

Fuzzy Product Rough Sets as a Special Case of Fuzzy T-Rough Sets

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Abstract

Fuzzy T-rough set consists of a set X and a T -similarity relation on X , where T is a lower semi-continuous triangular norm. In this paper, axiomatic definition for fuzzy Π -rough sets and its upper approximation operator were proposed. The method employed was by relaxing the arbitrary T and adopting its special case T_P (product triangular norm). The results obtained suggests an easier way of being specific to the product case of fuzzy rough sets and computations regarding its upper approximation operators. Some important propositions and examples were also provided.

Keywords: Fuzzy sets, rough sets, triangular norm, similarity relation, approximation operator.

INTRODUCTION

The notion of Rough sets was originally proposed by Pawlak (1982). It passed through a number of extensions and generalizations. The notion was severally compared to fuzzy set which was proposed by Zadeh (1965) to check the notion that is more general than the other. Dubois and Prade (1990) found it more natural to combine the two notions of uncertainty rather than to have them compete on the same problem. Consequently, they proposed a fuzzy rough set and rough fuzzy sets which involve the use fuzzy set and rough set within a single framework. They have also shown that the latter is a special case of the former. Morsi and Yakout (1998) studied the fuzzy T -rough set with respect to a T -similarity relation R on a universe X . They generalized the Farinas, L., Prada, H. (1986) definition for the upper approximation operator $\bar{A}: I^X \rightarrow I^X$ of a fuzzy T -rough set (X, R) , given originally for the special case $T = \text{Min}$, to the case of arbitrary T . They also proposed a new definition for the lower approximation operator $\underline{A}: I^X \rightarrow I^X$ of a fuzzy T -rough set (X, R) . Motivated by these developments, we study the fuzzy T -rough set as a special case T_P , i.e. $T = \text{product } (\Pi)$, defined as, $\forall \alpha, \beta \in I, \alpha \Pi \beta = \alpha\beta$, where I is a unit interval.

NOTATIONS

Notations	Meaning
Π	Product triangular norm
I^X	Fuzzy set on X
$I^{X \times X}$	Fuzzy binary relation on X
\vee	Supremum
$\underline{\alpha}$	Constant fuzzy set

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METHODOLOGY

Definition 3.1. (Schweizer and Sklar, 1983). A triangular norm (briefly, t-norm) is a binary operation on the unit interval $I = [0,1]$ that is associative, symmetric, monotone in each argument and has 1 as a neutral element, i.e., it is a function $T: [0,1]^2 \rightarrow [0,1]$, such that $\forall x, y, z \in [0,1]$:

- (T1). $T(x, y) = T(y, x)$,
- (T2). $T(x, T(y, z)) = T(T(x, y), z)$,
- (T3). $T(x, y) \leq T(x, z)$ whenever $y \leq z$,
- (T4). $T(x, 1) = x$.

Clement, Mesiar and Pap (2004) the following are the four basic t-norms; the minimum T_M , the product T_P , the Łukasiewicz-t-norm T_L and the drastic product T_D which are respectively defined by;

$$T_M = \min(x, y) \dots\dots\dots(1)$$

$$T_P(x, y) = x \cdot y \dots\dots\dots(2)$$

$$T_L(x, y) = \max(x + y - 1, 0) \dots\dots\dots(3)$$

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0,1]^2 \\ \min(x, y) & \text{otherwise} \end{cases} \dots\dots\dots(4)$$

The drastic product T_D and the minimum T_M are the smallest and the largest T -norms respectively (with respect to the point-wise order). The minimum T_M is the only T -norm where each $x \in I$ is an idempotent element. We have the following strict inequalities between the four basic T -norms:

$$T_D < T_L < T_P < T_M \dots\dots\dots(5)$$

Definition 2.1. (Ovchinnikov, 1991). Given a lower semi-continuous triangular norm T , a fuzzy binary relation R on X ($R \in I^{X \times X}$) is said to be a T -similarity relation if the following conditions are satisfied, $\forall x, y, z \in X$:

- i. $R(x, x) = 1$ reflexivity
- ii. $R(x, y) = R(y, x)$ symmetry
- iii. $R(x, z) T(x, y) \leq R(z, y)$ T -transitivity

Definition 2.2. (Pawlak, 1982). A fuzzy T -rough set (or a fuzzy T -approximation space) is a pair (X, R) where R is a T -similarity relation on X .

