Communication in Physical Sciences, 2024, 11(4): 785-798

On the Exponentiated Type II Generalized Topp-Leone-G Family of Distribution: Properties and Applications

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Received: 02 May 2024/Accepted: 24 August 2024/Published: 09 September 2024

***Abstract**: In the modern era of statistics, distribution theory plays a crucial role in accurately modelling phenomena across various scientific fields. Traditional statistical distributions often fall short of adequately representing complex lifetime data. To address this limitation, this study introduces the Exponentiated Type II Generalized Topp-Leone-G (ET2GTL-G) family of distributions. This study employs the maximum likelihood estimation (MLE) method to estimate the parameters of the ET2GTL-G family and illustrates its application to two real-life datasets: (1) civil engineering hailing times, and (2) failure and service times for a windshield. Comparative analyses with existing distributions, such as the Kumaraswamy Extension Exponential (KEED), Kumaraswamy Exponential (KED), Exponential Generalized Exponentiated Exponential (EGEE), and Exponentiated Weibull-Exponential (EWED) distributions, highlight the superior goodness-of-fit and empirical flexibility of the ET2GTL-G distribution. For the first dataset, the ET2GTL-G distribution reported a minimum Akaike Information Criterion (AIC) value of 274.7174, compared to the next best-fit KED with an AIC value of 275.0377. For the second dataset, the ET2GTL-G distribution achieved an AIC value of 204.2458, outperforming the EGEE distribution which had an AIC value of 206.9956. These results underscore the potential of the ET2GTL-G family to improve the modelling of lifetime data, thereby contributing significantly to the fields of medicine, engineering, and beyond.*

***Keywords:** T-X, Exponentiated, MLE, ET_2GTL-G, hybridization*

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1.0 Introduction

The role of distribution theory is particularly powerful in the modern period of statistics. Without selecting the right mathematical form of a model, statistical modelling of the phenomenon, applications, or data validity might be a little bit challenging. In many fields of pure science and social science, traditional statistical distributions have been utilized to derive inferences and model data such as modelling environment pollution in environmental science, modelling duration without claims in actuarial science, modelling machines' life cycle in engineering, modelling patient survival times after surgery in medical sciences, the modelling failure rate of software in computer science, and measuring average time from marriage to divorce in social sciences. According to Yahaya and Doguwa (2021), many problems abound in various fields of human endeavour cannot be perfectly and adequately handled by most commonly known conventional probability distributions available for modelling lifetime data sets such as Normal, Weibull, Pareto, Gomepertz, Rayleigh, Exponential, e.t.c, and there is a clear need for an extended version of these classical distributions to enhance their capability while also improving their goodness of fit. Some of well- the well-known modified families of distributions in the literature proposed by different researchers to improve the standard theoretical distribution but are not limited to the Beta-G by Eugene *et al*.(2002), Weibull-X family of distributions of Alzaatreh *et al* (2013), Exponentiated Generalized class of Cordeiro *et al*.(2013), Logistics-G introduced by Torabi and Montzeri (2014), Gamma-X family of Alzaatreh *et al.* (2014), Odd Generalized Exponential-G of Tahir *et al*. (2015), Type I half- logistic family of Cordeiro *et al*. (2016), Kumaraswamy-Weibull-Generated family of Hassan and Elgarhy (2016), New Weibull-G family of Tahir *et al* (2016), Generalized Transmuted-G of Nofal *et al*.(2017), New Generalized family of distributions of Ahmad (2018), Topp-Leone Kumaraswamy- G family of distribution by Ibrahim *et al*. (2020), *Rayleigh*-Exponentiated Odd Generalized-*X* Family by Yahaya and Doguwa (2021), Type I Half Logistic Exponentiated-G family by Bello *et al* (2021). According to Hussain *et al*. (2023), the Generalized distribution can be used effectively in fitting lifetime datasets because it can accommodate monotonic and non-monotonic data characteristics. Since there is a clear need for generalized families of distributions to offer greater distributional flexibility and be able to model lifetime with monotonically and non-monotonically increasing, decreasing and constant or more importantly with bathtub-shaped failure rates, Hence, this study proposed Exponentiated Type II Generalized Topp-Leone-G family of distribution capable of modelling with monotonic and non-monotonic hazard functions that can provide better fits to medical and Engineering data.

