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The Use of Implicit Single-Step Linear Block Method on Third Order Ordinary Differential Equations by Interpolation and Collocation Procedure

[1]Althemai, J. M., [2]Sabo, J., [1]Yaska Mutah

[2]Department of Mathematics,
Adamawa State University, Mubi,
Adamawa State-Nigeria.

[1]Department of Mathematics and Statistics,
Federal Polytechnic, Mubi,
Adamawa State-Nigeria.

Email: sabojohn630@yahoo.com

Abstract
This research showed the use of implicit single-step linear block method on third order ordinary differential equations by interpolation and collocation procedure. The block method is found using power series as a basic function through interpolation and collocation. The properties of the block method were considered. The outcome achieves on the application of new block method on selected tested modeled third order linear problems was found to give better approximation than the current methods in literature.

Keywords: Collocation, Implicit One-step, Interpolation, Linear Block Method, Third Order and Ordinary Differential Equations,

INTRODUCTION
Real life problems particularly in sciences and engineering can be expressed as differential equations in order to analyze and understand the physical phenomena (Omar 1999). Numerous problems in real life situations contain rates of change of one or more independent variables. These rates of change can be stated in terms of derivatives which tip to differential equations.

The numerical solution to third order ordinary differential equations is conventionally answered by a decreasing to a system of first order ordinary differential equations and then appropriate numerical method for first order would be used to solve the system system (Skwame *et al*, 2019). However, the major hindrances for this method are computational problem which affects the accurateness of the method in terms of the error, complications in writing computer program for the method and consumption of human energy. In order to overcome this setbacks or difficulties, this research will proposed the use of implicit single-step linear block method on higher order initial value problems (specifically, third order initial value problems) by interpolation and collocation procedure. In developing these methods, a power series of the form

**Author for Correspondence*

$$y(g)=\sum_{j=0}^{r+s-1} a_j g^j \quad (1)$$

will be used as an approximate solution to initial value problems of ordinary differential equations of the form

$$y'''(g)=f(g, y(g)),\ y(g_0)=y_0,\ y'(g_0)=y'_0,\ y''(g_0)=y''_0 \quad (2)$$

Block methods will then introduced with the aim of approximating numerical solutions, (Omar 2004, Kuboye & Omar 2015 and Adeyeye & Omar 2019a). Meanwhile, the block method which was first proposed by Milne according to (Olabode 2007), has been studied by some scholars such as (Abdelrahim & Omar 2017, Fasasi 2018, Adeyeye & Omar 2018, Sunday 2018, Skwame, Dalatu, Sabo & Mathew 2019, Abdulsalam, Sanu, Majid 2019, Adeyeye & Omar 2019a, 2919b, 2019c, Raymond, Skwame & Adiku 2021, Sabo, Althemai & Hamadina 2021, Sabo, Bakari & Babuba, 2021, Abdelrahim 2021), among others, have developed block hybrid methods for direct solution of higher order ordinary differential equations without reduction methods, whereby the accuracy of the methods is better than when it is reduced to system of first order ordinary differential equations.

Mathematical Formulation of the Method

The approximation of equation (1) as a basic function was consider, where r and s are the number of different interpolation and collocation respectively.
Differentiating equation (1) three times and substitute into equation (2) to yield,

$$f(g, y, y', y'')=\sum_{j=0}^{r+s-1} j(j-1)(j-2)a_j g^{j-3} \quad (3)$$

A mesh of single-step length is considered with a constant step size h given by $h=g_{n+j}-g_n,\ j=0,1$ and four off-step points at $g_{n+\frac{1}{8}},\ g_{n+\frac{1}{4}}, g_{n+\frac{3}{8}}$ *and* $g_{n+\frac{1}{2}}$.

Interpolating (1) at point $g_{n+r},\ r=\frac{1}{8}\left(\frac{1}{8}\right)\frac{3}{8}$ and collocating (3) at points $g_{n+s},\ s=0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},1$, gives a system of nonlinear equation of the form,

$$AG=K \quad (4)$$

where,

$A=[a_0\ \ a_1\ \ a_2\ \ a_3\ \ a_4\ \ a_5\ \ a_6\ \ a_7\ \ a_8]^T$,

$K=\left[y_{n+\frac{1}{8}}\ \ y_{n+\frac{1}{4}}\ \ y_{n+\frac{3}{8}}\ \ f_n\ \ f_{n+\frac{1}{8}}\ \ f_{n+\frac{1}{4}}\ \ f_{n+\frac{3}{8}}\ \ f_{n+\frac{1}{2}}\ \ f_{n+1}\right]^T$ and