Definition 2.3. (Morsi and Yakout, 1998). An operator \bar{A} on I^X (from I^X to I^X) is said to be a fuzzy T -upper approximation operator on X if it satisfies the following axioms for all $\mu \in I^X$, ($\mu_j \in I^X$ for all $j \in J$, where J is an index set), for all $x, y \in X$ and all $\underline{\alpha} \in I$:

- (U1). $\bar{A}\mu \geq \mu$
- (U2). $\bar{A}\bar{A}\mu = \bar{A}\mu$
- (U3). $\bar{A}(\bigvee_{j \in J} \mu_j) = \bigvee_{j \in J} \bar{A}\mu_j$
- (U4). $\bar{A}(1_x)(y) = \bar{A}(1_y)(x)$
- (U5). $\bar{A}(\underline{\alpha} T \mu) = \underline{\alpha} T \bar{A}\mu$

Proposition 2.4.(Morsi and Yakout, 1998). Let R be a T -similarity relation on X . Define an operator \bar{A}_R on I^X by,
 $\bar{A}_R\mu(x) = \sup_{u \in X} (R(u, x) T \mu(u)) \dots\dots\dots(6)$

for all $\mu \in I^X$ and $x \in X$. Then \bar{A}_R is an upper approximation operator.

In next section, the arbitrary T used in Definitions 3.2, 3.3 and 3.4 was relaxed and adopted a product triangular norm T_P (also denoted by Π). Consequently, new definitions were established, namely; Π -similarity relation, fuzzy Π -rough sets and fuzzy Π -upper approximation operator.

RESULTS AND DISCUSSION

Definition 3.1. A fuzzy binary relation R on X ($R \in I^{X \times X}$) is said to be a Π -similarity relation if $\forall a, b, c \in X$ the following conditions are satisfied:

- (S1). $R(a, a) = 1$ Reflexive
- (S2). $R(a, b) = R(b, a)$ Symmetric
- (S3). $R(a, c) \Pi R(a, b) \leq R(c, b)$ Π -transitive

Definition 3.2. A fuzzy Π -rough set (or a fuzzy Π -approximation space) is a pair (X, R) where R is a Π -similarity relation on X .

Definition 3.3. An operator \bar{A} on I^X is said to be a fuzzy Π -upper approximation operator on X if the following axioms are satisfied $\forall \mu \in I^X, a, b \in X, \underline{\alpha} \in I$:

- (U1). $\bar{A}\mu \geq \mu$
- (U2). $\bar{A}\bar{A}\mu = \bar{A}\mu$
- (U3). $\bar{A}(\bigvee_{j \in J} \mu_j) = \bigvee_{j \in J} \bar{A}\mu_j$
- (U4). $\bar{A}(1_a)(b) = \bar{A}(1_b)(a)$
- (U5). $\bar{A}(\underline{\alpha} \Pi \mu) = \underline{\alpha} \Pi \bar{A}\mu$

Proposition 3.4. Suppose R is a Π -similarity relation on X , then $\forall \mu \in I^X, a \in X$ an operator \bar{A}_R on I^X defined by:

$$\bar{A}_R\mu(a) = \bigvee_{u \in X} (R(u, a) \Pi \mu(u)) \dots \dots \dots (7)$$

is a fuzzy Π -upper approximation operator on X .

Proof:

Let $a, b \in X, \mu \in I^X, \mu_j \in I^X$ for all $j \in J$, then

- i. $\bar{A}_R\mu(a) \geq R(a, a) \Pi \mu(a) = 1 \Pi \mu(a) = \mu(a)$ by (S1)
- ii. $\bar{A}_R\bar{A}_R\mu(a) = \bigvee_{u \in X} (R(u, a) \Pi \bar{A}_R\mu(u))$
 $= \bigvee_{u \in X} \bigvee_{v \in X} (R(u, a) \Pi R(v, u) \Pi \mu(v)), \Pi$ is associative
 $\leq \bigvee_{v \in X} (R(a, a) \Pi R(v, a) \Pi \mu(v))$ By (S3)
 $= \bigvee_{v \in X} (1 \Pi R(v, a) \Pi \mu(v))$
 $= \bigvee_{v \in X} (R(v, a) \Pi \mu(v))$
 $= \bar{A}_R\mu(a)$

Therefore, $\bar{A}_R\bar{A}_R\mu \leq \bar{A}_R\mu$, and by (i) equality follows, i.e. $\bar{A}_R\bar{A}_R\mu = \bar{A}_R\mu$

- iii. $\bar{A}_R(\bigvee_{j \in J} \mu_j)(a) = \bigvee_{u \in X} (R(u, a) \Pi \bigvee_{j \in J} \mu_j(u))$
 $= \bigvee_{u \in X} \bigvee_{j \in J} (R(u, a) \Pi \mu_j(u))$
 $= \bigvee_{j \in J} (\bigvee_{u \in X} (R(u, a) \Pi \mu_j(u)))$
 $= \bigvee_{j \in J} \bar{A}_R\mu_j(a)$
- iv. $\bar{A}_R(1_a)(b) = \bigvee_{u \in X} (R(u, b) \Pi (1_a)(u)) = R(a, b) = \bar{A}_R(1_b)(a)$ By symmetry of R
Therefore, $R(a, b) = R(b, a)$

$$\begin{aligned} \text{v. } \bar{A}_R(\underline{\alpha}\Pi\mu)(a) &= \bigvee_{u \in X} (R(u, a) \Pi (\underline{\alpha}\Pi\mu)(u)) \\ &= \underline{\alpha} \Pi (\bigvee_{u \in X} (R(u, a) \Pi \mu(u))) \\ &= \underline{\alpha} \Pi \bar{A}_R\mu(a) \end{aligned}$$

Hence, \bar{A}_R having satisfied (U1) - (U5) is a fuzzy Π -upper approximation operator on I^X .

