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Application of Jacobi and Gauss–Seidel Numerical Iterative Solution Methods for the Stationary Distribution of Markov Chain

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Abstract
The Physical or Mathematical behaviour of this model may be represented by describing all the different states it may occupy and by indicating how it moves among these states. In this study, the stationary distribution of Markov chains was solved using iterative methods that begin with an initial estimate of the solution vector and then modified it in a way that brings it closer and closer to the real solution with each step or iteration. These methods also involved matrix operations like multiplication with one or more vectors, which preserves the transition matrices while speeding up the process. We computed the solutions using Jacobi iterative method and Gauss-Seidel iterative method in order to shed more light on the solutions of stationary distribution in Markov chain. This was done with the aid of several already-existing laws, theorems, and formulas of Markov chain and the application of normalization principle and matrix operations such as lower, upper, and diagonal matrices. The stationary distribution vector's π_i, $i = 1, 2, \dots, 4$ *are obtained for the illustrative example one as* $\pi^{(3)} = (0.078125, 0.109375, 0.21875, 0.59375)$ *as well as the four eigenvalues of the matrix as* $\lambda_1 = 1.0$, $\lambda_2 = -0.7718$, $\lambda_{3,4} = -0.1141 \pm 0.5576i$ *using Jacobi iterative technique, and for illustrative example two using Gauss-Siedel method as* $\boldsymbol{\pi^{(3)}}$ *= (0.090909, 0.181818, 0.363636, 0.363636). The research shown that Gauss Siedel method converged faster than Jacobi method.*

Keywords: Gauss-Seidel, Gerschgorin's theorem, Jacobi, compact storage, Successive over-relaxation

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INTRODUCTION

The two different types of solution methods used in the field of numerical analysis are iterative and direct. The iterative methods begin with an initial estimation of the solution vector and then modified it so that with each step or iteration, it gets more and more close to the real solution and it eventually arrives to the actual solution. If an initial approximation is unknown, an estimate is made or an arbitrary initial vector is utilized in its place and the answer must be computed when a predetermined number of clearly defined stages have been completed.

Iterative methods, in one form or another, are the most often used techniques for obtaining the stationary probability vector from either the stochastic transition probability matrix or the infinitesimal generator, and there were many factors considered when making this choice. In the conventional iterative techniques, matrices are only used in multiplication with one or more vectors, which leaves the transition matrices unaltered. Thus, it is necessary to design compact storage solutions that decrease the amount of memory required to store the matrix while also being suitable for matrix multiplication Stewart (2009).

Iterative techniques provide benefits when doing a sequence of connected tests, and it is very helpful to use good starting approximations of the solution vector. Also, rounding error buildup is essentially nonexistent with iterative techniques because the matrix is never modified. For these reasons, iterative procedures have typically been favoured to direct ones. In the application of direct approaches, the coefficient matrix data structure is a source of concern as Markov models produce matrices that are frequently too big to fit into a standard two-dimensional array for storage in computer memory. Also, due to the volume of fill-in that can quickly overwhelm available storage capacity, direct approaches are typically not recommended when the transition matrix is large and not bounded.

Romanovsky (1970) introduced the Markov chain as a mathematical representation of the movement of a tracer throughout the cardiovascular system (CVS), and assumed the existence of a steady state condition for the CVS results in a Markov chain of finite order with constant coefficients, while Saff (1973) showed the degree of the greatest logical approximation to the exponential function in Markov chain. Moler and Van Loan (1978) explained the nineteen problematic techniques to compute the exponential of a matrix in order to calculate the transient solution of continuous time Markov processes. The Ramaswami's formula is a relationship that may be used to determine the consecutive components of stationary distribution π after the matrix G has been computed.

The Ramaswami's initial probabilistic technique is given by Ramaswami and Neuts (1980), while Ramaswami (1988) used an algebraic approach, and Philippe and Sidje (1993) took advantage of the knowledge contained in infinitesimal generators of Markov processes to compute efficiently and economically the transient solution of continuous time Markov chain. Stewart (1994, 2009) provided a methodical and in-depth analysis of the numerical solution of a Markov chain with applications to biological systems, computer system, economical system and educational system.