$$G=\begin{bmatrix} 1 & g_{n+\frac{1}{8}} & g^2_{n+\frac{1}{8}} & g^3_{n+\frac{1}{8}} & g^4_{n+\frac{1}{8}} & g^5_{n+\frac{1}{8}} & g^6_{n+\frac{1}{8}} & g^7_{n+\frac{1}{8}} & g^8_{n+\frac{1}{8}} \\ 1 & g_{n+\frac{1}{4}} & g^2_{n+\frac{1}{4}} & g^3_{n+\frac{1}{4}} & g^4_{n+\frac{1}{4}} & g^5_{n+\frac{1}{4}} & g^6_{n+\frac{1}{4}} & g^7_{n+\frac{1}{4}} & g^8_{n+\frac{1}{4}} \\ 1 & g_{n+\frac{3}{8}} & g^2_{n+\frac{3}{8}} & g^3_{n+\frac{3}{8}} & g^4_{n+\frac{3}{8}} & g^5_{n+\frac{3}{8}} & g^6_{n+\frac{3}{8}} & g^7_{n+\frac{3}{8}} & g^8_{n+\frac{3}{8}} \\ 0 & 0 & 0 & 6 & 24g_n & 60g_n^2 & 120g_n^3 & 210g_n^4 & 336g_n^5 \\ 0 & 0 & 0 & 6 & 24g_{n+\frac{1}{8}} & 60g^2_{n+\frac{1}{8}} & 120g^3_{n+\frac{1}{8}} & 210g^4_{n+\frac{1}{8}} & 336g^5_{n+\frac{1}{8}} \\ 0 & 0 & 0 & 6 & 24g_{n+\frac{1}{4}} & 60g^2_{n+\frac{1}{4}} & 120g^3_{n+\frac{1}{4}} & 210g^4_{n+\frac{1}{4}} & 336g^5_{n+\frac{1}{4}} \\ 0 & 0 & 0 & 6 & 24g_{n+\frac{3}{8}} & 60g^2_{n+\frac{3}{8}} & 120g^3_{n+\frac{3}{8}} & 210g^4_{n+\frac{3}{8}} & 336g^5_{n+\frac{3}{8}} \\ 0 & 0 & 0 & 6 & 24g_{n+\frac{1}{2}} & 60g^2_{n+\frac{1}{2}} & 120g^3_{n+\frac{1}{2}} & 210g^4_{n+\frac{1}{2}} & 336g^5_{n+\frac{1}{2}} \\ 0 & 0 & 0 & 6 & 24g_{n+1} & 60g^2_{n+1} & 120g^3_{n+1} & 210g^4_{n+1} & 336g^5_{n+1} \end{bmatrix}$$

Solving (4) for $a_j,\ j=0(1)8$ which are constants to be obtained and substitute into (2) to give a one-step continuous block method of the scheme,

$$y(g)=\alpha_{\frac{1}{8}}(g)y_{n+\frac{1}{8}}+\alpha_{\frac{1}{4}}(g)y_{n+\frac{1}{4}}+\alpha_{\frac{3}{8}}(g)y_{n+\frac{3}{8}}+h^3\left[\sum_{j=0}^{1}\beta_j(g)f_{n+j}+\beta_s(g)f_{n+s}\right],\ s=\frac{1}{8}\left(\frac{1}{8}\right)\frac{1}{2},1 \qquad (5)$$

where $\alpha_r(g)$, $\beta_j(g)$ *and* $\beta_s(g)$ are presented as functions of x with

$$x=\frac{g-g_n}{h} \qquad (6)$$

where

$$\left.\begin{aligned}
\alpha_{\frac{1}{8}} &= 32t^2-20t+3\\
\alpha_{\frac{1}{4}} &= -64t^2+32t-3\\
\alpha_{\frac{3}{8}} &= 32t^2-12t+1\\
\beta_0 &= -\frac{32}{63}t^8+\frac{64}{35}t^7-\frac{23}{9}t^6+\frac{11}{6}t^5-\frac{53}{72}t^4+\frac{1}{6}t^3-\frac{51529}{2580480}t^2+\frac{1787}{1720320}t-\frac{7}{655360}\\
\beta_{\frac{1}{8}} &= \frac{1024}{441}t^8-\frac{17408}{2205}t^7+\frac{448}{45}t^6-\frac{1856}{315}t^5+\frac{32}{21}t^4-\frac{16027}{188160}t^2+\frac{3679}{225792}t-\frac{401}{430080}\\
\beta_{\frac{1}{4}} &= -\frac{256}{63}t^8+\frac{4096}{315}t^7-\frac{664}{45}t^6+\frac{328}{45}t^5-\frac{4}{3}t^4-\frac{449}{35840}t^2+\frac{7199}{645120}t-\frac{257}{245760}\\
\beta_{\frac{3}{8}} &= \frac{1024}{315}t^8-\frac{1024}{105}t^7+\frac{448}{45}t^6-\frac{64}{15}t^5+\frac{32}{45}t^4-\frac{757}{80640}t^2+\frac{3}{17920}t+\frac{1}{20480}\\
\beta_{\frac{1}{2}} &= -\frac{64}{63}t^8+\frac{128}{45}t^7-\frac{118}{45}t^6+\frac{47}{45}t^5-\frac{1}{6}t^4-\frac{893}{430080}t^2+\frac{1}{73728}t-\frac{13}{983040}\\
\beta_1 &= \frac{32}{2205}t^8-\frac{64}{2205}t^7+\frac{1}{45}t^6-\frac{1}{126}t^5+\frac{1}{840}t^4-\frac{89}{6021120}t^2+\frac{11}{36126720}t+\frac{1}{13762560}
\end{aligned}\right\} \qquad (7)$$

