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Analysis of a New Numerical Approach to Solutions of Heat Conduction Equations Arising from Heat Diffusion

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Abstract
In this work, a new numerical finite difference scheme with the aim of obtaining a new numerical scheme that will be used to solve for the solution of Partial Differential Equations (PDE) arising from heat conduction equation is developed. This is significant because in recent times there is a growing interest in literatures to obtain a continuous numerical method for solving PDE. The numerical accuracy of this new approach is also studied. Detailed numerical results have shown that the new method provides better results than the known explicit finite difference method by Schmidt. And in terms of stability, the new scheme has been able to clearly shown that it is more stable than the old Schmidt explicit method. There is no semi-discretization involved and no reduction of PDE to a system of ODEs in the new approach, but rather a system of algebraic equations is directly obtained. MATLAB software was used to solve for the desired solutions and the results obtained has shown that the method is near exact solutions.

Keywords: Lines; Multistep collocation; Parabolic; Taylor's polynomial

INTRODUCTION
In this study, we intent to obtain a scheme that will estimate the approximate solutions of parabolic partial differential equation in one space variable of the form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 \le x \le b,\ 0 \le t \le T$$

(1.0)
Subject to the initial and boundary conditions

$U(x,0) = f(x),\ 0 \le x \le b,$

$U(0,t) = g_1(t),\ t \ge 0$

$U(b,t) = g_2(t),\ t \ge 0$

Where t and x are the time and space coordinates respectively, and the quantities h and k are the mesh sizes in the space and time directions.
Of recent, there is a growing interest concerning continuous numerical methods of solution for Ordinary Differential Equations (ODEs). In this work, therefore, we extend a known

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continuous numerical method of solving ODEs to solve for the solution of heat conduction equations in the form of eqn. (1.0). This is achieved by imploring the collocation and interpolation method on PDEs directly over multi steps along lines which results in the reduction of the PDEs to a system of ODEs. We intend to solve this large system of coupled ODEs arising from the reduction method by a semi – discretization (Adam & David, 2002; Awoyemi, 2002; Awoyemi, 2003)

The Solution Method

To set up the solution method, we follow Odekunle (2008), where he postulated that in doing so you subdivide the interval $0 \le x \le b$ into N equal subintervals by the grid points $x_m = mh, \quad m = 0,..., N$ where $Nh = b$. On these meshes we seek $l-$step approximate solutions $U(x,t)$ of the form

$$U(x,t) = \sum_{r=0}^{p-2} a_r Q_r(x,t), \qquad x \in [x_m, x_{m+l}] \tag{2.0}$$

such that $0 = x_0 < ... < x_m < ... < x_N = b.$ To do this, we must implore a basis function say $Q_r(x,t), \; r = 0,..., p-2$ which are assumed known, and a_r are constants that we need to determine and we assume that $p \le l + s$, where s is the number of collocation points. The equality holds if the number of interpolation points used is equal to l (Bao, Jaksch, & Markowich, 2003). There will be flexibility in the choice of the basis function $Q_r(x,t)$ as may be desired for specific application. For this work, we consider the Taylor's polynomial $Q_r(x,t) = x^r t^r$. The interpolation values $U_{m,n},...,U_{m+l-1,n}$ are assumed to have been determined from previous steps, while the method seeks to obtain $U_{m+l,n}$ (Benner & Mena ,2004; Bensoussan, Prato, Delfour & Mitter, 2007).

