https://dx.doi.org/10.4314/dujopas.v7i4b.4
ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 7 No. 4b December 2021*

The Performance Measure Analysis on the States Classification in Markov Chain

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Abstract
The system being modelled is assumed to occupy one and only one state at any moment in time and its evolution is represented by transitions from state to state. Also, the physical or mathematical behaviour of this system may be represented by describing all the different states it may occupy and by indicating how it moves among these states. In this work, the concept of the classification of groups of states, between states that are recurrent, meaning that the Markov chain is guaranteed to return to these states infinitely often, and states that are transient, meaning that there is a nonzero probability that the Markov chain will never return to such a state are investigated, in order to provide some insight into the performance measure analysis such as the mean first passage time, R_{ij}, *the mean recurrence time of state* R_{jj} *as well as recurrence iterative matrix* $R^{(k+1)}$. *Our quest is to demonstrate with illustrative examples on Markov chains with different classes of states, and the following results are obtained, the mean recurrence time of state 1 is infinite, as well as the mean first passage times from states 2 and 3 to state 1. The mean first passage time from state 2 to state 3 or vice versa is given as 1, while the mean recurrence time of both state 2 and state 3 is given as 2.*

Keywords: Embedded Markov chain, ergodic Markov chain, mean first passage time, single-step transition, k-dependent Markov chains, recurrence iterative matrix.

INTRODUCTION
The transitions in Markov chain are assumed to occur instantaneously and the future evolution of the system depends only on its current state and not on its past history, then the system may be represented by a Markov process. Even when the system does not possess this Markov property explicitly, it is often possible to construct a corresponding implicit representation. Examples of the use of Markov processes may be found extensively throughout the biological, physical, and social sciences as well as in business and engineering. The information we would like to obtain concerning a system is the probability of being in a given state or set of states at a certain time after the system becomes operational. Other measures of interest include the time taken until a certain state is reached for the first time. The values assumed by the random variables $X(t)$ are called states. The set of all possible states forms the state space of the process and this may be discrete or continuous. If the state space is discrete, the process is referred to as a chain and the states are usually identified with the set of natural numbers $\{0, 1, 2, \dots\}$. An example of a discrete

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state space is the number of customers at a service facility. An example of a continuous state space is the length of time a customer has been waiting for service. Other examples of a continuous state space could include the level of water in a dam or the temperature inside a nuclear reactor where the water level/temperature can be any real number in some continuous interval of the real axis. Markov chains are frequently illustrated graphically. Circles or ovals are used to represent states. Single-step transition probabilities are represented by directed arrows, which are frequently, but not always, labeled with the values of the transition probabilities. The absence of a directed arrow indicates that no single-step transition is possible. Romanovsky (1970) introduced the application and simulation of a discrete Markov Chains and this was extended to the introduction of Numerical Solutions of Markov Chains by Stewart (1994, 2009), while the suitability of the Markov chain approach is demonstrated in the wind feed in Germany by Pesch *et al.* (2015). Uzun and Kiral (2017) carried out the study to predict the direction of the gold price movement, and to determine the probabilistic transition matrix of the closing returns of gold prices, using the Markov chain model of fuzzy state, while the application of Markov chain using a data mining approach to get a prediction of the monthly rainfall data is shown by Aziza *et al.* (2019). The application of Markov chain on the spread of disease infection which shown that Hepatitis B was more infectious overtime than tuberculosis and HIV is demonstrated by Clement (2019), while the application of Markov chain to Journalism is demonstrated by Vermeer and Trilling (2020), but in this study, the performance measures of mean first passage time, R_{ij}, the mean recurrence time of state, R_{jj}, as well as recurrence iterative matrix, $R^{(k+1)}$ are analysed, for Markov chains with different classes of states, and these are demonstrated with illustrative examples.