Evaluating (5) to obtain the continuous form as

$$\begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+1} \end{bmatrix} - \begin{bmatrix} 3 & -3 & 1 \\ 1 & -3 & 3 \\ 15 & -35 & 21 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{8}} \end{bmatrix} = h^3 \begin{bmatrix} -\frac{7}{655360} & -\frac{401}{430040} & -\frac{257}{245760} & \frac{1}{20480} & -\frac{13}{983040} & \frac{1}{13762560} \\ \frac{11}{1966080} & -\frac{3}{143360} & \frac{247}{245760} & \frac{59}{61440} & \frac{1}{327680} & \frac{1}{13762560} \\ \frac{3943}{393216} & -\frac{4757}{86016} & \frac{2249}{16384} & -\frac{1361}{12288} & \frac{16943}{196608} & \frac{871}{917504} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \quad (8)$$

Differentiating (5) once, we have

$$y'(g) = \alpha'_{\frac{1}{8}}(g) y_{n+\frac{1}{8}} + \alpha'_{\frac{1}{4}}(g) y_{n+\frac{1}{4}} + \alpha'_{\frac{3}{8}}(g) y_{n+\frac{3}{8}} + h^3 \left[\sum_{j=0}^{1} \beta'_j(g) f_{n+j} + \beta'_s(g) f_{n+s} \right], \; s = \frac{1}{8}\left(\frac{1}{8}\right)\frac{1}{2}, 1 \quad (9)$$

Evaluating (9) at g_n, $g_{n+\frac{1}{8}}$, $g_{n+\frac{1}{4}}$, $g_{n+\frac{3}{8}}$, $g_{n+\frac{1}{2}}$ *and* g_{n+1}, we have

$$\begin{bmatrix} hy'_n \\ hy'_{n+\frac{1}{8}} \\ hy'_{n+\frac{1}{4}} \\ hy'_{n+\frac{3}{8}} \\ hy'_{n+\frac{1}{2}} \\ hy'_{n+1} \end{bmatrix} - \begin{bmatrix} -20 & 32 & -12 \\ -12 & 16 & -4 \\ -4 & 0 & 4 \\ 4 & -16 & 12 \\ 12 & -32 & 20 \\ 44 & -96 & 52 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \end{bmatrix} = h^3 \begin{bmatrix} \frac{1787}{1720320} & \frac{3678}{225792} & \frac{7199}{645120} & \frac{3}{17920} & -\frac{1}{73728} & \frac{11}{36126720} \\ -\frac{773}{10321920} & \frac{1003}{752640} & \frac{5339}{1290240} & -\frac{83}{322560} & \frac{121}{1720320} & -\frac{31}{72253440} \\ \frac{1}{129024} & -\frac{13}{80640} & -\frac{247}{107520} & -\frac{13}{80640} & \frac{1}{129024} & 0 \\ \frac{29}{1146880} & -\frac{209}{2257920} & \frac{5029}{1290240} & \frac{23}{15360} & -\frac{619}{5160960} & \frac{31}{72253440} \\ \frac{19}{1032192} & \frac{19}{376320} & \frac{7309}{645120} & \frac{2609}{161280} & \frac{307}{286720} & -\frac{11}{36126720} \\ \frac{446413}{5160960} & -\frac{534377}{1128960} & \frac{77283}{71680} & -\frac{166841}{161280} & \frac{1575353}{2580480} & \frac{136181}{12042240} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \quad (10)$$

Differentiating (5) twice, we have

$$y''(g) = \alpha''_{\frac{1}{8}}(g) y_{n+\frac{1}{8}} + \alpha''_{\frac{1}{4}}(g) y_{n+\frac{1}{4}} + \alpha''_{\frac{3}{8}}(g) y_{n+\frac{3}{8}} + h^3 \left[\sum_{j=0}^{1} \beta''_j(g) f_{n+j} + \beta''_s(g) f_{n+s} \right], \; s = \frac{1}{8}\left(\frac{1}{8}\right)\frac{1}{2}, 1 \quad (11)$$