Maryam and Ali (2013) proposed that the experiment has the acceptable speed without suffering from the problem of computational complexity, while Pesch *et al.* (2015) demonstrated the appropriateness of the Markov chain technique in the wind feed in Germany. Meanwhile, Swanirbhar (2015) reviewed the Jacobi iterative solver and its

hardware based performance analysis. With special attention paid to 1-norms, Glaister (2016) provided a detailed investigation of the link between theoretical convergence results and observed monotonic and non-monotonic convergence. Uzun and Kiral (2017) used the Markov chain model of fuzzy state to anticipate the direction of gold price movement and to estimate the probabilistic transition matrix of gold price closing returns. Tesfaye (2017) demonstrated the usefulness of second degree refinement Jacobi (SDRJ) in contrast to other techniques by presenting a second degree for solving the system of linear equations $ay = b$ and taking into account the spectral radius.

Azizah *et al.* (2019) predicted monthly rainfall data using the Markov chain model of fuzzy state, and Clemence (2019) used the Markov chain to show that Hepatitis B became more contagious over time than tuberculosis and HIV when it comes to the transmission of infectious diseases. Additionally, Zakaria *et al.* (2019) developed Markov chain model for forecasting Miri's air pollution index, while Vermeer and Trilling (2020) used the Markov chain to show how it may be used in journalism. Also, by taking into account consistently ordered 3-cyclic matrices that are obtained when a finite difference method is used to solve differential equations, Gashaye and Tesfaye (2021) presented a generalized refinement of the Gauss-Siedel method and demonstrated that a smaller number of iterations could be completed with a higher rate of convergence.

Agboola and Badmus (2021) analysed the distribution function of the renewal process and sequence $\{Xn, n \geq i\}$ using the concept of discrete time Markov chain to obtain performance measures, and concluded that, it is not possible for an infinite number of renewals to occur in a finite period of time. Agboola and Ayoade (2021) investigated a block structure that arises frequently when modeling queueing systems to provide some insight into the solutions of stationary distribution of Markov chain.

Agboola and Ayinde (2021) analysed the concept of the classification of groups of states, between states that are recurrent, transient to provide some insight into the performance measure analysis such as the mean first passage time, the mean recurrence time of state as well as recurrence iterative matrix. Agboola (2021) computed the solutions and algorithms for lower - upper triangular matrix approach and the Grassmann - Taksar - Heyman to obtained the stationary distribution vectors in Markov chain. Furthermore, Agboola and Ayinde (2022) demonstrated the use of successive over-relaxation algorithm using block numerical iterative solution methods to compute the stationary distributions vectors in Markov chain.

Agboola and Ayoade (2022) investigated a block structure that arises frequently when modeling G/M/1 queueing systems to provide some insight into the solutions of stationary distribution of Markov chain using block lower Hess Enberg numerical iterative methods on the structured Markov chain. Agboola (2022) estimated an approximate solution to the stationary probability vector and the global solution using decomposition and aggregation algorithmic numerical iterative methods by considering the following steps, the left-hand eigenvector of length corresponding to the Eigen value closest to 1 and the weights while Agboola and Nehad (2022) computed the solutions and algorithms for tiny state spaces utilizing matrix scaling and powering approaches.

However, in this study, application of Jacobi and Gauss–Seidel numerical iterative solution methods for the stationary distribution of Markov chain is demonstrated in order to shed more light on the solution of stationary distribution of Markov chain by using existing laws and theorems of Markov chain, and it was shown through the illustrative example 1 and 2

that Gauss-Siedel iterative methods converged faster than Jacobi iterative method. Also, in line with Gerschgorin's theorem, It was shown that the stationary probability vector is the eigenvector that corresponds to a dominant eigenvalue of iterative matrix H_J.