Nomenclature

$f_{jj}^{(n)}$ Conditional Probability that on leaving state j the first return to state j occurs n steps later

p_{ij} Probability of moving from state i to state j

$p_{jj}^{(n)}$ that the Markov chain is once again in state j , n time steps after leaving it

R_{ij} mean first passage time

R_{jj} mean recurrence time

$R^{(k+1)}$ recurrence iterative matrix

Material and Methodology

Markov chain $\{X_n,\ n = 0,1,2,\dots\}$ is a stochastic process that satisfies the following relationship, called the Markov property:

For all natural numbers n and all states x_n,

$$Prob\{X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0\}$$
$$= Prob\{X_{n+1} = x_{n+1} | X_n = x_n\}. \quad (1)$$

The state in which the system finds itself at time step $(n + 1)$ depends only on where it is at time step n. The fact that the Markov chain is in state x_n at time step n is the sum total of all the information concerning the history of the chain that is relevant to its future evolution.

For simplification, Eq. (1) is written as

$$p_{ij}(n) = Prob\{X_{n+1} = j | X_n = i\} \quad (2)$$

which are called the single-step transition probabilities, or just the transition probabilities, of the Markov chain. They give the conditional probability of making a transition from state $x_n = i$ to state $X_{n+1} = j$ when the time parameter increases from n to $(n + 1)$.

The matrix $P(n)$, formed by placing p_{ij} (n) in row i and column j, for all i and j , is called the transition probability matrix or chain matrix. We have

$$p(n) = \begin{pmatrix} p_{00}(n) & p_{01}(n) & p_{02}(n) & \cdots & p_{0j}(n) & \cdots \\ p_{10}(n) & p_{11}(n) & p_{12}(n) & \cdots & p_{1j}(n) & \cdots \\ p_{20}(n) & p_{21}(n) & p_{22}(n) & \cdots & p_{2j}(n) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i0}(n) & p_{i1}(n) & p_{i2}(n) & \cdots & p_{ij}(n) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (3)$$

Notice that the elements of the matrix $P(n)$ satisfy the following two properties:

$$\text{and, for all } i, \quad \begin{cases} 0 \leq p_{i\,j}(n) \leq 1 \\ \sum_{all\ j} p_{i\,j}(n) = 1 \end{cases} \quad (4)$$

A matrix that satisfies these properties is called a Markov matrix or stochastic matrix.

Homogeneous Markov Chain

A Markov chain is said to be homogeneous if for all states i and j

$$p_{ij} = Prob\{X_{n+1} = j|X_n = i\} = Prob\{X_{n+m+1} = j|X_{n+m} = i\}$$
$$For\ n = 0, 1, 2, \cdots\ and\ \ m \geq 0.$$

i.e., for homogenous Markov chain,

$$p_{ij} = Prob\{X_1 = j|X_0 = i\} = Prob\{X_2 = j|X_1 = i\} = Prob\{X_3 = j|X_2 = i\} = \cdots$$

We have replaced $p_{i\,j}(n)$ by p_{ij} since transitions no longer depend on n.

Therefore, when the Markov chain is homogeneous, we find

$$Prob\{X_{n+m} = a, X_{n+m-1} = b, \ \cdots, X_{n+2} = k,\ X_{n+1} = j|X_n = i\}$$
$$= p_{i\,j}\ p_{j\,k} \ldots p_{cb}\ p_{ba} \quad (5)$$

for all possible values of n.

Non –Homogenous Markov Chain

For non-homogenous Markov chain,

$$p_{ij} = Prob\{X_1 = j|X_0 = i\} \neq Prob\{X_2 = j|X_1 = i\}$$

The probability of being in state j at time step $(n + 1)$ and in state k at time step $(n + 2)$, given that the Markov chain is in state i at time step n, is

$$Prob\{X_{n+2} = k, X_{n+1} = j|X_n = i\}$$
$$= prob\{X_{n+2} = k|X_{n+1} = j, X_n = i\}\, prob\{X_{n+1} = j|X_n = i\}$$
$$= prob\{X_{n+2} = k|X_{n+1} = j\} prob\{X_{n+1} = j|X_n = i\}$$
$$= p_{i\,jk}(n+1) p_{i\,j}(n) \quad (6)$$

Which is the probability of the sample path i, j, k that begins in state i at time step n.