Evaluating (11) at g_n, $g_{n+\frac{1}{8}}$, $g_{n+\frac{1}{4}}$, $g_{n+\frac{3}{8}}$, $g_{n+\frac{1}{2}}$ *and* g_{n+1}, we have

$$\begin{bmatrix} h^2 y''_n \\ h^2 y''_{n+\frac{1}{8}} \\ h^2 y''_{n+\frac{1}{4}} \\ h^2 y''_{n+\frac{3}{8}} \\ h^2 y''_{n+\frac{1}{2}} \\ h^2 y''_{n+1} \end{bmatrix} - \begin{bmatrix} 64 & -128 & 64 \\ 64 & -128 & 64 \\ 64 & -128 & 64 \\ 64 & -128 & 64 \\ 64 & -128 & 64 \\ 64 & -128 & 64 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \end{bmatrix} = h^3 \begin{bmatrix} -\frac{51529}{1290240} & -\frac{16027}{94080} & -\frac{449}{17920} & -\frac{757}{40320} & \frac{893}{215040} & -\frac{89}{3010560} \\ -\frac{187}{86016} & -\frac{14537}{282240} & \frac{4441}{53760} & \frac{121}{13440} & -\frac{1339}{645120} & \frac{37}{3010560} \\ -\frac{227}{430080} & \frac{183}{31360} & \frac{43}{32256} & -\frac{103}{13440} & \frac{221}{215040} & -\frac{42}{9031680} \\ \frac{1013}{1290240} & -\frac{403}{94080} & \frac{1357}{17920} & \frac{2267}{40320} & -\frac{149}{43008} & \frac{37}{3010560} \\ -\frac{451}{430080} & \frac{419}{56448} & \frac{2237}{53760} & \frac{2137}{13440} & \frac{27767}{645120} & -\frac{89}{3010560} \\ \frac{207421}{430080} & -\frac{81801}{31360} & \frac{913463}{16280} & -\frac{76711}{13440} & \frac{120601}{43008} & \frac{1046261}{9031680} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \quad (12)$$

Combining and solving (8), (10) and (12) simultaneously, to give the explicit block method as

$$\begin{bmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{8}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{128} \\ 1 & \frac{1}{4} & \frac{1}{32} \\ 1 & \frac{3}{8} & \frac{9}{128} \\ 1 & \frac{1}{2} & \frac{1}{8} \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{bmatrix} = h^3 \begin{bmatrix} \frac{31849}{165150720} & \frac{1637}{7225344} & -\frac{3167}{20643840} & \frac{397}{5160960} & -\frac{1447}{82575360} & \frac{139}{1156055040} \\ \frac{1289}{1290240} & \frac{11}{5040} & -\frac{31}{32256} & \frac{1}{2016} & -\frac{73}{645120} & \frac{1}{1290240} \\ \frac{44577}{18350080} & \frac{27297}{4014080} & -\frac{3159}{2293760} & \frac{99}{81920} & -\frac{2511}{9175040} & \frac{243}{128450560} \\ \frac{181}{40320} & \frac{31}{2205} & -\frac{1}{2520} & \frac{1}{315} & -\frac{1}{2016} & \frac{1}{282240} \\ \frac{73}{2520} & \frac{32}{2205} & \frac{44}{315} & -\frac{32}{315} & \frac{53}{630} & \frac{17}{17640} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} y'_{n+\frac{1}{8}} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{3}{8}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{8} \\ 1 & \frac{1}{4} \\ 1 & \frac{3}{8} \\ 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y'_n \\ hy''_n \end{bmatrix} = h^2 \begin{bmatrix} \frac{20017}{5160960} & \frac{715}{112896} & -\frac{2509}{645120} & \frac{31}{16128} & -\frac{1123}{2580480} & \frac{107}{36126720} \\ \frac{361}{40320} & \frac{461}{17640} & -\frac{29}{4032} & \frac{11}{2520} & -\frac{41}{40320} & \frac{1}{141120} \\ \frac{8007}{573440} & \frac{29790}{62720} & \frac{153}{71680} & \frac{15}{1792} & -\frac{477}{286720} & \frac{9}{802816} \\ \frac{191}{10080} & \frac{152}{2205} & \frac{4}{315} & \frac{8}{315} & -\frac{1}{1008} & \frac{1}{70560} \\ \frac{79}{630} & -\frac{704}{2205} & \frac{344}{315} & -\frac{64}{63} & \frac{191}{315} & \frac{5}{441} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} y''_{n+\frac{1}{8}} \\ y''_{n+\frac{1}{4}} \\ y''_{n+\frac{3}{8}} \\ y''_{n+\frac{1}{2}} \\ y''_{n+1} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} y''_n \end{bmatrix} = h \begin{bmatrix} \frac{3881}{92160} & \frac{599}{5040} & -\frac{221}{3840} & \frac{1}{36} & -\frac{287}{46080} & \frac{3}{71680} \\ \frac{227}{5760} & \frac{37}{210} & \frac{19}{720} & \frac{1}{90} & -\frac{1}{320} & \frac{1}{40320} \\ \frac{417}{10240} & \frac{93}{560} & \frac{129}{1280} & \frac{3}{40} & -\frac{39}{5120} & \frac{3}{71680} \\ \frac{7}{180} & \frac{8}{45} & \frac{1}{15} & \frac{8}{45} & \frac{7}{180} & 0 \\ \frac{47}{90} & -\frac{256}{105} & \frac{256}{45} & -\frac{256}{45} & \frac{14}{5} & \frac{73}{630} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \tag{15}$$

The Properties of the Block Method

In this section, the basic properties of the method will be recognized.

