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On a Coupled Fixed Point Theorem for a Mapping Satisfying a Contraction Of Rational Type in Partially Ordered S-Metric Space

***[1]Mashina M. S., [2]Zakariyya, A and [3]Umar, U. I.**
[1, 2, 3]Department of Mathematics,
Federal College of Education,
Zaria

Email: mohsanmash@gmail.com

Abstract
In this paper, the concept of S-metric is used as a generalizion of the ordinary metric space to obtain a coupled fixed point result for a mapping satisfying a contraction of rational type in partially ordered S-metric space. Since such results has already been proved in ordinary metric space, similar procedure was followed in stating and proving our result in the setting of S-metric space. The result presented here modify and generalize the existing result of Circ et al (2012) which was stated in partially ordered metric space.

Keywords: *Coupled fixed point, contractive condition, partially ordered set, rational type, S-metric space*

INTRODUCTION

A mapping $T: X\to X$ has a fixed point if there exist $x \in X$ such that $Tx = x$ The Banach contraction principle (or Banach's theorem) states that every contraction mapping defined on a complete metric space has a unique fixed point. That is for a complete metric space (X, d) and a mapping $T: X\to X$ satisfying $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$ with $0 < \alpha < 1$, then T has unique fixed point. Some authors like Agarwal *et al*. (2008), Ariza-Ruiz & Jimnez-Melado (2009), and Harjani & Sadrangani (2009). have expanded, developed and established Banach's contraction principle in many ways.

Bhaskar & Lakshmikantham (2006) presented the concept of coupled fixed point and and some important concepts as follows

Definition 1: Suppose that (X, d) is a complete metric space and $(X, \leq)$ is a partially ordered set. For the product space $X \times X$, and $(x, y), (u, v) \in X \times X$, $(u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v$.

Definition 2: Suppose $F: X \times X \to X$ and $(X, \leq)$ is a partially ordered set. Then F has the mixed monotone property if $F$is
monotone non decreasing in x, that is for any $x, y, x_1, x_2 \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and
monotone non increasing in y, that is for any $x, y, y_{1,} y_2 \in X$,

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

**Author for Correspondence*

Definition 3: Let $F: X \times X \to X$. An element $(x, y) \in X \times X$ is called a coupled fixed point of F if
$F(x, y) = x$ and $F(y, x) = y$

Since then some authors like Lakshmikantham & Ciric (2009), Mehta and Joshi (2010) and Ciric *et al*. (2012) have demonstrated some results on coupled fixed point in partially ordered metric space using various forms of contractive condition.
Harjani *et al*. (2010) proved a fixed point theorem for mapping satisfying a contractive condition of rational type in partially ordered metric space stating the following theorem.

Theorem 1 (Harjani *et al*. (2010))**:** Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X \to X$ be a continuous and non decreasing mapping such that

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx). d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for $x, y \in X, x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If there exists $x_0 < Tx_0$, then T has a fixed point.

Motivated by the result of Harjani *et al*. (2010) (Theorem 1) Circ *et al.* (2012) established and proved Theorem 1 using the concept of coupled fixed point as follows

Theorem 2: Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X \times X \to X$ be a continuous and mapping which has the mixed monotone property such that for some $\alpha, \beta \in [0,1)$ $\forall\, x, y, u, v \in X, x \neq u, v \neq y$, we have

$$d(T(x, y), T(u, v)) \leq \alpha \frac{d\big(x, T(x, y)\big). d\big(u, T(u, v)\big)}{d(x, u)} + \beta d(x, u)$$

with $\alpha + \beta < 1$. Then T has a coupled fixed point.
For many years now, to modify and extend a domain to a more general space has been a keen and vigorous research in fixed point theory. A number of generalizations of metric space have been stated in several ways by some authors. For example, 2-metric space introduced by Gähler (1963), D-metric space by Dhage (1992) G-metric space Mustafa & Sims (2006) and D^*-metric space by Sedghi *et al* (2007).
Sequel to these generalizations, Sedghi *et al* (2012) presented a new generalization of ordinary metric space as follows.