In this work, a new generalized family of distribution is proposed called the Exponentiated Type II Generalized Topp-Leone-G family (ET_2GTL-G) from the Cumulative Distribution Function (cdf) of the Exponentiated-G family defined by Guptal *et al.* (1998) as

$$F_{EG}\left(x;\theta,\xi\right)=G^{\theta}(x;\xi) \quad (1)$$

With the pdf of;

$$f_{EG}\left(x;\theta,\xi\right)=\theta g(x;\xi)G^{\theta-1}(x;\xi) \quad (2)$$

Where θ is a shape parameter belonging to a set of positive real numbers (i.e. $\theta>0$).

According to Alzaatreh *et al.* (2013), the cdf of the T-X family of distribution is given as

$$F\left(x\right)=\int_{\alpha_1}^{W[G(x)]} r(t)dt=R\left[W\left[G\left(x\right)\right]\right] \quad (3)$$

where $W\left[G(x)\right]$ must be found to satisfy the following conditions

(i) $W\left[G(x)\right]\in\left[\alpha_1,\alpha_2\right]$

(ii) $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing, and (4)

(iii) $W\left[G(x)\right]\to\alpha_1$ as $x\to-\infty$ and $W\left[G(x)\right]\to\alpha_2$ as $x\to\infty$

Let $r(t)$ be the pdf of a random variable $T\in\left[\alpha_1,\alpha_2\right]$ for $-\infty\le\alpha_1<\alpha_2<\infty$ and $W\left[G(x)\right]$ be a function of the cdf of a random variable X.

Then the pdf corresponding to equation (3.3) is given by;

$$f(x)=\left\{\frac{d}{dx}W\left[G(x)\right]\right\}r\{W\left[G(x)\right]\} \tag{5}$$

Let X be any arbitrary random variable with CDF: $G\left(x;\xi\right)$.Also, let $T\in\left(\alpha_1,\alpha_2\right)$ be a random variable with a PDF: $r(t)$. Furthermore, let our proposed link function be Type II Generalized Topp-Leone family of distribution, (Hassan *et al.*, 2019) and it is given as,

$$F_{T_2GTL-G}(t;\beta,\alpha,\xi)=1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha} \tag{6}$$

And the Probability Density Function (pdf) is given as

$$f_{T_2GTL-G}(t;\beta,\alpha,\xi)=2\alpha\beta h(t;\xi)H^{2\beta-1}(t;\xi)\left[1-H^{2\beta}(t;\xi)\right]^{\alpha-1} \tag{7}$$

where $\beta>0,\alpha>0\,;t>0$ and $F(t;\xi)$ *and* $f(t;\xi)$ are the cdf and pdf of the baseline distribution with parameter vector ξ.

2.0 Exponentiated Type II Generalized Topp-Leone-G Family Of Distribution (ET$_2$GTL-G)

The CDF of ET$_2$GTL-G family of distribution that generalized T$_2$GTL-G is given by:

$$F_{\mathrm{ET_2GTL-G}}(t,\beta,\alpha,\theta;\xi)=\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta} \tag{8}$$

And its pdf is derived as

$$f_{\mathrm{ET_2GTL-G}}(t,\beta,\alpha,\theta,\xi)=2\alpha\beta\theta h(t;\xi)H^{2\beta-1}\left(t;\xi\right)\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha-1}\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta-1} \tag{9}$$

Proof:

Let the Exponentiated-G be the baseline distribution with the cdf and pdf defined in equations (1) and (2) and let equation (6) be the link function, we can write;

$$F_{\mathrm{ET_2GTL-G}}(t;\beta,\alpha,\theta,\xi)=\int_0^{F_{T_2GTL-G}(t;\beta,\alpha,\xi)} f_{EG}(x;\theta,\xi)dx \tag{10}$$

$$F_{\mathrm{ET_2GTL-G}}(t;\beta,\alpha,\theta,\xi)=\int_0^{1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}} \theta g\left(x;\xi\right)G^{\theta-1}\left(x;\xi\right)dx \tag{11}$$

$$Let\ y=G\left(x;\xi\right),\ when\ x\to 0,\ y\to 0,\ when\ x\to 1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha},\ y\to\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]$$

$$\frac{dy}{dx}=g\left(x;\xi\right)\quad dx=\frac{dy}{g(x;\xi)}$$

$$\therefore F_{\mathrm{ET_2GTL-G}}(t;\beta,\alpha,\theta,\xi)=\int_0^{\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]} \theta g\left(x;\xi\right)y^{\theta-1}\frac{dy}{g(x;\xi)}$$

