \lambda^2 e^{(T-1)} \left[e^{(|k(x_i,t_j)|)} + 1\right], \qquad i \geq 0$$

Hence, a second order convergence follows

Numerical Results and Discussion

In this section, we used the scheme discussed in the previous section for solving some problem of Volterra integral equations of second kind in which the integrals were approximated by the quadrature rule and maple 13 for computations.

Problem 1 *Consider the Volterra integral equation of the seccon kind*

$$u(x) = \lambda \int_0^x k(x,t)u(t)dt + g(x)$$

where $g(x) = (1-x)e^{(-x)}$ *and* $k(x,t) = e^{(t-x)}$ *with the exact solution* $u(x) = e^{(-x)}$. *The approximate solution* u_i *with* $i = 10$ *and the absolute error* $|e_i| = |u(x) - u_i|$ *are presented in Table*1

Table1: The results obtained by the numerical scheme (6) on problem1.

x	u_i	$\lvert e_i \rvert$
0	0.999990000	$1.0000E-5$
1	0.367870260	$9.1812E-6$
2	0.135335014	$2.6924E-7$
3	0.0497870120	$5.6368E-8$
4	0.0183156301	$8.7887E-9$
5	0.0067379400	$7.0000E-9$
6	0.0024785212	$8.0001E-10$
7	0.0009118815	$5.0000E-10$
8	0.0003354625	$9.0000E-11$
9	0.0001234789	$9.8000E-12$

Table (1) Shows that the performance of the scheme (6) obtained by using problem1 in which an approximate solution is obtained.

Problem 2 *Consider the Volterra integral equation of the seccon kind*

$$u(x) = \int_0^x k(x,t)u(t)dt + g(x)$$

where $g(x) = 1$ *and* $k(x,t) = (t-x)$ *with the exact solution* $u(x) = \cos(x)$. *The approximate solution* u_i *with* $i = 10$ *and the absolute error* $|e_i| = |u(x) - u_i|$ *are presented in Table* 2.

Table2: The results obtained by the numerical scheme (6) on problem2.

x	u_i	$\lvert e_i \rvert$
0	0.9872365202	$1.2763E-2$
1	0.54030000001	$2.3058E-6$
2	0.4161462130	$6.2400E-7$
3	0.9899920100	$4.8700E-7$
4	0.6536436000	$2.1001E-8$
5	0.28036621000	$8.5500E-8$
6	0.9601702812	$5.5001E-9$
7	0.7539022502	$4.1000E-9$
8	0.1455000321	$1.9000E-9$
9	0.9111302614	$6.0001E-10$

Table (2) Shows that the performance of the scheme (6) obtained by using problem 2 in which an approximate solution is obtained.

Problem 3 *Consider the Volterra integral equation of the seccon kind*

$$u(x) = \int_0^x k(x,t)u(t)dt + g(x)$$

where $g(x) = 1 + x - x^2$ *and* $k(x,t) = 1$ *with the exact solution* $u(x) = 1 + 2x$. *The approximate solution* u_i *with* $i = 10$ *and the absolute error* $|e_i| = |u(x) - u_i|$ *are presented in Table* 3.

Table3: The results obtained by the numerical scheme (6) on problem3.

x	u_i	$\mid e_i\mid$
0	0.9999564820	$4.3518E-5$
1	2.9999252600	$7.4740E-5$
2	4.9999625540	$3.7446E-5$
3	6.9999925500	$7.4500E-6$
4	8.9999996210	$3.7900E-7$
5	10.9999993236	$6.7641E-7$
6	12.9999992115	$7.8850E-7$
7	14.9999999201	$7.9900E-8$
8	16.9999999941	$5.8999E-9$
9	18.9999999992	$8.9999E-10$

Table (3) Shows that the performance of the scheme (6) obtained by using problem3 in which an approximate solution is obtained.

Conclusion

In this paper, we presented a scheme for solving the second kind linear Volterra integral equations where both kernel and function under integrals were approximated by the quadrature rule. We established the error bound analysis for the convergence of the scheme. The approximate results were obtained by comparing with the exact solutions by the use of some problems so that to test the efficiency, accuracy and effectiveness of the scheme.

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