Definition 4: Let X be a nonempty set. An S-metric space on X is a function $S : X^3 \to [0, \infty)$ such that for each $x, y, z, a \in X$, the following conditions are satisfied.
1. $S(x, y, z) \geq 0$ for all $x, y, z \in X$
2. $S(x, y, z) = 0$ if and only if $x = y = z$
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X,S) is called an S-metric space.
Immediate examples of an S-metric space are presented by Sedghi *et al* (2012) as follows.

Example 1: Let $X = \mathbb{R}^n$ and $\| . \|$ a norm on X. Then $S(x, y, z) = \| x + z - 2x \| + \| y - z \|$ is an S-metric on X.

Example 2: Let $X = \mathbb{R}^2$ and d an ordinary metric on X. Then $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ is an S-metric on X.
Again Sedghi *et al* (2012) stated the following important definitions and lemma as follows

Definition 5: Let (X,S) be an S-metric space.

1. A sequence $\{x_n\}$ converges to x if for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ for all $x \in X$. That is if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$ and write $\lim_{n\to\infty} x_n = x$.
2. $\{x_n\}$ is called a Cauchy sequence if for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ $\in$ such that for each $n, m \geq n_0$, $S(x_n, x_n, x_m) < \epsilon$.
3. (X,S) is said to be complete if every Cauchy sequence is convergent.

Lemma 1: Let (X,S) be an S-metric space. Then $S(x,x,y) = S(y,y,x)$

Using the notion of S-metric space, concept of coupled fixed point, we extended and generalized the result of Circ *et al.* (2012) and proved a coupled fixed point theorem for mapping satisfying a contractive condition of rational type in partially ordered S-metric space.

MAIN RESULT

The result and proof in this section is a modification of Theorem 2 using similar procedures in the setting of S-metric space.

Theorem 3: Let $(X, \leq, S)$ be a partially ordered complete S-metric space. Let $T: X \times X \to X$ be a continuous mapping which has the mixed monotone property such that

$$S(T(x,y),(T(x,y),T(u,v)) \leq p\frac{S\big(x,x,T(x,y)\big).S\big(u,u,T(u,v)\big)}{S(x,x,u)} + qS(x,x,u) \qquad (1)$$

$\forall\, x,y,u,v \in X,\ x \neq u,\ y \neq v,$ and for some $p, q \in [0,1)$ with $p + q < 1$, then T has a coupled fixed point.

Proof: Suppose that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$ for some $(x_0, y_0) \in X \times X$.
In general, $x_{n+1} = T(x_0, y_0)$ and $y_{n+1} = T(y_0, x_0)$
For $x_2 = T(x_1, y_1)$ and $x_2 = T(x_1, y_1)$
We now set

$$T^2(x_0,y_0) = T(T(x_0,y_0)\,, T(y_0,x_0)\,) = T(x_1,y_1) = x_2$$
$$T^2(y_0,x_0) = T(T(y_0,x_0)\,, T(x_0,y_0)\,) = T(y_1,x_1) = y_2$$

Following this, and from Definition 2, we write

$$x_1 = T(x_0,y_0) \leq T(x_1,y_1) = T^2(x_0,y_0) = x_2$$
$$y_1 = T(y_0,x_0) \geq T(y_1,x_1) = T^2(y_0,x_0) = y_2$$

Hence for $n \geq 1$

$$x_{n+1} = T^{n+1}(x_0,y_0) = T(T^n(x_0,y_0)\,, T^n(y_0,x_0)\,)$$
$$y_{n+1} = T^{n+1}(y_0,x_0) = T(T^n(y_0,x_0)\,, T^n(x_0,y_0)\,)$$

This implies that

$$x_0 \leq T(x_0,y_0) = x_1 \leq T^2(x_0,y_0) = x_2 \leq \cdots \leq T^n(x_0,y_0) = x_n \leq \cdots$$
$$y_0 \geq T(y_0,x_0) = y_1 \geq T^2(y_0,x_0) = y_2 \geq \cdots \geq T^n(y_0,x_0) = y_n \leq \cdots$$