More generally,

$$Prob\{X_{n+m} = a, X_{n+m-1} = b, \ \cdots, X_{n+2} = k,\ X_{n+1} = j|X_n = i\}$$
$$= prob\{X_{n+m} = a|X_{n+m-1} = b\} prob\{X_{n+m-1} = b|X_{n+m-2} = c\} \cdots prob\{X_{n+2} = k|X_{n+1}$$
$$= j\} prob\{X_{n+1} = j|X_n = i\}$$
$$= p_{ba}(n+m-1) p_{cb}(n+m-2) \cdots p_{jk}(n+1) p_{ij}(n) \quad (7)$$

for all possible values of n. Stewart (2009)

k-Dependent Markov Chains

A stochastic process is not a Markov chain if its evolution depends on more than its current state, for instance, if transitions at step $(n + 1)$ depend not only on the state occupied at time step n, but also on the state occupied by the process at time step $(n - 1)$. Therefore, the technique of converting a non-Markovian process to a Markov chain by incorporating additional states may be generalized. If a stochastic process has s states and is such that transitions from each state depend on the history of the process during the two prior steps, then a new process consisting of s^2 states may be defined as a Markov chain. If a stochastic

process that consists of s states is such that transitions depend on the state of the system during the prior k time steps, then we may construct a Markov chain with s^k states. This could be stated formally as follows:
Let $\{X_n, n \geq 0\}$ be a stochastic process such that there exists an integer k for which

$$prob\,\{X_{n+1} = x_{n+1} | X_n = x_n, \cdots, X_{n-k+1} = x_{n-k+1}, X_{n-k} = x_{n-k}, \cdots, X_0 = x_0\}$$
$$prob\,\{X_{n+1} = x_{n+1} | X_n = x_n, \cdots, X_{n-k+1} = x_{n-k+1}\} \quad (8)$$

for all $n \geq k$. Thus, the future of the process depends on the previous k states occupied. Such a process is said to be a K-dependent process. If $k = 1$, then X_n is a Markov chain. For $k > 1$, a new stochastic process $\{Y_n, n \geq 0\}$, with $Y_n = (X_n, X_{n+1}, \ldots, X_{n-k+1})$, is a Markov chain. If the set of states of X_n is denoted by S, then the states of the Markov chain Y_n are the elements of the cross product

$$\underbrace{S \times S \times S \times \cdots \times S}_{k-times} \quad (9)$$ Stewart (1994)

Classification of States

Given $p_{jj}^{(n)}$ to be the probability that the Markov chain is once again in state *j* , *n* time steps after leaving it. In these intervening steps, it is possible that the process visited many different states as well as state *j* itself.

let $f_{jj}^{(n)}$ define a new conditional probability, that on leaving state *j* the first return to state *j* occurs *n* steps later. Therefore,

$$f_{jj}^{(n)} = \text{Prob \{first return to state } j \text{ occurs exactly } n \text{ steps after leaving it\}}$$

$$= Prob\{X_n = j, X_{n-1} \neq j,\ X_{n+2} \neq j, \cdots, X_1 \neq j | X_0 = j\} \quad for\ n = 1, 2, 3, \cdots$$

To relate $p_{jj}^{(n)}$ and $f_{jj}^{(n)}$, we shall construct a recurrence relation that permits us to compute $f_{jj}^{(n)}$.

to compute $p_{jj}^{(n)}$ by taking higher powers of the single-step transition probability matrix p such that

$$f_{jj}^{(i)} = p_{jj}^{(i)} = p_{jj} \quad (10)$$

Which is the probability that the first return to state *j* occurs one step after leaving it

Since $$p_{jj}^{(0)} = 1.$$
Therefore
$$P_{jj}^{(1)} = f_{jj}^{(1)} p_{jj}^{(0)}.$$

Consider $P_{jj}^{(2)}$, the probability of being in state *j* two time steps after leaving it. This can happen because the Markov chain simply does not move from state *j* at either time step or else because it leaves state *j* on the first time step and returns on the second. In order to fit our analysis to the recursive formulation, we interpret these two possibilities as follows.
i. The Markov chain "leaves" state *j* and "returns" to it for the first time after one step, which has probability $f_{jj}^{(1)}$, and then "returns" again at the second step, which has probability $P_{jj}^{(1)}$.
ii. The Markov chain leaves state *j* and does *not* return for the first time until two steps later, which has probability $f_{jj}^{(2)}$.