To obtain the required scheme we follow the work of Biazar & Ebrahimi (2005) which pointed out that we apply the above interpolation conditions on eqn. (2.0) to obtain

$$a_0 Q_0(x_{m+g}, t_n) + ... + a_{p-2} Q_{p-2}(x_{m+g}, t_n) = U(x_{m+g}, t_n), \quad \text{where } g = -\frac{1}{15}\left(\frac{1}{15}\right)l - \frac{44}{15} \tag{2.1}$$

We then write eqn. (2.1) as a simple matrix equation in the augmented form as,

$$\begin{bmatrix} Q_0\left(x_{m-\frac{1}{15}}, t_n\right) & \cdots & Q_{p-2}(x_{m-\frac{1}{15}}, t_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ Q_0\left(x_{m+l-\frac{44}{15}}, t_n\right) & \cdots & Q_{p-2}(x_{m+l-\frac{44}{15}}, t_n) \end{bmatrix} \begin{bmatrix} a_0 \\ \cdots \\ \cdots \\ \cdots \\ a_{p-2} \end{bmatrix} = \begin{bmatrix} U(x_{m-\frac{1}{15}}, t_n) \\ \cdots \\ \cdots \\ \cdots \\ U(x_{m+l-\frac{44}{15}}, t_n) \end{bmatrix} \tag{2.2}$$

Using three interpolation points and one collocation point, implies that $s = 1$, $p = 4$, $l = 3$ and $r = 0,1,2$. Substituting p in eqn. (2.1) we obtain,

$$a_0 Q_0(x_{m+g}, t_n) + a_1 Q_1(x_{m+g}, t_n) + a_2 Q_2(x_{m+g}, t_n) = U_{m+g,n} \tag{2.3}$$

Substituting l in g, we have $g = -\frac{1}{15}, 0, \frac{1}{15}$

Putting the values of g in eqn. (2.3) and writing it as matrix in an augmented form we have,

$$\begin{bmatrix} Q_0\left(x_{m-\frac{1}{15}},t_n\right) & Q_1\left(x_{m-\frac{1}{15}},t_n\right) & Q_2(x_{m-\frac{1}{15}},t_n) \\ Q_0(x_m,t_n) & Q_1(x_m,t_n) & Q_2(x_m,t_n) \\ Q_0\left(x_{m+\frac{1}{15}},t_n\right) & Q_1\left(x_{m+\frac{1}{15}},t_n\right) & Q_2(x_{m+\frac{1}{15}},t_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} U(x_{m-\frac{1}{15}},t_n) \\ U(x_m,t_n) \\ U(x_{m+\frac{1}{15}},t_n) \end{bmatrix} \tag{2.4}$$

From eqn. (2.4) we obtain the following values

$$\left.\begin{matrix} Q_0\left(x_{m-\frac{1}{15}},t_n\right)=1 & Q_1\left(x_{m-\frac{1}{15}},t_n\right)=x_{m-\frac{1}{15}}t_n & Q_2(x_{m-\frac{1}{15}},t_n)=x^2{}_{m-\frac{1}{15}}t^2{}_n \\ Q_0(x_m,t_n)=1 & Q_1(x_m,t_n)=x_m t_n & Q_2(x_m,t_n)=x^2{}_m t^2{}_n \\ Q_0\left(x_{m+\frac{1}{15}},t_n\right)=1 & Q_1\left(x_{m+\frac{1}{15}},t_n\right)=x_{m+\frac{1}{15}}t_n & Q_2(x_{m+\frac{1}{15}},t_n)=x^2{}_{m+\frac{1}{15}}t_n{}^2 \end{matrix}\right] \tag{2.6}$$

Putting the values in eqn. (2.6) in eqn. (2.4) we obtain

$$\begin{bmatrix} 1 & x_{m-\frac{1}{15}}t_n & x^2{}_{m-\frac{1}{15}}t^2{}_n \\ 1 & x_m t_n & x^2{}_m t^2{}_n \\ 1 & x_{m+\frac{1}{15}}t_n & x^2{}_{m+\frac{1}{15}}t^2{}_n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} U_{m-\frac{1}{15},n} \\ U_{m,n} \\ U_{m+\frac{1}{15},n} \end{bmatrix} \tag{2.7}$$

We solve eqn. (2.7) by Gaussian elimination method to obtain the value of a_2 as

$$a_2 = \frac{225\left(U_{m+\frac{1}{15},n} + U_{m-\frac{1}{15},n} - 2U_{m,n}\right)}{2h^2t^2{}_n},$$