$$F_{\mathrm{ET_2GTL-G}}(t,\beta,\alpha,\theta;\xi)=\theta\int_0^{\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]} y^{\theta-1}dy$$

$$F(t,\beta,\alpha,\theta;\xi)=\theta\left[\frac{y^{\theta-1+1}}{\theta-1+1}\right]_0^{\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]}$$

$$=y^{\theta}\Big|_0^{\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]}$$

$$=\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta}-0$$

$$F_{\mathrm{ET_2GTL-G}}(t;\beta,\alpha,\theta,\xi)=\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta}$$

3.0 Important Representation

The binomial expansion to expand cdf of ET_2GTL-G is given below as;

$$\left[1-t\right]^{b}=\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{b}{i}t^{i} \tag{12}$$

Using the series expansion in equation (12), then equation (8) becomes

$$\left[F\left(t\right)\right]^{h}$$

$$\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]^{\theta h}=\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{\theta h}{i}\left[1-H(t;\xi)^{2\beta}\right]^{\alpha_i}$$

Consider;

$$\left[1-H(t;\xi)^{2\beta}\right]^{\alpha i}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\binom{\alpha_i}{j}H\left(t;\xi\right)^{2\beta_j}$$

Therefore,

$$\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]^{\theta h}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(-1\right)^{i+j}\binom{\theta h}{i}\binom{\alpha_i}{j}H\left(t;\xi\right)^{2\beta j}$$

$$\left[F\left(t\right)\right]^{h}=\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\binom{\theta h}{i}\binom{\alpha_i}{j}H\left(t;\xi\right)^{2\beta j}$$

$$\left[F\left(t\right)\right]^{h}=\sum_{i,j=0}^{\infty}B_q H\left(t;\xi\right)^{2\beta j} \tag{13}$$

where; $B_q=\left(-1\right)^{i+j}\binom{\theta h}{i}\binom{\alpha_i}{j}$

Also, we have expansion for pdf as;

$$f(t,\alpha,\beta,\theta,\xi)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\gamma_q h(t;\xi)H(t;\xi)^{2\beta(n+1)-1} \tag{14}$$

where; $\gamma_q=2\alpha\beta\theta\left(-1\right)^{p+n}\binom{\theta-1}{p}\binom{\alpha(1+k)-1}{n}$

4.0 Statistical Properties of ET2GTL-G Family of Distribution

In this section, some statistical properties of the ET2GTL-G family of distributions were derived.

4.1 Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, r^{th} the moment for the variable $T \sim ET_2GTL-G$ says $\mu^1{}_r$ is derived as follows;

$$\mu_r^1 = E\left(t^r\right) = \int_{-\infty}^{\infty} t^r f\left(t\right) dt \tag{15}$$

By using important representation in equation (14), we have

$$= \int_0^1 t^r 2\alpha\beta\theta h(t;\xi) H\left(t;\xi\right)^{2\beta(1+n)-1} \sum_{p,n=0}^{\infty} (-1)^{p+n} \binom{\theta-1}{p} \binom{\alpha(1+k)-1}{n} dt \tag{16}$$

$$= \int_0^1 t^r \sum_{p,n=0}^{\infty} \gamma_q h\left(t;\xi\right) H\left(t;\xi\right)^{2\beta(n+1)-1} dt \tag{17}$$

4.2 Moment Generating Function (MGF)

The Moment Generating Function $T \sim ET_2GTL-G$ is given as:

$$M_x\left(t\right) = \int_{-\infty}^{\infty} e^{tx} f\left(t\right) dt \tag{18}$$

By using important representation in (14), we have

$$= \int_0^{\infty} e^{tx} \sum_{p,n=0}^{\infty} \gamma_q h\left(t;\xi\right) H\left(t;\xi\right)^{2\beta(n+1)-1} dt \tag{19}$$

$$M_x\left(t\right) = \sum_{p,n=0}^{\infty} \gamma_q B_r \tag{20}$$

where, $B_r = \int_0^1 e^{tx} \gamma_q h\left(t;\xi\right) H\left(t;\xi\right)^{2\beta(n+1)-1} dt$