Thus,

$$P_{jj}^{(2)} = f_{jj}^{(1)} P_{jj}^{(1)} + f_{jj}^{(2)} = f_{jj}^{(1)} P_{jj}^{(1)} + f_{jj}^{(2)} P_{jj}^{(0)} \quad (11)$$

Where $f_{jj}^{(2)}$ may be computed as

$$f_{jj}^{(2)} = P_{jj}^{(2)} - f_{jj}^{(1)} P_{jj}^{(1)} \quad (12)$$

In a similar manner,

$$P_{jj}^{(3)} = f_{jj}^{(1)} P_{jj}^{(2)} + f_{jj}^{(2)} P_{jj}^{(1)} + f_{jj}^{(3)} P_{jj}^{(0)}$$

From which $f_{jj}^{(3)}$ is computed as

$$f_{jj}^{(3)} = P_{jj}^{(3)} - P_{jj}^{(2)} - f_{jj}^{(2)} P_{jj}^{(1)}$$

By continue in this version and by also using the theorem of total probability

$$P_{jj}^{(n)} = \sum_{l=1}^{n} f_{jj}^{(l)} P_{jj}^{(n-l)}, \quad n \geq 1. \quad (13)$$

Recursively,

$$f_{jj}^{(n)} = \sum_{l=1}^{n-1} f_{jj}^{(l)} P_{jj}^{(n-l)}, \qquad n \geq 1. \quad (14)$$

The probability of ever returning to state *j* is denoted by $f_{jj}^{(n)}$ and is given by

$$f_{jj} = \sum_{n=1}^{\infty} f_{jj}^{(n)} \quad (15)$$

If $f_{jj} = 1$, then state j is said to be *recurrent*. In other words, state j is recurrent if and only if, beginning in state j , the probability of returning to j is 1, i.e., the Markov chain is guaranteed
to return to this state in the future. In this case, we have $P_{jj}^{(n)} > 0,\ for\ some\ n > 0$.
Thus the expected number of visits that the Markov chain makes to a recurrent state *j* given that it
starts in state *j* is infinite. We now show that the expected number of visits that the Markov chain
makes to state *j* given that it starts in state *j* is equal to

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty \text{ for recurrent state } j.$$

$$\text{Let } I_n = \begin{cases} 1 & \text{if the Markov chain is in state j at time step } n \\ 0 & \text{otherwise} \end{cases}$$

Then

$\sum_{n=0}^{\infty} I_n$ is the total number of time steps that state j is occupied.

Conditioning on the fact that the Markov chain starts in state j, the expected number of time steps it is in state *j* is

$$E[\sum_{n=0}^{\infty} I_n \,|X_0 = j] = \sum_{n=0}^{\infty} E[I_n|X_0 = j] = \sum_{n=0}^{\infty} prob[X_n|X_0 = j] = \sum_{n=1}^{\infty} p_{jj}^{(n)}$$

(16)

Thus, when state j is recurrent,

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty.$$

When state j is transient,

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty.$$

When state j is recurrent, i.e. when $f_{jj} = 1$,
The mean recurrence time R_{jj} of state j is given as

$$R_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}. \quad (17)$$

This is the average number of steps taken to return to state *j* for the first time after leaving it. A recurrent state *j* for which R_{jj} is finite is called a positive recurrent state or a recurrent non null state.