Order and Error constant of the Method

Consider the linear operator $L\{y(x):h\}$ defined by

$$L\{y(x):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{k} \frac{jh^{(i)}}{i!} y_n^{(i)} - h^{(3-1)}\left[d_i f(y_n) + b_i F(Y_m)\right] \tag{16}$$

Applying the Taylor series expansion of Y_m *and* $F(Y_m)$ and compare the coefficients of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots \tag{17}$$

The linear operator L and the associate block method are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = C_{p+2} = 0, \quad C_{p+3} \neq 0.$ C_{p+3} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + 0h^{p+4}$

Expanding the block in Taylor series gives

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{8}\right)^{j}}{j!}-y_n-\frac{1}{8}hy'_n-\frac{1}{128}h^2y''_n-\frac{31849}{165150720}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}-\frac{1637}{7225344}\left(\frac{1}{8}\right)+\frac{3167}{20643840}\left(\frac{1}{4}\right)\\ -\frac{397}{5160960}\left(\frac{3}{8}\right)+\frac{1447}{82575360}\left(\frac{1}{2}\right)\\ -\frac{139}{1156055040}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^{j}}{j!}-y_n-\frac{1}{4}hy'_n-\frac{1}{32}h^2y''_n-\frac{1289}{1290240}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}-\frac{11}{5040}\left(\frac{1}{8}\right)+\frac{31}{32256}\left(\frac{1}{4}\right)-\frac{1}{2016}\left(\frac{3}{8}\right)\\ +\frac{73}{645120}\left(\frac{1}{2}\right)-\frac{1}{1290240}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{8}\right)^{j}}{j!}-y_n-\frac{3}{8}hy'_n-\frac{9}{128}h^2y''_n-\frac{44577}{18350080}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}-\frac{27297}{4014080}\left(\frac{1}{8}\right)+\frac{3159}{2293760}\left(\frac{1}{4}\right)\\ -\frac{99}{81920}\left(\frac{3}{8}\right)+\frac{2511}{9175040}\left(\frac{1}{2}\right)-\\ \frac{243}{128450560}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{j}}{j!}-y_n-\frac{1}{2}hy'_n-\frac{1}{8}h^2y''_n-\frac{181}{40320}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}-\frac{31}{2205}\left(\frac{1}{8}\right)+\frac{1}{2520}\left(\frac{1}{4}\right)-\frac{1}{315}\left(\frac{3}{8}\right)\\ +\frac{1}{2016}\left(\frac{1}{2}\right)-\frac{1}{282240}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{(1)^{j}}{j!}-y_n-hy'_n-\frac{1}{2}h^2y''_n-\frac{73}{2520}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}-\frac{32}{2205}\left(\frac{1}{8}\right)-\frac{44}{315}\left(\frac{1}{4}\right)+\frac{32}{315}\left(\frac{3}{8}\right)-\frac{53}{630}\left(\frac{1}{2}\right)\\ -\frac{17}{17640}(1)\end{bmatrix}
\end{bmatrix} \quad (18)$$

Comparing the coefficient of h, according to (Sunday 2018), the order p of our method is $p = [4 \quad 4 \quad 4 \quad 4 \quad 4]^T$ and the error constant are given respectively by
$C_{p+3} = [-1.1274\times10^{-8} \quad -7.3165\times10^{-9} \quad -1.0644\times10^{-8} \quad -4.0367\times10^{-9} \quad -3.2294\times10^{-6}]$

Consistency of the Method

A method is supposed to be steady or consistent, in case it has order greater than one. So from the above study, it is apparent that our method is steady (Omar 2004).

Zero Stability of the Method

A block method is said to be zero stable as $h \to 0$ if the roots of the first characteristics polynomial $\rho(r) = 0$ satisfy $\left\|\left[\sum A^0 R^{k-1}\right]\right\| \le 1$, and those roots with $R = 1$ must be simple.

Now,

$$\rho(r) = \left| r\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right| = \left|\begin{bmatrix}r&0&0&0&-1\\0&r&0&0&-1\\0&0&r&0&-1\\0&0&0&r&-1\\0&0&0&0&r-1\end{bmatrix}\right| = r^4(r-1)$$

Then, solving for z in

$$r^4(r-1) \quad (19)$$

gives $r = 0, 0, 0, 0, 1$. Hence the block method is said to be zero stable.

Convergence of the Block Method

Theorem: the essential and adequate conditions for a linear multistep method to be convergent are that it must be consistent and zero-stable. Hence our method derived is consistent (Skwame *et al,* 2019).