METHODOLOGY

This study adopted the computation and comparison of Jacobi and Gauss–Seidel numerical iterative solution methods for the stationary distribution of Markov chain. We started with the analysis of the Jacobi iterative method.

The Iterative Methods of Jacobi

Building iterative solution methods involves describing a problem as an equation of the form $f(y) = 0$. The formula $f(y) = Ay - B$ is utilized if the function $f(y)$ is nonlinear, or even a system of linear equation. An iterative technique can be produced by setting $f(y) = 0$ as $y = g(y)$, and then establishing the iterative process.

$$y^{k+1} = g\left(y^{(k)}\right). \qquad (1)$$

$y^{(0)}$ is an initial approximation. Alternatively, the value from the preceding iteration is copied and pasted onto the right-hand side of the current iteration. Jacobi, Gauss-Seidel, and sequential over-relaxation (SOR) are among the traditional and well-known iterative techniques for resolving systems of linear equations. These techniques are founded on a set of nonhomogeneous linear equations.

$$Ay = B \qquad \text{or} \qquad Ay - B = 0,$$

an iterative formula of the form

$$y^{k+1} = H\left(y^{(k)}\right) + C, \quad k = 0, 1, \ldots \qquad (2)$$

This is achieved by separating the coefficient matrix A.

$$A = M - N$$

with nonsingular *M*, we have

$$(M - N)y = B$$

Or

$$My = N\,y + B,$$

Consequently, the iterative operation is started.

$$y^{(k+1)} = M^{-1}Ny^{(k)} + M^{-1}B = H\left(y^{(k)}\right) + C, \qquad k = 0, 1, \ldots \qquad (3)$$

The eigenvalues of the iteration matrix, which is given as $H = M^{-1}N$, are used to calculate the rate of convergence of the iterative method. M and N are chosen using various strategies via the Jacobi and Gauss-Seidel methods. By Starting with Jacobi's approach and considering the nonhomogeneous system of linear equations $Ay = B$, where $A \epsilon \Re^{(n\times n)}$ is nonsingular and $B \neq 0$. The nonhomogeneous system of linear equations is written as.

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \cdots + a_{1n}y_n &= b_{1,} \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \cdots + a_{2n}y_n &= b_{2,} \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \cdots + a_{3n}y_n &= b_3, \\ &\\ a_{n1}y_1 + a_{n2}y_2 + a_{n3}y_3 + \cdots + a_{nn}y_n &= b_n. \end{aligned}$$

The result is obtained by moving all terms with off-diagonal components a_{ij}, $i \neq j$ to the right as

$$\begin{aligned} a_{11}y_1 &= b_1 \quad -a_{12}y_2 - a_{13}y_3 - \cdots - a_{1n}y_n, \\ a_{22}y_2 &= b_2 - a_{21}y_1 \quad - a_{23}y_3 - \cdots - a_{2n}y_n, \\ a_{33}y_3 &= b_3 - a_{31}y_1 - a_{32}y_2 \quad - \cdots - a_{3n}y_n, \end{aligned}$$

$$\vdots$$
$$a_{nn}y_n = b_n - a_{n1}y_1 - a_{n2}y_2 - a_{n3}y_3 - \cdots - \quad .$$

The method should now be iterated upon. Using the prior values (the values obtained at iteration k), we can assign new values to the right-hand side components of y as follows:

$$\begin{aligned} a_{11}y_1^{(k+1)} &= \quad -a_{12}y_2^k - a_{13}y_3^k - \cdots - a_{1n}y_n^k + b_1\,, \\ a_{22}y_2^{(k+1)} &= -a_{21}y_1^k \quad - a_{23}y_3^k - \cdots - a_{2n}y_n^k + b_2, \\ a_{33}y_3^{(k+1)} &= -a_{31}y_1^k - a_{32}y_2^k \quad - \cdots - a_{3n}y_n^k + b_3, \\ &\vdots \\ a_{nn}y_n^{(k+1)} &= -a_{n1}y_1^k - a_{n2}y_2^k - a_{n3}y_3^k - \cdots - \quad + b_n. \end{aligned} \qquad (4)$$