We then substitute $r = 0,1,2$ in eqn. (2.0) to obtain

$$U(x,t) = a_0Q_0 + a_1Q_1 + a_2Q_2 \tag{2.8}$$

By substituting Q_0, Q_1 *and* Q_2 in eqn. (2.8) we obtain

(2.9)

Substituting the value of a_2 in eqn. (2.9) we have

$$U(x,t) = a_0 + a_1xt + x^2t^2\left(\frac{225U_{m+\frac{1}{15},n} + 225U_{m-\frac{1}{15},n} - 450U_{m,n}}{2h^2t^2{}_n}\right) \tag{2.10}$$

Taken the first and second derivatives of eqn. (2.10) with respect to x we have

$$U'(x,t) = aa_1t + 2xt^2\left(\frac{225U_{m+\frac{1}{15},n} + 225U_{m-\frac{1}{15},n} - 450U_{m,n}}{2h^2t^2{}_n}\right)$$

$$U''(x,t) = 2t^2\left(\frac{225U_{m+\frac{1}{15},n} + 225U_{m-\frac{1}{15},n} - 450U_{m,n}}{h^2t^2{}_n}\right) \tag{2.11}$$

we collocate eqn. (2.11) at $t = t_n$ to arrive at

$$U''(x,t) = 2\left(\frac{225U_{m+\frac{1}{15},n} + 225U_{m-\frac{1}{15},n} - 450U_{m,n}}{h^2}\right) \tag{2.12}$$

Similarly, we reverse the roles of x and t in eqn. (2.0), and we also subdivide the interval $0 \le t \le T$ into y equal subintervals by the grid points $t_n = nk, \quad n = 0,...,y$ where $yk = T$ (Dehghan, 2003). On these meshes we seek $l-$step approximate solution to $U(x,t)$ of the form

$$U(x,t) = \sum_{r=0}^{p-2} a_r q_r(x,t), \qquad t \in [t_n, t_{n+l}] \tag{2.13}$$

Such that $0 = t_0 < ... < t_n < ... < t_y = T$. Again we implore the services of another basis function say $q_r(x,t), \; r = 0,..., p-2$ which are assumed known, a_r are constants to be determined. We assume $p \le l + s$, where s is the number of collocation points. The equality in equation holds only if the number of interpolation points used is equal to l. There will be flexibility in the

choice of the basis function $q_r(x,t)$ as may be desired for specific application. For this method, we consider the Taylor's polynomial giving by $q_r(x,t) = x^r t^r$. The interpolation values $U_{m,n},...,U_{m,n+l-1}$ are assumed to have been determined from previous steps, while the method seeks to obtain $U_{m,n+l}$ (Eyaya, 2010; Penzl, 2000; Pierre, 2008).

Richard et al., (2001) suggested that to obtain our required scheme we apply the above interpolation conditions on eqn. (2.11) to obtain

$a_0 q_0(x_m, t_{n+f}) + ... + a_{p-2} q_{p-2}(x_m, t_{n+f}) = U(x_m, t_{n+f})$, we assume $f = 0\left(\frac{1}{15}\right)l - \frac{29}{15}$ (2.14)

We can write (2.14) as a simple matrix equation in an augmented form as

$$\begin{bmatrix} q_0(x_m,t_n) & \cdots & q_{p-2}(x_m,t_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ q_0(x_m,t_{n+l-\frac{29}{15}}) & \cdots & q_{p-2}(x_m,t_{n+l-\frac{29}{15}}) \end{bmatrix} \begin{bmatrix} a_0 \\ \cdots \\ \cdots \\ \cdots \\ a_{p-2} \end{bmatrix} = \begin{bmatrix} U(x_m,t_n) \\ \cdots \\ \cdots \\ \cdots \\ U(x_m,t_{n+l-\frac{29}{15}}) \end{bmatrix} \tag{2.15}$$