If $R_{jj} = \infty$, we say that state j is a null recurrent state.
Transition between two different states

Let $f_{ij}^{(n)}$, $for\ i \neq j$ be the probability that, starting from state i, the first passage to state j occurs
in exactly n steps. Then,

$$f_{ij}^{(1)} = p_{ij}$$

From Eq. (13), we derive

$$P_{ij}^{(n)} = \sum_{l=1}^{n} f_{ij}^{(l)} P_{jj}^{(n-l)}, \quad n \geq 1. \tag{18}$$

This is re-arranged to obtain

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{l=1}^{n-1} f_{ij}^{(l)} P_{jj}^{(n-l)}, \qquad n \geq 1. \tag{19}$$

Let the probability that state j is ever reached from state i be given by f_{ij}. Therefore,

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

Such that when:
$f_{ij} < 1$, the process starting from state i may never reach state j
$f_{ij} = 1$, the expected value of the sequence $f_{ij}^{(n)}$, $n = 1, 2, \cdots$ of first passage probabilities for a fixed pair i and j $(i \neq j)$ is called the mean first passage time and its denoted by R_{ij}. Therefore,

$$R_{ij} = n \sum_{n=1}^{\infty} f_{ij}^{(n)}, \ for\ i \neq j \tag{20}$$

Such that

$$R_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} \left(1 + R_{kj}\right) = 1 + \sum_{k \neq j} p_{ik} R_{kj}$$

since the process in state i either goes to state j in one step (with probability p_{ij}), or else goes first to some intermediate state k in one step (with probability p_{ik}) and then eventually on to j in an additional R_{kj} steps. If $i = j$, then $R_{ij}$is the mean recurrence time of state i and Equation 20 continues to hold.
Let e to denote a (column) vector whose components are all equal to 1 and whose length is determined by its context. Likewise, we shall use E to denote a square matrix whose elements are all equal to 1. Notice that $E = ee^T$. Letting $diag\{R\}$ be the diagonal matrix whose i^{th} diagonal element is R_{ii} , it follows that

$$R_{ij} = 1 + \sum_k p_{ik} R_{kj} + p_{ij} R_{jj} \tag{21}$$

In matrix form,

$$R = E + P(R - dig(R))$$

Where the diagonal elements of R are the mean recurrence times, the off-diagonal elements are the mean first passage times. The matrix R may be obtained iteratively from the equation

$$R^{(k+1)} = E + P(R^{(k)} - dig\left(R^{(k)}\right)) \quad \text{with} \quad R^{(0)} = E \tag{22}$$

Results

Illustrative Example on Recurrent State
Consider the discrete-time Markov chain whose transition diagram is shown in Figure below:

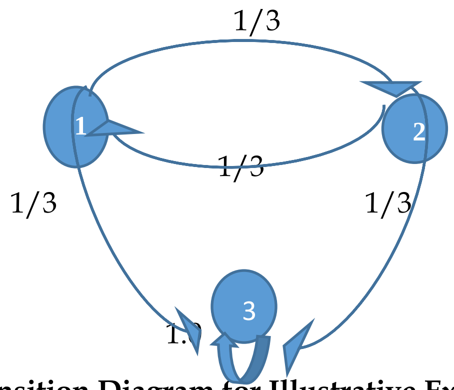

Figure 1: Transition Diagram for Illustrative Example on Recurrent State

Its transition probability matrix is given by

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The successive power of P are given by

$$P^k = \begin{cases} \begin{pmatrix} 0 & (1/3)^k & 1-(1/3)^k \\ (1/3)^k & 0 & 1-(1/3)^k \\ 0 & 0 & 1 \end{pmatrix} & if\ k = 1, 3, 5 \\ \begin{pmatrix} (1/3)^k & 0 & 1-(1/3)^k \\ 0 & (1/3)^k & 1-(1/3)^k \\ 0 & 0 & 1 \end{pmatrix} & if\ k = 2, 4, 6 \end{cases}$$

It follows that

$$f_{11}^{(1)} = p_{11}^{(1)} = 1.$$

$$f_{11}^{(2)} = P_{11}^{(2)} - f_{11}^{(1)} P_{11}^{(1)} = \left(\frac{1}{3}\right)^2 - 0 = \left(\frac{1}{3}\right)^2,$$

$$f_{11}^{(3)} = P_{11}^{(3)} - P_{11}^{(2)} - f_{11}^{(2)} P_{11}^{(1)} = 0 - \left(\left(\frac{1}{3}\right)^2 \times 0\right) - \left(0 \times \left(\frac{1}{3}\right)^2\right) = 0$$

$$f_{11}^{(4)} = P_{11}^{(4)} - f_{11}^{(3)} P_{11}^{(1)} - f_{11}^{(2)} P_{11}^{(2)} - f_{11}^{(1)} P_{11}^{(3)} = \left(\frac{1}{3}\right)^4 - 0 - \left(\left(\frac{1}{3}\right)^2 \times \left(\frac{1}{3}\right)^2\right) - 0 = 0.$$