4.3 Probability Weighted Moments (PWM)

The class of moment used to describe inverse form estimators for the parameters and quantiles of a distribution is known as Probability Weighted Moments (PWMs) and was proposed by Greenwood *et al*., (1979). The PWMs, represented by $\tau r, s$ can be derived for a random variable T using the following relationship.

$$\tau r, s = E\left[T^r F(t)^s\right] = \int_{-\infty}^{\infty} T^r f\left(t\right)\left(F\left(t\right)\right)^s dt \tag{21}$$

The PMWs are derived by substituting equations (13) and (14) into equation (21) replacing h with s, we have

$$\tau r, s = \int_0^1 T^r 2\alpha\beta\theta h\left(t;\xi\right)$$

$$\sum_{i,j=0}^{\infty} \sum_{p,n=0}^{\infty} (-1)^{p+n} (-1)^{i+j} \binom{\theta-1}{p} \binom{\alpha(1+k)-1}{n} \binom{\theta s}{i} \binom{\alpha_i}{j} H\left(t;\xi\right)^{2\beta(n+j+1)-1} dt \tag{22}$$

$$\tau r,s=\int_0^1 T^r\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_q\beta_k h(t;\xi)H(t;\xi)^{2\beta(n+j+1)-1}dt \quad (23)$$

$$=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_p\beta_k\int_0^1 T^r h(t;\xi)H(t;\xi)^{2\beta(n+j+1)-1}dt \quad (24)$$

$$\tau r,s=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_p\beta_k .Tr,2\beta(n+j+1)-1 \quad (25)$$

where, $\tau r,2\beta(n+j+1)-1=\int_0^1 T^r h(t;\xi)H(t;\xi)^{2\beta(n+j+1)-1}$

4.4 Quantile Function of ET$_2$GTL-G

The quantile function is an important tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For t, the quantile function is $F(t)=U$, where U is distributed as $U(0,1)$. The ET$_2$GTL-G family is easily simulated by inverting equation (8) which yields the Quantile function $Q(U)$ defined as

$$Q(u)=H(t;\xi)^{-1}\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}} \quad (26)$$

where $H(t;\xi)^{-1}$ is the quantile function of the baseline cdf $G(t;\xi)$. The first quartile, the median and the third quartile are obtained by setting $U=0.25$, $U=0.5$, $U=0.75$, respectively in equation (26)

where $t=Q^{-1}$ is the quantile function of the baseline distribution.

4.5 Reliability Function

The Probability of an item not failing before some time is known as the survival or reliability function and it is defined as;

$$R(t;\theta,\alpha,\xi)=P(T>t)=1-F(t;\xi)$$

$$R(t;\theta,\alpha,\xi)=P(T>t)=1-\left[1-\left[1-H^{2\beta}(t;\xi)\right]^{\alpha}\right]^{\theta} \quad (27)$$

4.6 Hazard Function

The hazard function is the probability of an event of interest occurring within a relatively short period and it is given as;

$$T(t;\alpha,\beta,\theta;\xi)=\frac{f(t;\alpha,\beta,\theta;\xi)}{R(t;\alpha,\beta,\theta;\xi)}$$

$$T(t;\alpha,\beta,\theta;\xi)=\frac{2\alpha\beta\theta h(t;\xi)H(t;\xi)^{2\beta-1}\left[1-H(t;\xi)^{2\beta}\right]^{\alpha-1}\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]^{\theta-1}}{1-\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]^{\theta}} \quad (28)$$

4.7 Distribution of Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $T_1,T_2...T_n$ be independent and identically distributed $(i.i.d)$ random variables with their corresponding continuous distribution function $F(t)$. Let $T_{1:n}<T_{2:n}<...<T_{n:n}$ the corresponding ordered random sample be from a population of size n.

Let $F_{r:n}(t)$ and $f_{r:n}(t), r=1,2,3...,n$ denote the cdf and pdf of the r^{th} order statistics $T_{r:n}$ respectively. David (1970) gave the probability density function of $T_{r:n}$ as;

$$f_{r:n}\left(t\right)=\frac{1}{B\left(r,n-r+1\right)}F^{r-1}\left(t\right)\left[1-F\left(t\right)\right]^{n-r}f\left(t\right) \tag{29}$$