Putting $(x,y) = (x_n, y_n)$, $(u,v) = (x_{n-1}, y_{n-1})$, using equation (1) and lemma 1 we get

$$\begin{aligned}
S(x_{n+1},x_{n+1},x_n) &= S(T(x_n,y_n),T(x_n,y_n),T(x_{n-1},y_{n-1}) \\
&\leq p\frac{S\big(x_n,x_n,T(x_n,y_n)\big).S\big(x_{n-1},x_{n-1},T(x_{n-1},y_{n-1})\big)}{S(x_n,x_n,x_{n-1})} + qS(x_n,x_n,x_{n-1}) \\
&= p\frac{S(x_n,x_n,x_{n+1}).S(x_{n-1},x_{n-1},x_n)}{S(x_n,x_n,x_{n-1})} + qS(x_n,x_n,x_{n-1}) \\
&= pS(x_n,x_n,x_{n+1}) + qS(x_n,x_n,x_{n-1})
\end{aligned}$$

$$S(x_{n+1},x_{n+1},x_n) \leq \left(\frac{q}{1-p}\right) S(x_n,x_n,x_{n-1}) \qquad (2)$$

In a similar fashion and using equation (1), we get

$$S(y_{n+1}, y_{n+1}, y_n) \leq \left(\frac{q}{1-p}\right) S(y_n, y_n, y_{n-1}) \tag{3}$$

Thus from equation (2) and (3)

$$S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_n) \leq \left(\frac{q}{1-p}\right) \left(S(x_n, x_n, x_{n-1}) + S(y_n, y_n, y_{n-1})\right)$$

Or by lemma 1

$$S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \leq \left(\frac{q}{1-p}\right) \left(S(x_{n-1}, x_{n-1}, x_n) + S(y_{n-1}, y_{n-1}, y_n)\right) \tag{4}$$

Let $I_n = S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_n)$

$$A = \left(\frac{q}{1-p}\right)$$

By equation (4), we get

$$I_n \leq AI_{n-1} \leq A^2 I_{n-1} \leq \cdots \leq A^n I_0 \tag{5}$$

If $I_n = 0$, then (x_0, y_0) is a coupled fixed point of T

For $I_n \neq 0$, then for each $m \in \mathbb{N}$ we get by equation (5) and applying condition 3 of Definition 4,

$$\begin{aligned}
&S(x_n, x_n, x_{n+m}) + S(y_n, y_n, y_{n+m}) \\
&\leq \left(2S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+1}) + 2S(y_n, y_n, y_{n+1}) + S(y_n, y_n, y_{n+1})\right) \\
&\leq \left(2S(x_n, x_n, x_{n+1}) + 2S(x_n, x_n, x_{n+1}) + 2S(y_n, y_n, y_{n+1}) + 2S(y_n, y_n, y_{n+1})\right) \\
&+ \cdots + \left(2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_{m-1}, x_{m-1}, x_m)\right) \\
&+\left(2S(y_{m-2}, y_{m-2}, y_{m-1}) + S(y_{m-1}, y_{m-1}, y_m)\right) \\
&\leq \left(2S(x_n, x_n, x_{n+1}) + 2S(x_n, x_n, x_{n+1})\right) + \cdots + \left(2S(x_{m-1}, x_{m-1}, x_m) + 2S(y_{m-1}, y_{m-1}, y_m)\right) \\
&\leq 2[I_n + I_{n+1} + \cdots + I_{m-1}](S(x_0, x_0, x_1) + S(y_0, y_0, y_1)) \\
&\leq \frac{2A^n}{1-A}(S(x_0, x_0, x_1) + S(y_0, y_0, y_1))
\end{aligned}$$

For $A < 1$ and taking limit as $n, m \to \infty$, we have

$S(x_n, x_n, x_{n+m}) + S(y_n, y_n, y_{n+m}) \to 0$

Thus $\lim_{n,m\to\infty} S(x_n, x_n, x_{n+m}) = 0$ and $\lim_{n,m\to\infty} S(y_n, y_n, y_{n+m}) = 0$

This implies $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

As (X, S) is a complete S-metric space, there exists $x, y \in X$ such that $x_n \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$

The continuity of T implies that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}, y_{n-1}) = T\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) = T(x, y)$$

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(y_{n-1}, x_{n-1}) = T\left(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}\right) = T(y, x)$$

Thus $x = T(x, y)$ and $y = T(y, x)$.

Hence T has a coupled fixed point.

CONCLUSION

As a generalized metric in 3-tuples, the notion of S-metric was used to state and prove a coupled fixed point theorem for mapping satisfying a contractive condition of rational type in partially ordered S-metric space.

The result presented here modify and generalize Theorem 2 in the frame work of S-metric space.

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