Region of Absolute Stability of our Method

Applying the boundary locus method, we obtain the stability polynomial as

$$\bar{h}(w) = h^{15}\left(-\frac{523}{25979622062080\,0}w^4 - \frac{1}{60610578481152000}w^5\right) + h^{12}\left(-\frac{10705061}{22728966930432000}w^4 - \frac{8429}{34093450395648000}w^5\right)$$
$$+ h^9\left(-\frac{463317839}{284112086630400}w^4 - \frac{2311}{5261334937600}w^5\right) + h^6\left(-\frac{7441243}{5284823040}w^4 - \frac{11}{880803840}w^5\right)$$
$$+ h^6\left(-\frac{184343}{860160}w^4 - \frac{27}{28672}w^5\right) + w^5 - \frac{5}{2}w^4 \quad (20)$$

Applying the stability polynomial, we obtain the region of absolute stability in Figure 1 below,

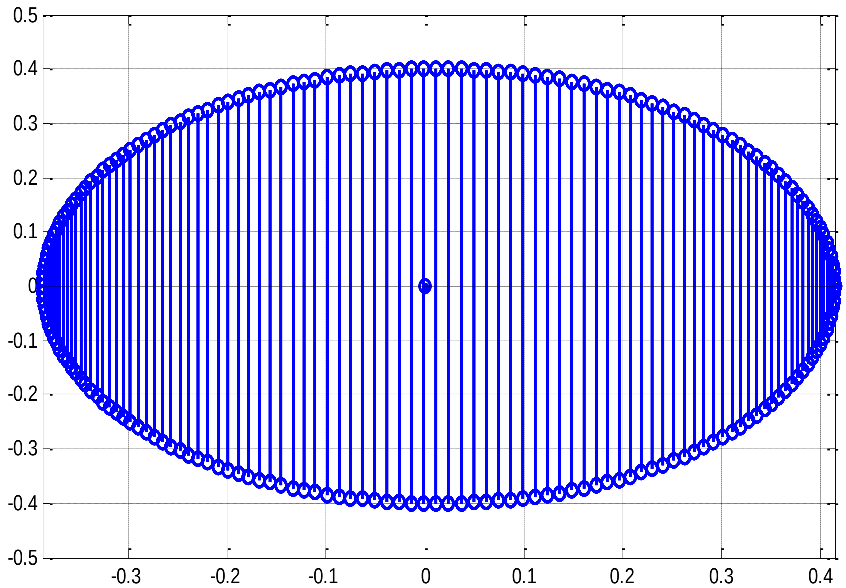

Figure 1: Region of absolute stability of our method

Mathematical Implementation

The following numerical problems are carefully considered in order to study the effectiveness and exactness of the new block method when related with the existing methods. The new block method was applied to solve third order linear problem problems.

Problem one:

Consider the third order linear problem

$y''' + 4y' - x = 0,\ y(0) = y'(0) = 0,\ y''(0) = -1,\ h = 0.1$

With the particular solution given by

$y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{x^2}{8}$

Source (Sunday 2018, Adeyeye & Omar 2019a and Aigbiremhon & Omole 2020)

Table 1. Showing the result for problem one

x-value	particular Solution	Commutated Solution	Error in new Method	Error in (Sunday 2018)	Error in (Adeyeye & Omar 2019a)	Error in (Aigbiremhon & Omole 2020)
0.1	0.00498751665476719416	0.00498751665476770617	5.1201e-16	8.3209e-13	0.2304e-14	2.8818e-09
0.2	0.01980106362445904698	0.01980106362447933580	2.0289e-14	3.4752e-12	0.1658e-13	3.2893e-08
0.3	0.04399957220443531927	0.04399957220456152103	1.2620e-13	7.8178e-12	0.4850e-13	1.1954e-07
0.4	0.07686749199740648358	0.07686749199784637519	4.3989e-13	1.3681e-11	0.1147e-12	2.8709e-07
0.5	0.11744331764972380299	0.11744331765085170446	1.1279e-12	2.0825e-11	0.2425e-12	5.5398e-07
0.6	0.16455792103562370419	0.16455792103801160326	2.3879e-12	2.8962e-11	0.4436e-12	9.2975e-07
0.7	0.21688116070620482401	0.21688116071063664648	4.4318e-12	3.7764e-11	0.7467e-12	1.4149e-06
0.8	0.27297491043149163616	0.27297491043895861507	7.4670e-12	4.6879e-11	0.1183e-11	1.9995e-06
0.9	0.33135039275495382287	0.33135039276663008973	1.1676e-11	5.5941e-11	0.1753e-11	2.6636e-06
1.0	0.39052753185258919756	0.39052753186978796647	1.7199e-11	6.4592e-11	0.2481e-11	3.3776e-06