Using two interpolation points and one collocation point in eqn. (2.15) implies that $p = 3, r = 0,1 \;\; l = 2$ and $f = 0, \frac{1}{15}$, and by substitution eqn.(2.15) becomes

$$\begin{bmatrix} q_0(x_m,t_n) & q_1(x_m,t_n) \\ q_0\left(x_m,t_{n+\frac{1}{15}}\right) & q_1\left(x_m,t_{n+\frac{1}{15}}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} U(x_m,t_n) \\ U(x_m,t_{n+\frac{1}{15}}) \end{bmatrix} \tag{2.16}$$

From eqn. (2.14) we obtain the following values:

$$\left.\begin{array}{ll} Q_0(x_m,t_n)=1 & Q_1(x_m,t_n)=x_m t_n \\ Q_0(x_m,t_{n+\frac{1}{15}})=1 & Q_1(x_m,t_{n+\frac{1}{15}})=x_m t_{n+\frac{1}{15}} \end{array}\right\} \quad (2.17)$$

Substituting the values of eqn. (2.17) into eqn. (2.16), we have this below matrix

$$\begin{bmatrix} 1 & x_m t_n \\ 1 & x_m t_{n+\frac{1}{15}} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}=\begin{bmatrix} U_{m,n} \\ U_{m,n+\frac{1}{15}} \end{bmatrix} \quad (2.18)$$

Solving eqn. (2.18) for value of a_1 we obtain

$$a_1=\frac{15U_{m,n+\frac{1}{15}}-15U_{m,n}}{kx_m}$$

Again, when we substitute $r=0,1,$ into eqn. (2.13), we obtain

$$U(x,t)=a_0q_0+a_1q_1 \quad (2.19)$$

By substituting the values of a_1,q_0,q_1 in equation (2.19) we have

$$U(x,t)=a_0+15xt\frac{\left(U_{m,n+\frac{1}{15}}-U_{m,n}\right)}{kx_m} \quad (2.20)$$

Taken the first derivatives of equation (2.20) with respect to t we obtain

$$U'(x,t)=15x\left(\frac{U_{m,n+\frac{1}{15}}-U_{m,n}}{kx_m}\right) \quad (2.21)$$

We collocate eqn. (2.21) at $x=x_m$ to yield

$$U'(x,t)=15\left(\frac{U_{m,n+\frac{1}{15}}-U_{m,n}}{k}\right) \quad (2.22)$$

But from eqn. (1.0) it is clear that eqn. (2.22) is equal to eqn. (2.10), which implies that

$15\frac{\left(U_{m,n+\frac{1}{15}}-U_{m,n}\right)}{k}=2\left(\frac{225U_{m+\frac{1}{15},n}+225U_{m-\frac{1}{15},n}-450U_{m,n}}{h^2}\right)$, manipulating mathematically and putting, $r=\frac{k}{h^2}$ we obtain

$$U_{m,n+\frac{1}{15}}=(1-30r)U_{m,n}+15r\left(U_{m+\frac{1}{15},n}+U_{m-\frac{1}{15},n}\right) \quad (2.23)$$

Eqn. (2.23) is a new numerical scheme for solving the heat equation.
To illustrate the viability of this method, we use it to solve two problems (5.1) and (5.2) respectively with known exact solutions.

Stability Analysis

To find the stability condition for eqn. (2.23) we follow Odekunle (2006), which say that in order for us to successfully obtain the stability condition, we let $Mh = 1$, and denote the errors at the grid points in the range $-1 \le x \le 1$, at $t = 0$ by $Z(mh) = Z_m{}^0 \; (m = 0,1,2,\ldots,M)$.