This is the Jacobi iterative method. In matrix form, A is split as $A = E - L - U$ where
- E is a diagonal matrix,
- L is a strictly lower triangular matrix,
- U is a strictly upper triangular matrix,

and so the method of Jacobi becomes equivalent to

$$EY^{(k+1)} = (L + U)Y^{(k)} + B,$$

or

$$Y^{(k+1)} = E^{-1}(L + U)Y^{(k)} + E^{-1}B. \qquad (5)$$

It is important to note that the diagonal matrix E must be nonsingular for this system to function. As a result, $M = E$ and $N = (L + U)$ are split according to Jacobi's method. The matrix of its iterations is as follows:

$$H_J = E^{-1}(L + U).$$

The system of equations whose solution we seek in the Markov chain setting is

$$\pi Q = 0, \text{ or, equivalently, } Q^T\pi^T = 0.$$

Setting $y = \pi^T$, and let $Q^T = E - (L + U)$. Due to the fact that $E_{jj} \neq 0$ and E^{-1} exists for all j, E is a nonsingular matrix. The next approximation is formed by solving the system of equations after the k^{th} approximation, $Y^{(k)}$, has been produced.

$$EY^{(k+1)} = (L + U)Y^{(k)},$$

or

$$Y^{(k+1)} = E^{-1}(L + U)Y^{(k)}.$$

In scalar form,

$$y_i^{(k+1)} = \frac{1}{e_{ii}}\left\{\sum_{i\neq j}\left(l_{ij} + u_{ij}\right)y_j^{(k)}\right\}, \quad i = 1, 2, \ldots, n. \qquad (6)$$

The Iterative Methods of Gauss–Seidel

When using the Jacobi technique to do the calculations outlined in equation (3), the components of the vector $Y^{(k+1)}$ are typically acquired one at a time as $y_1^{(k+1)}$, $y_2^{(k+1)}, \ldots, y_n^{(k+1)}$. Only components from the previous iteration $Y^{(k)}$ are used for evaluating $y_i^{(k+1)}$, and the Gauss-Seidel approach uses the most recent component approximations even though components from the current iteration $y_j^{(k+1)}$, for $j < 1$, are accessible and (hopefully) more accurate. This can be done by simply overwriting parts with the new approximation once it has been found. equation (4) can be rewritten using the most recent values to produce.

$$\begin{aligned} a_{11}y_1^{(k+1)} &= \quad -a_{12}y_2^k - a_{13}y_3^k - \cdots - a_{1n}y_n^k + b_1\,, \\ a_{22}y_2^{(k+1)} &= -a_{21}y_1^{(k+1)} \quad - a_{23}y_3^k - \cdots - a_{2n}y_n^k + b_2, \end{aligned}$$

$$a_{33}y_3^{(k+1)} = -a_{31}y_1^{(k+1)} - a_{32}y_2^{(k+1)} \quad - \cdots - a_{3n}y_n^k + b_3,$$
$$\vdots$$
$$a_{nn}y_n^{(k+1)} = -a_{n1}y_1^{(k+1)} - a_{n2}y_2^{(k+1)} - a_{n3}y_3^{(k+1)} - \cdots - a_{n(n)}y_n^{(k)} + b_n. \quad (7)$$

The second equation uses the value of the freshly computed first component, y_1, and reads $y_1^{(k+1)}$ rather than $y_1^{(k)}$. The new values of $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ are used in the third equation, and all components other than the last are used in the final equation. The $\boldsymbol{i^{th}}$ equation is expressed when there are n unknowns and n linear equations.

$$a_{ii}y_i^{(k+1)} = \left\{\sum_{j=1}^{i-1} a_{ij}y_j^{(k+1)} - \sum_{j=1+1}^{n} a_{ij}y_j^{(k)}\right\}, \quad i = 1, 2, \ldots, n. \quad (8)$$