Therefore,

$$f_{11}^{(k)} = 0, \qquad for\ all\ k \geq 3.$$

Also,

$$f_{33}^{(1)} = p_{33}^{(1)} = 1.$$

$$f_{33}^{(2)} = P_{33}^{(2)} - f_{33}^{(1)} P_{33}^{(1)} = 1 - 1 = 0,$$

$$f_{33}^{(3)} = P_{33}^{(3)} - P_{33}^{(2)} - f_{33}^{(2)} P_{33}^{(1)} = 1 - 1 - 0 = 0$$

We have,

$$f_{33}^{(0)} = 0, \qquad for\ all\ k \geq 2.$$

Thus, $\quad f_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)} = 1.$

Mean recurrence time is

$$R_{33} = \sum_{n=1}^{\infty} n f_{33}^{(n)} = 1.$$

Illustrative Example on Transient State

The Markov chain with transition probability P given below has two transient states 1 and 2, and two ergodic states 3 and 4. The matrix P and $\lim_{n\to\infty} P^n$ are

$$P = \begin{pmatrix} 0.4 & 0.5 & 0.1 & 0.0 \\ 0.3 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 & 0.2 \end{pmatrix},$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0.0 & 0.0 & 4/9 & 5/9 \\ 0.0 & 0.0 & 4/9 & 5/9 \\ 0.0 & 0.0 & 4/9 & 5/9 \\ 0.0 & 0.0 & 4/9 & 5/9 \end{pmatrix} \qquad (23)$$

By using the following result when j is a state of a discrete Markov chain

$$\begin{cases} \lim_{n\to\infty} P_{ij}^{(n)} = 0, & \text{if state j is a null recurrent or transient state} \\ \lim_{n\to\infty} P_{ij}^{n} > 0,\ if\ \ j & \text{is ergodic} \\ \lim_{n\to\infty} P_{ij}^{(n)} = f_{ij} \lim_{n\to\infty} P_{jj}^{(n)}, & \text{for other state i, positive recurrent, transient or otherwise} \end{cases} \qquad (24)$$

Therefore,

Since states 1 and 2 are transient, $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for $i = 0, 1, 2, 3, 4$ and $j = 1, 2$.

Since state 3 and 4 are ergodic, $\lim_{n\to\infty} P_{ij}^{n} > 0$ for $j = 3, 4$.

Since $f_{ij} = 1$ for $i = 1, 2, 3, 4,\ \ j = 1, 2$ and $\forall\ i \neq j$.

Illustrative Example on Recurrence, Ergodic and Transient States

Consider a homogeneous discrete-time Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

With $0 < a < 1.$

In this example

$$P_{11}^{(n)} = a^n \text{ for } \ n = 1, 2, 3, \cdots,$$
$$P_{22}^{(n)} = P_{33}^{(n)} = 0 \text{ for } \ n = 1, 3, 5, \cdots,$$
$$P_{22}^{(n)} = P_{33}^{(n)} = 1 \text{ for } \ n = 0, 2, 4, \cdots$$

And that

$$P_{12}^{(n)} = aP_{12}^{(n-1)} + \left(b \times 1_{\{n\ is\ odd\}}\right) + \left(c \times 1_{\{n\ is\ even\}}\right), \qquad (25)$$

Where $1_{\{.\}}$ Is an indicator function, which has the value 1 when the condition inside the braces is true and the value 0 otherwise. Then

$$f_{11}^{(1)} = P_{11}^{(1)} = a$$
$$f_{11}^{(2)} = P_{11}^{(2)} - f_{11}^{(1)} p_{11}^{(1)} = a^2 - (a \times a) = 0$$
$$f_{11}^{(3)} = P_{11}^{(3)} - f_{11}^{(1)} p_{11}^{(2)} - f_{11}^{(2)} p_{11}^{(1)} = P_{11}^{(3)} - f_{11}^{(1)} p_{11}^{(2)} = 0 \qquad (26)$$