Now, since $Z_m{}^n$ satisfies the original difference equation, we get

$$Z_{m,n+\frac{1}{15}} = (1-30r)Z_{m,n} + 15r\left(Z_{m+\frac{1}{15},n} + Z_{m-\frac{1}{15},n}\right) \tag{3.1}$$

Let the solution of the finite difference equation which reduces to $e^{i\beta x}$ be,

$$Z_m{}^n = e^{\alpha nk} e^{i\beta mh} \tag{3.2}$$

Substituting eqn. (3.2) in eqn. (3.1) and carrying out mathematical manipulations we obtain,

$$e^{\frac{\alpha k}{15}} = 1 - 30r + 15r\left(e^{\frac{i\beta h}{15}} + e^{-\frac{i\beta h}{15}}\right) \tag{3.3}$$

Let $e^{\frac{\alpha k}{15}} = \xi$, then by manipulation again, we obtain

$$\xi = 1 - 60r\sin^2\frac{\beta h}{30} \tag{3.4}$$

Eqn. (3.4) is called the amplification error of the equation (Yildiz & Subasi, 2001).

Thus for stability, $|\xi| \le 1$, hence we have $-1 \le 1 - 60r\sin^2\frac{\beta h}{30} \le 1$, and by manipulation, we have

$r\sin^2\frac{\beta h}{30} \le \frac{1}{30}$, and $r \le \frac{1}{30}$, since $\sin^2\frac{\beta h}{30} \le 1$.

Hence the equation is conditionally stable if $r \le \frac{1}{30}$

Error Analysis

From our equation we have

$$U_{m,n+\frac{1}{15}} - (1-30r)U_{m,n} - 15r\left(U_{m+\frac{1}{15},n} + U_{m-\frac{1}{15},n}\right) = 0 \tag{4.1}$$

From eqn. (4.1) we find Taylor's expansion of $U_{m,n+\frac{1}{15}}$, $U_{m+\frac{1}{15},n}$, $U_{m-\frac{1}{15},n}$ and by substitution into eqn. (4.1) and manipulating mathematically we obtain

$$U_{m,n+\frac{1}{15}} - (1-30r)U_{m,n} - 15r\left(U_{m+\frac{1}{15},n} + U_{m-\frac{1}{15},n}\right) = \frac{k^2}{450}\left(\frac{\partial^2 U}{\partial t^2} - \frac{1}{90r}\frac{\partial^4 U}{\partial t^4}\right)_{m,n} + \ldots \tag{4.2}$$

Also, we have the difference equation to be,

$$\overline{U}_{m,n+\frac{1}{15}} - (1-30r)\overline{U}_{m,n} - 15r\left(\overline{U}_{m+\frac{1}{15},n} + \overline{U}_{m-\frac{1}{15},n}\right) = 0 \tag{4.3}$$

Subtracting eqn. (4.3) from eqn. (4.2) we obtain the error equation to be,

$$Z_{m,n+\frac{1}{15}} - (1-30r)Z_{m,n} - 11r\left(Z_{m+\frac{1}{15},n} + Z_{m-\frac{1}{15},n}\right) = \frac{k^2}{450}\left(\frac{\partial^2 U}{\partial t^2} - \frac{1}{90r}\frac{\partial^4 U}{\partial t^4}\right)_{m,n} + \ldots \tag{4.4}$$

The quantity $\frac{k^2}{450}\left(\frac{\partial^2 U}{\partial t^2} - \frac{1}{90r}\frac{\partial^4 U}{\partial t^4}\right)_{m,n} + \ldots$ in eqn. (4.4) is called the local truncation error of the difference formula

$$\overline{U}_{m,n+\frac{1}{15}} = (1-30r)\overline{U}_{m,n} + 15r\left(\overline{U}_{m+\frac{1}{15},n} + \overline{U}_{m-\frac{1}{15},n}\right)$$, while

$\frac{k^2}{450}\left(\frac{\partial^2 U}{\partial t^2} - \frac{1}{90r}\frac{\partial^4 U}{\partial t^4}\right)_{m,n}$ is the principal part of the truncation error (Zheyin & Qiang, 2012)

The method is of order $k^2 + kh^2$

Numerical Examples

In this section we give some numerical examples to compute approximate solutions for equation (1.0) by the method discussed in this paper. This is in order to test the numerical accuracy of the new method. To achieve this, we follow Richard et al., (2001) and Saumaya et al., (2012), we truncate the Taylor's polynomial after second order and use it as the basis function for the computation. The resultant interpolant is used to solve the following test problems.