Rearranging these equations so that all new values appear on the left-hand side, we find

$$a_{11}y_1^{(k+1)} = \quad -a_{12}y_2^k - a_{13}y_3^k - \cdots - a_{1n}y_n^k + b_1\,,$$
$$a_{22}y_2^{(k+1)} + a_{21}y_1^{(k+1)} = \quad -a_{23}y_3^k - \cdots - a_{2n}y_n^k + b_2,$$
$$a_{33}y_3^{(k+1)} + a_{31}y_1^{(k+1)} + a_{32}y_2^{(k+1)} = \quad - \cdots - a_{3n}y_n^k + b_3,$$
$$\vdots$$
$$a_{nn}y_n^{(k+1)} + a_{n1}y_1^{(k+1)} + a_{n2}y_2^{(k+1)} + a_{n3}y_3^{(k+1)} = \; - \cdots - \; a_{n(n)}y_n^{(k)} + b_n. \quad (9)$$

The Gauss–Seidel iterative approach is equal to using the same $E - L - U$ splitting as Jacobi.

$$(E - L)Y^{(k+1)} = UY^{(k)} + B \quad (10)$$

This is simply the system of equations of matrix in equation (8). It is possible to write it as

$$Y^{(k+1)} = E^{-1}\left(LY^{(k+1)} + UY^{(k)} + B\right),$$

Or

$$Y^{(k+1)} = (E - L)^{-1}UY^{(k)} + (E - L)^{-1}B. \quad (11)$$

Thus, the iteration matrix for the method of Gauss–Seidel is given by

$$H_{GS} = (E - L)^{-1}U. \quad (12)$$

This iterative technique, which only works with nonsingular matrices (E – L) is comparable to splitting $M = (E - L)$ and $N = U$ most of the time. equation (9) becomes equation (10) because homogeneous systems of equations such as those found in Markov chains have a right side that is zero.

$$Y^{(k+1)} = (E - L)^{-1}UY^{(k)} \text{ or } Y^{(k+1)} = H_{GS}Y^{(k)}. \quad (13) \text{ Stewart (2009)}$$

Furthermore, the inverse, $(E - L)^{-1}$, exists since all of E's diagonal elements are nonzero. The stationary probability vector $\pi = Y^T$ clearly satisfies $H_{GS}Y = y$, suggesting that y is the right-hand eigenvector corresponding to a unit eigenvalue of H_{GS}. The unit eigenvalue of the matrix H_{GS} is a dominant eigenvalue due to the Stein-Rosenberg theorem and the fact that the associated Jacobi iteration matrix H_J has a dominant unit eigenvalue. The H_{GS} power approach and the Gauss-Seidel method are comparable as a consequence.

Nomenclature

Q , infinitesimal generator matrix; π, stationary distribution; y, unknown variable; E, nom singular matrix;

L, lower triangular matrix; U, upper triangular matrix; H_{GS}, iteration matrix for Gauss-Siedel and λ_i, eigen vector for $i = 1, 2, \ldots, k$.

RESULTS AND DISCUSSION

This section discusses the solutions of stationary distributions, $\pi^{(i)}, i = 0, 1, \ldots, k$ and the eigenvector corresponding to a dominating eigenvalue of H_J, using various illustrative examples for both Jacobi and Gauss-Siedel iterative formulae

Illustrative example using Jacobi iterative formulae: Consider a four-state Markov chain with stochastic transition probability matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.125 & 0.125 & 0.25 & 0.5 \end{pmatrix}$$

Since we are given P rather than Q, we need to write $\pi P = \pi$ as $\pi(P - 1) = 0$ and take $Q = (P - 1)$:

$$Q = \begin{pmatrix} -0.5 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \\ 0.125 & 0.125 & 0.25 & -0.5 \end{pmatrix}$$