It is immediately follow that $f_{11}^{(n)} = 0 \;\forall\; n \geq 2$, and thus the probability of ever returning to state 1 is given as

$$f_{11} = a < 1.$$

State 1 is therefore a transient state. Also,

$$f_{22}^{(1)} = P_{22}^{(1)} = 0$$
$$f_{22}^{(2)} = P_{22}^{(2)} - f_{22}^{(1)}p_{22}^{(1)} = P_{22}^{(2)} = 1$$
$$f_{22}^{(3)} = P_{22}^{(3)} - f_{22}^{(1)}p_{22}^{(2)} - f_{22}^{(2)}p_{22}^{(1)} = P_{22}^{(3)} = 0$$

and again it immediately follows that $f_{22}^{(3)} = 0 \;\forall\; n \geq 3$, we then have

$$f_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = f_{22}^{(2)} = 1,$$

which means that state 2 is recurrent. Furthermore, it is positive recurrent, since

$$R_{22} = \sum_{n=1}^{\infty} n f_{22}^{(n)} = 2 < \infty$$

In a similar fashion, it may be shown that state 3 is also positive recurrent.
Now for $f_{12}^{(n)}$:

$$f_{12}^{(1)} = b,$$
$$f_{12}^{(2)} = P_{12}^{(2)} - f_{12}^{(1)}p_{22}^{(1)} = P_{12}^{(2)} = aP_{12}^{(1)} + c = ab + c$$
$$f_{12}^{(3)} = P_{12}^{(3)} - f_{12}^{(1)}p_{22}^{(2)} - f_{12}^{(2)}p_{22}^{(1)} = P_{12}^{(3)} - f_{12}^{(1)} = (a^2b + ac + b) - b = (a^2b + ac)$$

Continue in this fashion, we find that

$$f_{12}^{(4)} = (a^3b + a^2c),$$
$$f_{12}^{(5)} = (a^4b + a^3c), \text{ etc.}$$

and it is easy to show that in general we have

$$f_{12}^{n} = \left(a^{(n-1)}b + a^{(n-2)}c\right).$$

It follows that the probability that state 2 is ever reached from state 1 is

$$f_{12} = \sum_{n=1}^{\infty} f_{12}^{(n)} = \frac{b}{1-a} + \frac{c}{1-a} = 1. \tag{27}$$

Similarly, we may show that $f_{12} = 1$. Also, it is evident that

$$f_{23}^{(1)} = f_{32}^{(1)} = 1,$$
$$f_{23}^{(n)} = f_{32}^{(n)} = 0 \;\forall\; n \geq 2,$$

So that

$$f_{23} = f_{32} = 1.$$

Also

$$f_{21}^{(n)} = f_{31}^{(n)} = 0 \;\forall\; n \geq 1, \tag{28}$$

and so state 1 can never be reached from state 2 or from state 3.
Therefore, to provide solution to the given illustrative example, and to examine the matrix M of mean first passage times (with diagonal elements equal to the mean recurrence times), we shall give specific values to the variables a, b, and c.
Let

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Then, using the iterative formula (9.11) we find

$$R^{(1)} = \begin{pmatrix} 1.3 & 1.8 & 1.9 \\ 2.0 & 2.0 & 1.0 \\ 2.0 & 1.0 & 1.0 \end{pmatrix}, \quad R^{(2)} = \begin{pmatrix} 1.6 & 2.36 & 2.53 \\ 3.0 & 2.0 & 1.0 \\ 3.0 & 1.0 & 2.0 \end{pmatrix}, \text{ etc.}$$

The iterative process tends to the matrix

$$R^{(\infty)} = \begin{pmatrix} \infty & 11/3 & 12/3 \\ \infty & 2.0 & 1.0 \\ \infty & 1.0 & 2.0 \end{pmatrix} \tag{29}$$