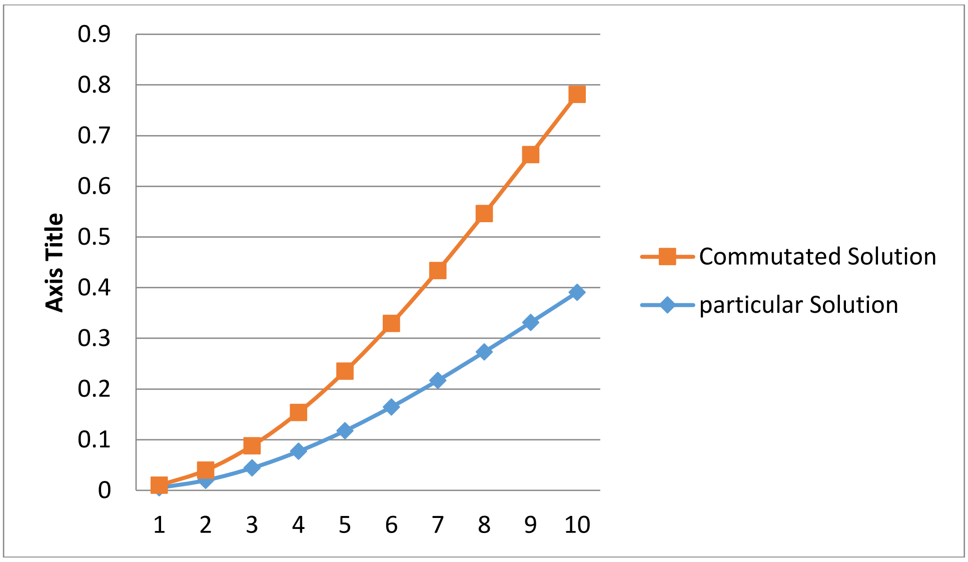

Figure 2. Showing the solution graph of problem one.

Problem two:

Consider the third order linear problem

$$y'''+y'=0,\ y(0)=0,\ y'(0)=1,\ y''(0)=2,\ h=0.1$$

With the particular solution given by

$$y(x)=2(1-\cos x)+\sin x$$

Source (Adesanya, Udoh & Ajileye 2013, Areo & Omojola 2017 and Sunday 2018)

Table 2. Showing the result for problem two

x-value	particular Solution	Commutated Solution	Error in new Method	Error in (Adesanya *et al*, 2013)	Error in (Areo & Omojola 2017)	Error in (Sunday 2018)
0.1	0.10982508609077662011	0.10982508609077668890	6.8790e-17	1.6613e-12	1.1177e-10	3.7470e-16
0.2	0.23853617511257795326	0.23853617511258045075	2.4975e-15	7.5411e-12	9.3348e-10	8.3267e-16
0.3	0.38484722841012753581	0.38484722841013871413	1.1178e-14	1.3843e-09	3.2775e-09	1.3878e-15
0.4	0.54729635430288032607	0.54729635430291085096	3.0525e-14	4.5006e-09	8.0524e-09	1.4433e-15
0.5	0.72426041482345756807	0.72426041482352295179	6.5384e-14	1.0520e-08	1.6249e-08	1.5543e-15
0.6	0.91397124357567876270	0.91397124357579969798	1.2094e-13	1.9715e-08	2.8912e-08	1.9986e-15
0.7	1.11453331266871420120	1.11453331266891678780	2.0259e-13	3.2968e-08	4.7125e-08	2.8866e-15
0.8	1.32394267220519191980	1.32394267220550777840	3.1586e-13	5.0419e-08	7.1985e-08	4.4409e-15
0.9	1.54010697308615447550	1.54010697308662074190	4.6627e-13	7.2608e-08	1.0458e-07	3.5527e-15
1.0	1.76086637307161707180	1.76086637307227627070	6.5920e-13	9.9511e-08	1.4596e-07	5.3291e-15

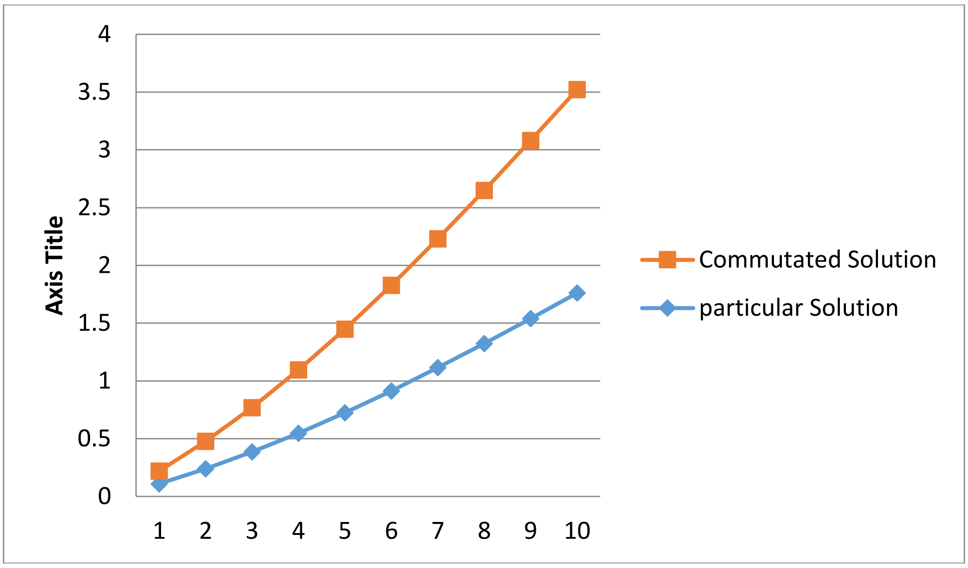

Figure 3. Showing the solution graph of problem two.