Transposing this, we obtain the system of equations

$$\begin{pmatrix} -0.5 & 0 & 0 & 0.125 \\ 0.5 & -0.5 & 0 & 0.125 \\ 0 & 0.5 & -0.5 & 0.250 \\ 0 & 0 & 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing this in full, we have

$$\begin{aligned} -0.5\pi_1 + \quad 0\pi_2 + \quad 0\pi_3 + \quad 0.125\pi_4 &= 0, \\ 0.5\pi_1 + \quad -0.5\pi_2 + \quad 0\pi_3 + \quad 0.125\pi_4 &= 0, \\ 0\pi_1 + \quad 0.5\pi_2 + \quad -0.5\pi_3 + \quad 0.250\pi_4 &= 0, \\ 0\pi_1 + \quad 0\pi_2 \; + \quad 0.5\pi_3 + \quad -0.5\pi_4 &= 0, \end{aligned}$$

or

$$\begin{aligned} -0.5\pi_1 \qquad\qquad\quad &= -0.125\pi_4, \\ -0.5\pi_2 \qquad\quad &= -0.5\pi_1 - 0.125\pi_4, \\ -0.5\pi_3 \quad &= -0.5\pi_2 - 0.250\pi_4, \\ -0.5\pi_4 &= -0.5\pi_3, \end{aligned}$$

From this we can write the iterative version,

$$\begin{aligned} -0.5\pi_1^{(k+1)} \qquad\qquad\quad &= -0.125\pi_4^{(k)}, \\ -0.5\pi_2^{(k+1)} \qquad\quad &= -0.5\pi_1^{(k)} - 0.125\pi_4^{(k)}, \\ -0.5\pi_3^{(k+1)} \quad &= -0.5\pi_2^{(k)} - 0.250\pi_4^{(k)}, \\ -0.5\pi_4^{(k+1)} &= -0.5\pi_3^{(k)}, \end{aligned}$$

which leads to

$$\begin{aligned} \pi_1^{(k+1)} &= 0.25\pi_4^{(k)}, \\ \pi_2^{(k+1)} &= \pi_1^{(k)} - 0.25\pi_4^{(k)}, \\ \pi_3^{(k+1)} &= \pi_2^{(k)} - 0.5\pi_4^{(k)}, \\ \pi_4^{(k+1)} &= \pi_3^{(k)}. \end{aligned}$$

We may now begin the iterative process. Starting with

$$\pi^{(0)} = (0.5 \quad 0.25 \quad 0.125 \quad 0.125)$$

we obtain

$$\pi_1^{(1)} = 0.25\pi_4^{(0)} = 0.25 \times 0.125 = 0.03125,$$
$$\pi_2^{(1)} = \pi_1^{(0)} - 0.25\pi_4^{(0)} = 0.5 - 0.25(0.125) = 0.53125,$$
$$\pi_3^{(1)} = \pi_2^{(0)} - 0.5\pi_4^{(0)} = 0.25 - 0.5(0.125) = 0.3125,$$
$$\pi_4^{(1)} = \pi_3^{(0)} = 0.125.$$

In this case, no further normalization is required because the total of the components of $\boldsymbol{\pi^{(1)}}$ equals 1. In fact, for any iteration, $k + 1$

$$\sum_{i=1}^{4} \pi_i^{(k+1)} = 0.25\pi_4^{(k)} + \pi_1^{(k)} + 0.25\pi_4^{(k)} + \pi_2^{(k)} + 0.50\pi_4^{(k)} + \pi_3^{(k)} = \sum_{i=1}^{4} \pi_i^{(k)}$$

If the initial approximation has components that add up to 1, the sum of the components of all approximations to the stationary distribution will always equal 1. By employing the iterative Jacobi method, the following series of approximations is produced:

$$\pi^{(0)} = (.50000, .25000, .12500, .12500),$$
$$\pi^{(1)} = (.03125, .53125, .31250, .12500),$$
$$\pi^{(2)} = (.03125, .06250, .59375, .31250),$$
$$\pi^{(3)} = (.078125, .109375, .21875, .59375).$$
$$\vdots$$

$(L + U)y = Ex$, is produced when $Q^T = E - (L + U)$ is substituted for $Q^T y = 0$, and since $\boldsymbol{E}$ is nonsingular, the eigenvalue equation is obtained.