By substituting equation (8) and equation (9) into equation (29), we have;

$$f_{r:n}\left(t\right)=\frac{1}{B\left(r,n-r+1\right)}\left[\left[1-\left[1-H\left(t;\xi\right)^{2\beta}\right]^{\alpha}\right]^{\theta}\right]^{r-1}\left[1-\left[1-\left[1-H\left(t;\xi\right)^{2\beta}\right]^{\alpha}\right]^{\theta}\right]^{n-r}$$
$$2\alpha\beta\theta h(t;\xi)H(t;\xi)^{2\beta-1}\left[1-H(t;\xi)^{2\beta}\right]^{\alpha-1}\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]^{\theta-1} \tag{30}$$

The pdf of the maximum order statistics is obtained by setting $r=n$ equation (30) as;

$$f_{n:n}\left(t;\alpha,\beta,\theta,\xi\right)=2n\alpha\beta\theta h(t;\xi)H^{2\beta-1}(t;\xi)\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha-1}\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta-1}$$
$$\left[\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta}\right]^{n-1} \tag{31}$$

Also, the pdf of the minimum order statistics is obtained by setting $r=1$ in equation (3.30)

$$f_{1:n}\left(t;\alpha,\beta,\theta,\xi\right)=2n\alpha\beta\theta h(t;\xi)H^{2\beta-1}(t;\xi)\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha-1}\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta-1}$$
$$\left[1-\left[1-\left[1-H^{2\beta}\left(t;\xi\right)\right]^{\alpha}\right]^{\theta}\right]^{n-1} \tag{32}$$

5.0 Sub-Model

In this section, we provide a sub-model of this family corresponding to the baseline Exponential (Ex) distribution to show the flexibility of the new family.

5.1 Exponentiated Type II Generalized Topp-Leone Exponential Distribution (ET$_2$GTLEx)

Let us consider the Exponential distribution which is the baseline distribution with parameters δ with cumulative distribution and probability density functions given, respectively by;

$$H_{Ex}(t;\delta)=1-e^{-\delta t} \tag{33}$$

and

$$h_{Ex}(t;\delta)=\delta e^{-\delta t} \tag{34}$$

where $t>0,\delta>0$

Then the cdf and pdf of the proposed ET$_2$GTLEx distribution with four parameters are obtained by inserting equation (33) into equation (8) and are respectively given as

$$F_{ET_2GTLEx}(t;\beta,\alpha,\theta,\delta)=\left[1-[1-[1-e^{-\delta t}]^{2\beta}]^{\alpha}\right]^{\theta} \tag{35}$$

and

$$f_{ET_2GTLEx}(t;\beta,\alpha,\theta,\delta,\xi)=2\beta\alpha\theta\delta e^{-\delta t}\left[1-e^{-\delta t}\right]^{2\beta-1}\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1} \tag{36}$$

where $t\geq 0,\alpha,\theta,and.\delta>0,$ are the shape parameters $\delta>0$ is a scale parameter

Furthermore, the following are the reliability function, hazard rate function and quantile function respectively

$$R(t;\alpha,\beta,\theta,\delta)=1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\theta} \tag{37}$$

$$r(t;\beta,\alpha,\theta,\delta)=\frac{2\beta\alpha\theta\delta e^{-\delta t}\left[1-e^{-\delta t}\right]^{2\beta-1}\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\theta-1}}{1-\left[1-\left[1-\left[1-e^{-\delta t}\right]^{2\beta}\right]^{\alpha}\right]^{\theta}} \tag{38}$$

$$t=Q(U)=\frac{1}{\delta}\left[-\log\left[1-\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}\right]\right] \tag{39}$$