Proposition 3.5. Let \bar{A} be a fuzzy Π -upper approximation operator on I^X . Define a fuzzy binary relation $R_{\bar{A}}$ on X by, $\forall a, b \in X$:

$$R_{\bar{A}}(a, b) = \bar{A}(1_a)(b) \dots \dots \dots (8)$$

Then, $R_{\bar{A}}$ is a Π -similarity relation.

Proof:

- i. $R_{\bar{A}}(a, a) = \bar{A}(1_a)(a) \geq (1_a)(a) = 1$ By (U1)
- ii. $R_{\bar{A}}(a, b) = \bar{A}(1_a)(b) = \bar{A}(1_b)(a) = R(b, a)$ By (U4)
- iii. $R_{\bar{A}}(a, b) = \bar{A}(1_a)(b) = \bar{A}(\bar{A}(1_a)(b))$ By (U2)
 $= \bigvee_{u \in X} (\bar{A}(1_u)(b) \Pi \bar{A}(1_a)(u))$ By (7)
 $= \bigvee_{u \in X} (R(u, b) \Pi R(a, u)).$

Hence (i) - (iii) shows that $R_{\bar{A}}$ is reflexive, symmetric and Π -transitive, which proves that $R_{\bar{A}}$ is a Π -similarity relation on X .

The results obtained would be an important tools for handling fuzzy rough sets based on the product triangular norm. The axiomatic definitions for fuzzy T -rough sets and its approximation operators provided in the existing literatures such as Morsi, N. N., Yakout M. M. (1998), Mi, J. S., Zhang, W. X. (2004), and Guilong, L. (2008) can only handle a fuzzy rough set based on the general case of the triangular norm without being specific to a particular one.

Example 3.6: Let $X = \{a, b, c\}$. Define a fuzzy binary relation on X by:

$$R = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 0.3 & 0.8 \\ 0.3 & 1 & 0.3 \\ 0.8 & 0.3 & 1 \end{pmatrix} \end{matrix}$$

- i. $R(a, a) = R(b, b) = R(c, c) = 1$ R is reflexive
- ii. $R(a, b) = R(b, a) = 0.3$
 $R(a, c) = R(c, a) = 0.8$
 $R(b, c) = R(c, b) = 0.3$ R is symmetric

- iii. $R(a, b) \Pi R(a, c) = 0.3 \Pi 0.8 = 0.24 < R(b, c),$
 $R(b, a) \Pi R(b, c) = 0.3 \Pi 0.3 = 0.09 < R(a, c),$
 $R(c, a) \Pi R(c, b) = 0.8 \Pi 0.3 = 0.24 < R(a, b)$ R is Π -transitive.

Hence R is a Π -similarity relation on X and the pair (X, R) is a fuzzy Π -rough set.

Now consider (7) and let $\mu = (a_{0.2}, b_{0.5}, c_{0.8})$, then

$$\begin{aligned} \bar{A}_R\mu(a) &= \bigvee (R(a, a) \Pi \mu(a), R(b, a) \Pi \mu(b), R(c, a) \Pi \mu(c)) \\ &= \bigvee (1 \Pi 0.2, 0.3 \Pi 0.5, 0.8 \Pi 0.8) \\ &= \bigvee (0.2, 0.15, 0.64) \\ &= 0.64 \\ \bar{A}_R\mu(b) &= \bigvee (R(a, b) \Pi \mu(a), R(b, b) \Pi \mu(b), R(c, b) \Pi \mu(c)) \\ &= \bigvee (0.3 \Pi 0.2, 1 \Pi 0.5, 0.3 \Pi 0.8) \\ &= \bigvee (0.06, 0.5, 0.24) \end{aligned}$$

$$\begin{aligned}
 &= 0.5 \\
 \bar{A}_R\mu(c) &= \vee(R(a, c) \Pi \mu(a), R(b, c) \Pi \mu(b), R(c, c) \Pi \mu(c)) \\
 &= \vee(0.8 \Pi 0.2, 0.3 \Pi 0.5, 1 \Pi 0.8) \\
 &= \vee(0.16, 0.15, 0.8) \\
 &= 0.8
 \end{aligned}$$

Therefore, $\bar{A}_R\mu = (a_{0.64}, b_{0.5}, c_{0.8})$

CONCLUSION

The concept of fuzzy T-rough sets was used to establish a fuzzy Π -rough set (also called a fuzzy Π -approximation space). Π -similarity relation was defined and used as the determinant of the fuzzy Π -rough set. Some important propositions were provided with counter examples. Meanwhile, the same concept can be extended to some other special cases of continuous triangular norm such as Minimum (denoted by m or \wedge), Lukasiewicz conjunction (denoted by W) etc.

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