Discussion

It may be readily verified that this matrix of Equation (29) satisfies Equation (22). Thus, the mean recurrence time of state 1 is infinite, as are the mean first passage times from states 2 and 3 to state 1. The mean first passage time from state 2 to state 3 or vice versa is given as 1, which must obviously be true since, on leaving either of these states, the process immediately enters the other. The mean recurrence time of both state 2 and state 3 is 2. States 2 and 3 are each periodic with period 2, since, on leaving either one of these states, a return to that same state is only possible in a number of steps that is a multiple of 2. These states are not ergodic. The first part of Equation (24) allows us to assert that the first column of $\lim_{n\to\infty} P^n$ contains only zero elements, since state 1 is transient. Also, since states 2 and 3 are both periodic, we are not in a position to apply the second part of the Equation.

Conclusion

The concept of the classification of groups of states, between states that are recurrent, meaning that the Markov chain is guaranteed to return to these states infinitely often, and states that are transient, meaning that there is a nonzero probability that the Markov chain will never return to such a state are investigated, in order to provide some insight into the performance measure analysis such as the mean first passage time, R_{ij}, the mean recurrence time of state R_{jj} as well as recurrence iterative matrix $R^{(k+1)}$. Our quest is to demonstrate with illustrative examples on Markov chains with different classes of states and to obtain results for performance measures.

References

Agboola, S. O. (2016). Repairman problem with multiple batch deterministic repairs, Unpublished Ph.D. Thesis, Obafemi Awolowo University, Ile-Ife, Nigeria, 256pp.

Azizah, A. , Welastica, R., Nur, F., Ruchjana, B. and Abdullah, A. (2019). An application of Markov chain for predicting rainfall data at West Java using data mining approach, *Earth and Environmental Science*, **303**(1): 203 – 216.

Clemence. T. (2019). Markov chain modelling of HIV, Tuberculosis, and Hepatitis B transmission in Ghana, *Hindawi, Interdisciplinary Perspective on Infectious Disease*, **27**(1)**:** 204 – 214.

Dayar., T. (1998). Permuting Markov chains to nearly completely decomposable form. *Technical Report BU-CEIS-9808*, Department of Computer Engineering and Information Science, Bilkent University, Ankara, Turkey. pp. 18 – 31.

Pesch, T., Schroder, S., Allelein, H. and Hake, J. (2015). A new Markov chain related statistical approach for modelling synthetic wind power time series, *New Journal of Physics, Dentsche Physikalishe,* **35**(2)**:** 64 – 85.

Philippe, B and Sidje, B. (1993) *Transient Solution of Markov Processes by Krylov Subspaces*, Technical Report, *IRISA*—Campus de Beaulieu, Rennes, France. pp. 11 - 24.

Ramaswami, V. (1988). A Stable Recursion for the Steady State Vector in Markov chains of M/G/1 type. *Communication in Statist. Stochastic Models*, **4**(1)**:** 183–188.

Ramaswami, V and Neuts, M. F. (1980). Some explicit formulas and computational methods for infinite server queues with phase type arrivals. *Journal of Applied Probability,* **17**(1)**:** 498–514.

Romanovsky, V.I. (1970). *Discrete Markov Chains*, Wolters-Noord off, Groningen, Netherlands.pp.23 – 44.

Stewart, W. J. (1994). *Introduction to the Numerical Solution of Markov Chains*. Princeton University Press, Princeton, N.J. pp. 14 – 38.

Stewart, W. J. (2009). *Probability, Markov Chain, Queues and Simulation*, Princeton University Press, United Kingdom. pp. 1- 42.

Uzun, B. and Kiral, E. (2017). Application of Markov chain-fuzzy states to gold price, *Science Direct. ELSEVIER,* **120**(1): 365 – 371.

Vermeer, S. And Trilling, D. (2020): Toward a better understanding of a new user journeys: A Markov chain approach. *Journalism Journal,* **21**(1): 879 – 894.
Zakaria, N. N., Mahmod, O., Rajalmgan, S., Hamita, D., Lazim, A. and Evizal, A. (2019). Markov chain model development for forecasting air pollution index of Miri, Sarawak, *Sustainability* **11(1):** 5190 -5202.