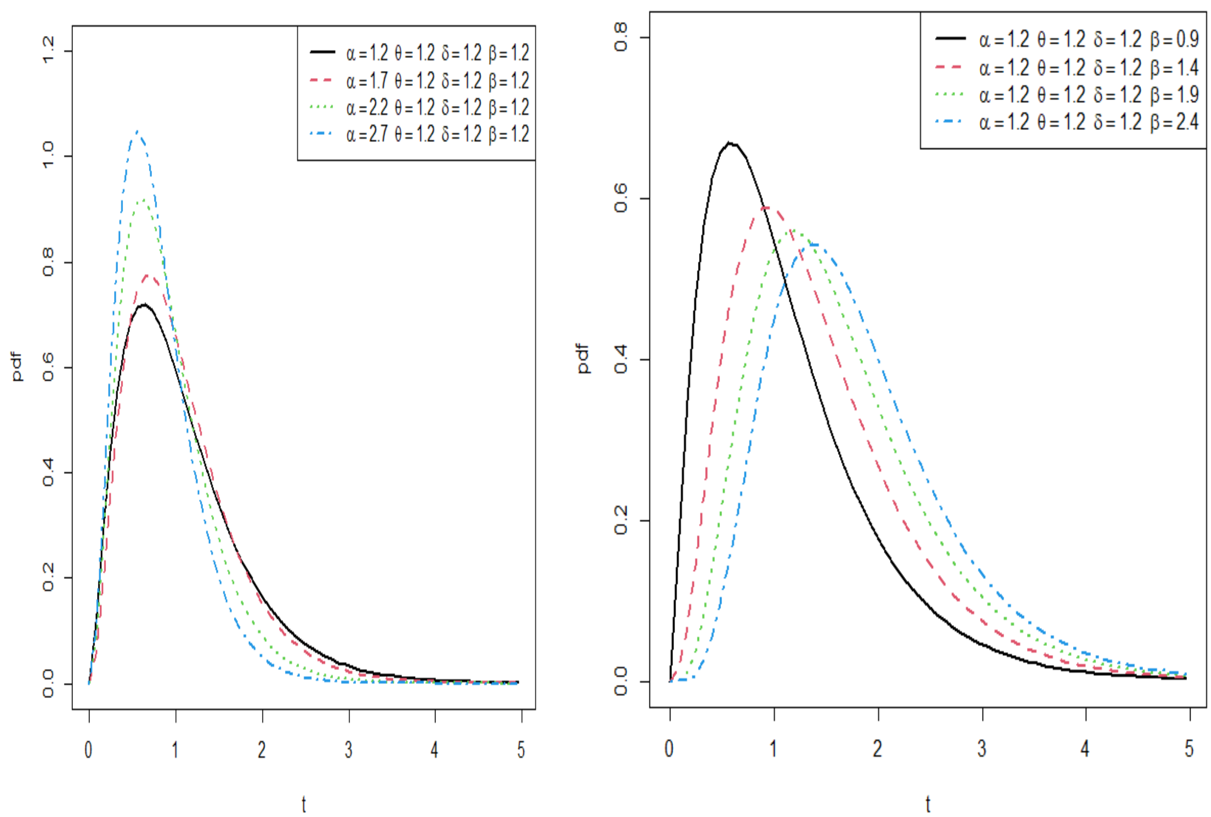

Fig. 1. Plots of Pdf of the ET2GTLEx distribution for different parameter values.

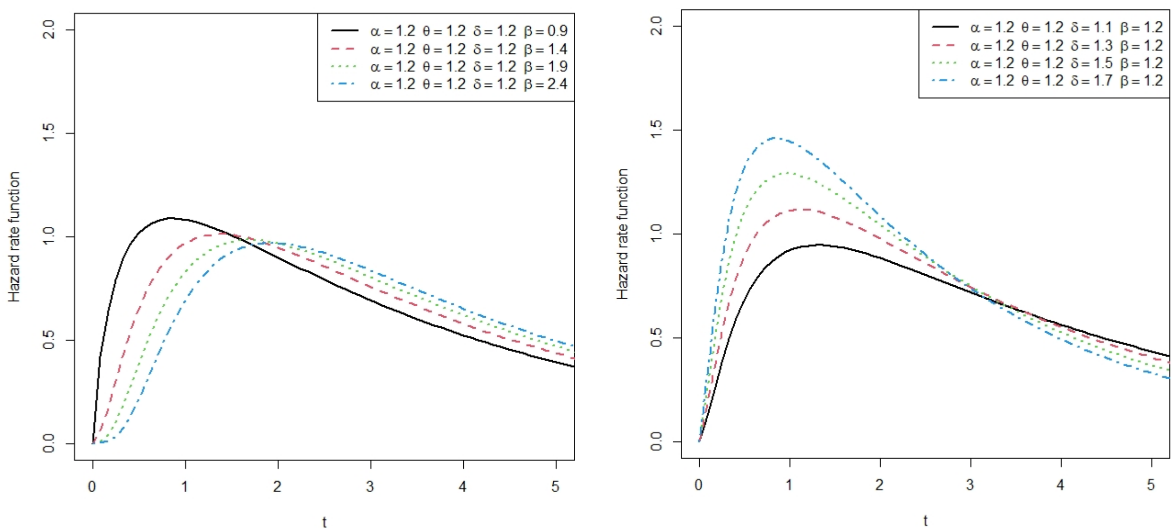

Fig. 2. Plots of Hazard function of the ET$_2$GTLEx distribution for different parameter values.