$$E^{-1}(L + U)y, \qquad (14)$$

where the right-hand eigenvector of the matrix $E^{-1}(L + U)$ is y and the unit eigenvalue $\boldsymbol{y}$ is an eigenvalue. The Jacobi iterative method matrix, H_J, will be evident right away. The equation (6) shows that the eigenvalue of H_J is unitary and there is also $\boldsymbol{Q^T}$'s zero-column-sum attribute. Therefore, the diagonal matrix E is written as

$$e_{ij} = \sum_{i=1, i\neq j}^{n} (l_{ij} + u_{ij}), \; j = 1, 2, ..., \qquad (15)$$

Gerschgorin's theorem, which states that no H_J eigenvalue may have a modulus greater than one, leads to the statement that for all $i, j, \; i \neq j, l_{ij}, \; u_{ij} \leq 0$. This theorem states that the union of the n circular disks with centers $c_i = a_{ii}$ and radii $r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$ contains the eigenvalues of any square matrix A of order n. In this way, the stationary probability vector is the eigenvector that corresponds to a dominant eigenvalue of H_J, and the Jacobi method is comparable to the power method used with the iteration matrix H_J.

Consequently, the Jacobi iteration matrix is represented as

$$H_J = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & -0.125 \\ -5 & 0 & 0 & -0.125 \\ 0 & -0.5 & 0 & -0.250 \\ 0 & 0 & -0.5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0.25 \\ 1.0 & 0 & 0 & 0.25 \\ 0 & 1.0 & 0 & 0.50 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The four eigenvalues of this matrix are

$$\lambda_1 = 1.0, \qquad \lambda_2 = -0.7718, \qquad \lambda_3 = -0.1141 \pm 0{,}5576i$$

Illustrative example using Gauss–Seidel iterative formulae: by considering the example previously solved using Jacobi's approach.

$$\begin{aligned} -0.5\pi_1 &= -0.125\pi_4, \\ -0.5\pi_2 &= -0.5\pi_1 - 0.125\pi_4, \\ -0.5\pi_3 &= -0.5\pi_2 - 0.250\pi_4, \\ -0.5\pi_4 &= -0.5\pi_3, \end{aligned}$$

We can write the iterative version from this.

$$\begin{aligned} -0.5\pi_1^{(k+1)} &= -0.125\pi_4^{(k)}, \\ -0.5\pi_2^{(k+1)} &= -0.5\pi_1^{(k+1)} - 0.125\pi_4^{(k)}, \\ -0.5\pi_3^{(k+1)} &= -0.5\pi_2^{(k+1)} - 0.250\pi_4^{(k)}, \\ -0.5\pi_4^{(k+1)} &= -0.5\pi_3^{(k+1)}, \end{aligned}$$

$$\begin{aligned} \pi_1^{(k+1)} &= 0.25\pi_4^{(k)}, \\ \pi_2^{(k+1)} &= \pi_1^{(k+1)} - 0.25\pi_4^{(k)}, \\ \pi_3^{(k+1)} &= \pi_2^{(k+1)} - 0.5\pi_4^{(k)}, \\ \pi_4^{(k+1)} &= \pi_3^{(k+1)}. \end{aligned}$$

Now is the time to start the iterative process. To begin with,

$$\pi^{(0)} = (0.5 \quad 0.25 \quad 0.125 \quad 0.125)$$

we obtain

$$\begin{aligned} \pi_1^{(1)} &= 0.25\pi_4^{(0)} = 0.25 \times 0.125 = 0.03125, \\ \pi_2^{(1)} &= \pi_1^{(1)} - 0.25\pi_4^{(0)} = 0.03125 - 0.25(0.125) = 0.0625, \\ \pi_3^{(1)} &= \pi_2^{(1)} - 0.5\pi_4^{(0)} = 0.0625 - 0.5(0.125) = 0.1250, \\ \pi_4^{(1)} &= \pi_3^{(1)} = 0.1250. \end{aligned}$$