Example 5.1

Use the scheme to approximate the solution to the heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \ 0 < x < 1 \quad 0 < t$$

$$U(0,t) = U(1,t) = 0, \ t > 0$$

$$U(x,0) = \sin \pi x, \qquad 0 \le x \le 1$$

Table 1: Results of Eqn. (2.23) on problem 5.1

x	Computed solution $U(x,t)$	Exact solution $U(x,t)$	Schmidt Method $U(x,t)$	Errors	
				New Method	Schmidt Method
0	0	0	0	0	0
0.1	0.308008706	0.308002141	0.307963277	6.6 X E-6	2.1 X E-4
0.2	0.585867367	0.585854886	0.58577788	1.2 X E-5	4.0 X E-4
0.3	0.806377253	0.806360073	0.806254085	1.7 X E-5	5.6 X E-4
0.4	0.947953314	0.947932118	0.947808521	2.0 X E-5	6.6 X E-4
0.5	0.996737101	0.996715865	0.996584857	2.1 X E-5	1.2 X E -4
0.6	0.947953314	0.947932118	0.947808521	2.0 X E-5	6.6 X E-4
0.7	0.806377253	0.806360073	0.806254085	1.7 X E-5	5.6 X E-4
0.8	0.585867367	0.585854886	0.58577788	1.2 X E-5	4.0 X E-4
0.9	0.398221058	0.308002141	0.307963277	6.6 X E-6	2.1 X E-4
1.0	0	0	0	0	0

Example 5.2

Use the scheme to approximate the solution to the heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \ 0 < t$$

$$U(-1,t) = U(1,t) = 0\,,\ t > 0$$

$$U(x,0) = \cos\left(\frac{\pi x}{2}\right),\ -1 \le x \le 1, t = 0$$

Table 2: Results of Eqn. (2.23) on problem 5.2

x	Exact Solution $U(x,t)$	Computed Solution $U(x,t)$	Schmidt method	Errors	
				New Method	Schmidt Method
-1.0	0	0	0	0	0
-0.75	0.380721639	0.380741429	0.380659316	1.9 X E-5	4.2 X E-4
-0.50	0.703481860	0.703518427	0.703366704	3.7 X E-5	7.9 X E -4
-0.25	0.919143346	0.919191122	0.918992885	4.8 X E-5	1.0 X E-3
0	0.994873588	0.994925302	0.995899602	5.2 X E-5	2.3 X E- 3
0.25	0.919143346	0.911191122	0.918992885	4.8 X E-5	1.0 X E-3
0.50	0.703481860	0.703518427	0.703366704	3.7 X E-5	7.9 X E -4
0.75	0.380721639	0.380741429	0.380659316	1.9 X E-5	4.2 X E-4
1.00	0	0	0	0	0

Discussion of Results

From tables I and II, which are the results of action of the new numerical scheme it can clearly be seen that the new approach produced more accurate results than the results of the known explicit method by Schmidt when applied to solving heat equations subject to some initial and boundary conditions. The results obtained have shown also that the method is very effective in solving parabolic partial differential equations arising from heat conditions. And in terms of stability, eqn. (3.4) has clearly shown that the equation is more stable than the Schmidt explicit method

Recommendations

We suggest research and investigation be carried – out into higher fractional and off – grid mesh points that might very easily and possibly produce better and more accurate or even exact solutions to PDEs arising from heat diffusion. Also, we suggest that research be conducted into the possibility of varying the number of collocation points, since we have been in this work able to effect the variation of interpolation points.

Conclusion

A continuous numerical interpolant is proposed for solving parabolic partial differential equation in one space variable by descretization. To check the strength, efficiency, viability and the accuracy of the numerical method, it is applied to solve two different test problems with known exact solutions. The numerical results have confirmed the effectiveness of new numerical scheme in solving the heat equations and suggested that it is an interesting and viable numerical method which involves the reduction of the PDEs to a system of ODEs.

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