Problem three:

Consider the third order linear problem

$$y'''+e^{x}=0,\ y(0)=1,\ y'(0)=-1,\ y''(0)=3,\ h=0.1$$

With the particular solution given by

$$y(x)=2\left(1+x^{2}\right)-e^{x}$$

Source (Omar & Abdelrahim 2016, Kayode & Obarhua 2017 and Adeyeye & Omar 2019b)

Table 3. Showing the result for problem three

x-value	Particular Solution	Commutated Solution	Error in new Method	Error in (Omar & Abdelrahim 2016)	Error in (Kayode & Obarhua 2017)	Error in (Adeyeye & Omar 2019b)
0.1	0.91482908192435237 52	0.91482908192435230895	6.6250e-17	1.1102e-14	1.8241e-13	1.6209e-14
0.2	0.85859724183983016 61	0.85859724183982776596	2.4001e-15	1.6076e-13	1.6708e-12	6.6058e-14
0.3	0.83014119242399689 60	0.83014119242398628100	1.0615e-14	6.3105e-13	6.0014e-12	1.5277e-13
0.4	0.82817530235872968 22	0.82817530235870097786	2.8704e-14	1.6232e-12	1.4860e-11	2.7955e-13
0.5	0.85127872929987185 32	0.85127872929981077174	6.1082e-14	3.3591e-12	3.0121e-11	4.5020e-13
0.6	0.89788119960949102 51	0.89788119960937840127	1.1262e-13	6.0841e-12	5.3842e-11	6.6835e-13
0.7	0.96624729252952347 84	0.96624729252933475621	1.8872e-13	1.0070e-11	8.8316e-11	9.3925e-13
0.8	1.05445907150753239 54	1.05445907150723706150	2.9533e-13	1.5616e-11	1.3606e-10	1.2670e-12
0.9	1.16039688884305033 62	1.16039688884261129340	4.3904e-13	2.3054e-11	1.9987e-10	1.6578e-12
1.0	1.28171817154095476 46	1.28171817154032763920	6.2713e-13	3.2750e-11	2.8281e-10	2.1174e-12

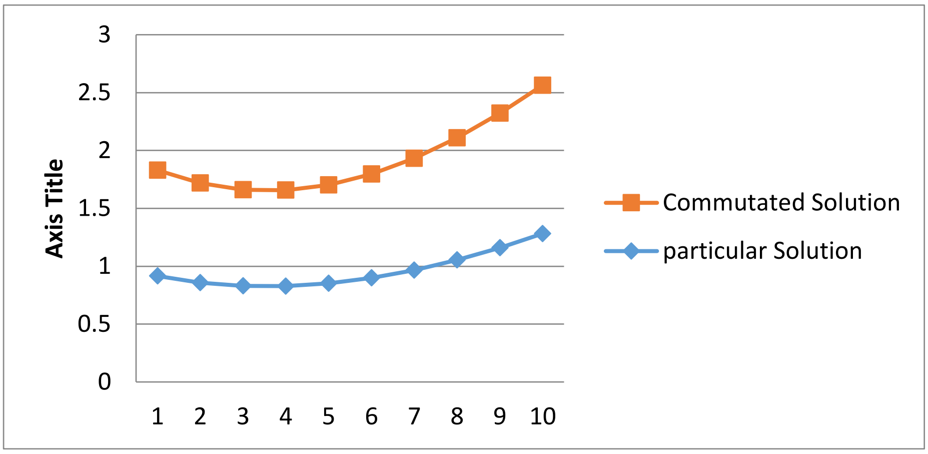

Figure 4. Showing the solution graph of problem three.

Conclusion

The numerical solution to higher order ordinary differential equations are conventionally solved by a reduction to a system of first order ordinary differential equations and then suitable numerical method for first order would be used to solve the system (Sabo, Bakari & Babuba). This method computes the numerical solution at one point at a time. However, the major setbacks for this method are computational burden which affects the accuracy of the method in terms of the error, difficulties in writing computer program for the method and wastage of human effort. In order to overcome these challenges and bring improvement on numerical method, we have exposed the direct solution of higher order initial value problems of ordinary differential equations using power series on single-step third derivative block hybrid method in this research. The new block method is derived using interpolation and collocation as a basic function and the basic properties of the block method which include order, error constant, consistency and zero stability are also analyzed. The new block method is been applied to solve third order initial value problems of ordinary differential equations without reducing the equations to their equivalent systems of first order ordinary differential equations. The outcome showed on the application of new block method on some sampled modeled third order linear problems was found to give better approximation than the existing methods. The direct method developed using interpolation and collocation procedure has been recommended for scholars, students and researchers.

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