It's worth noting that the sum of the components in $\boldsymbol{\pi^{(1)}}$ does not equal 1, implying that normalization is required.

$$\left\|\pi^{(1)}\right\|_1 = 0.34375$$

so dividing each element by 0.34375, we obtain

$$\left\|\pi^{(1)}\right\| = (0.090909, 0.181818, 0.363636, .363636) = \frac{1}{11}(1, 2, 4, 4)$$

The sequence of approximations below is computed:

$\boldsymbol{\pi^{(1)}}$ = (0.090909, 0.181818, 0.363636, .363636),
$\boldsymbol{\pi^{(2)}}$ = (0.090909, 0.181818, 0.363636, .363636),
$\boldsymbol{\pi^{(3)}}$ = (0.090909, 0.181818, 0.363636, .363636),

Gauss–Seidel converges in only one iteration in this example and to understand this, we must look at the iteration matrix.

$$H_{GS} = (E - L)^{-1}U. \qquad (16)$$

In the example considered above, we have

$$E = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{pmatrix}, \; L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \end{pmatrix}, \; U = \begin{pmatrix} 0 & 0 & 0 & -0.125 \\ 0 & 0 & 0 & -0.125 \\ 0 & 0 & 0 & -0.250 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H_{GS} = (E - L)^{-1}U = \begin{pmatrix} 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given that U has non-zero values only in the last column and that H_{GS} must have non-zero values only in the same column, the final diagonal element may be the only non-zero eigenvalue of H_{GS}. The diagonal elements of the matrix are often identical to the eigenvalues of upper and lower triangular matrices. Convergence must occur immediately following the

first iteration, as we can see in this example, because the magnitude of the subdominant eigen value (here set to 0) regulates the pace of convergence of the power technique when applied to Markov chain issues.

As indicated in Equation 1, the Gauss–Seidel technique corresponds to computing the $\boldsymbol{i^{th}}$ component of the current approximation from $i = 1, 2, \dots, n,$, i.e., from top to bottom (3).
To illustrate the solution direction, this is frequently referred to as forward Gauss–Seidel. A backward Gauss–Seidel repetition takes the following shape:

$$(E - U)Y^{(k+1)} = LY^{(k)}, \quad k = 0, 1, \dots, \qquad (17)$$

In a Jacobi arrangement, the updating process only employs the components from the previous iteration, rendering forward and backward iterations useless. Since the iterative method in this case essentially works with the inverse of the lower triangular portion of the matrix, $(E - L)^{-1}$, and intuitively, the closer this is to the inverse of the entire matrix, the faster the convergence. A forward iterative method is typically advised when the elemental mass preponderance is found below the diagonal. With the exception of the fact that M^{-1} is simple to detect, a splitting should be constructed so that M is as similar to Q^T as possible. Using the inverse of the top triangle section, $(E - U)^{-1}$, a backward iterative method is used when the majority of the non-zero mass is above the diagonal, and it shown that Gauss Siedel method converged faster than Jacobi method.

CONCLUSION

In this work, Jacobi iterative method and Gauss-Seidel iterative method are used to compute the solutions of stationary distribution in order to shed more light on the Markov chain's stationary distribution solutions. This is done with the aid of several already-existing laws, theorems, and formulas of Markov chain and the application of normalization principle and matrix operations such as lower, upper and diagonal matrices. The stationary distribution vector's π_i, $i = 1, 2, \dots, 4$ are obtained for the illustrative example one as $\pi^{(3)} = (0.078125, 0.109375, 0.21875, 0.59375)$ as well as the four eigenvalues of the matrix as $\lambda_1 = 1.0$, $\lambda_2 = -0.7718$, $\lambda_{3,4} = -0.1141 \pm 0.5576i$ using Jacobi iterative technique, and for illustrative example two using Gauss-Siedel method as $\boldsymbol{\pi^{(3)}}$ = (0.090909, 0.181818, 0.363636, 0.363636). The research shown that Gauss Siedel method converged faster than Jacobi method

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