6.0 Parameter Estimation

In this work, the maximum likelihood Estimate is used to estimate the unknown parameter of the ET$_2$GTL-G family for complete data. Let $t_1, t_2 \ldots t_n$ be a random sample of size n from the ET$_2$GTL-G family. Then, the likelihood function based on the observed sample for the vector ϕ of the parameter $(\alpha, \beta, \theta, \xi)^T$ is given by;

$$\log(\phi) = n\log(2) + n\log(\alpha) + n\log(\beta) + n\log(\theta) + \sum_{i=1}^{n}\log\left[h(t;\xi)\right] + (2\beta-1)\sum_{i=1}^{n}\log\left[H(t;\xi)\right] + (\alpha-1)\sum_{i=1}^{n}\log\left[1-H(t;\xi)^{2\beta}\right] + (\theta-1)\sum_{i=1}^{n}\log\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right] \quad (47)$$

The components of the score Vector $U = (U_\alpha, U_\beta, U_\theta, U_\xi)^T$ are given as

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-H(t;\xi)^{2\beta}\right] + (\theta-1)\sum_{i=1}^{n}\frac{\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\log\left[1-H(t;\xi)^{2\beta}\right]}{\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]} \quad (48)$$

$$U_\beta = \frac{n}{\beta} + 2\sum_{i=1}^{n}\log\left[H(t;\xi)\right] + (\alpha-1)\sum_{i-1}^{n}\frac{H(t;\xi)^{2\beta}\log H(t;\xi)^{2}}{1-H(t;\xi)^{2\beta}} - \alpha(\theta-1)\sum_{i=1}^{n}\frac{\left[1-H(t;\xi)^{2\beta}\right]^{\alpha-1}H(t;\xi)^{2\beta}\log H(t;\xi)^{2}}{1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}} \quad (49)$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right] \quad (50)$$

$$U_{\xi}=\sum_{i=1}^{n}\frac{h(t;\xi)^{\xi}}{h(t;\xi)}+(2\beta-1)\sum_{i=1}^{n}\frac{H(t;\xi)^{\xi}}{H(t;\xi)}-2\beta(\alpha-1)\sum_{i-1}^{n}\frac{H(t;\xi)^{2\beta-1}H(t;\xi)^{\xi}}{1-H(t;\xi)^{2\beta}}+$$
$$2\beta\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left[1-H(t;\xi)^{2\beta}\right]^{\alpha-1}H(t;\xi)^{2\beta-1}H(t;\xi)^{\xi}}{\left[1-\left[1-H(t;\xi)^{2\beta}\right]^{\alpha}\right]} \quad (51)$$

Equating $U_{\alpha},U_{\beta},U_{\theta}=0$ and solving these equations simultaneously yields the MLEs. Equation (48), (49), (50) and equation (51) cannot be solved analytically, and analytical software is required to solve them numerically.

7. 0 Application to Real-Life Dataset

In this section, we fit the ET$_2$GTLEx distribution to two real data sets and give a comparative study with the fits of the Kumaraswamy Extension Exponential distribution (KEED) by Elbatal, *et al*., (2018). Kumaraswamy Exponential distribution (KED) by Adepoju and Chukwu (2015), The Exponential Generalized Exponentiated Exponential distribution (EGEE) by Bukoye and Oyeyemi, (2018), The Exponentiated Weibull-Exponential distribution (EWED) by Elgarhy, *et al*. (2017). as comparator distributions for illustrative purposes. This application proved empirically the flexibility of the new distribution in modeling real-life data. All the computations are performed using the AdequacyModel package in R software.

The KEED by Elbatal, *et al*., (2018) has a probability density function given as:

$$f(x)=\gamma\beta\alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}}\left[1-e^{1-(1+\lambda x)^{\alpha}}\right]^{\gamma-1}\left[1-\left[1-e^{1-(1+\lambda x)^{\alpha}}\right]^{\gamma}\right]^{\beta-1} \quad (52)$$

The KED by Adebayo and Chukwu (2015) has pdf defined as:

$$f(x)=\alpha\beta\lambda e^{-\beta x}\left[1-e^{-\beta x}\right]^{\alpha-1}\left[1-\left[1-e^{-\beta x}\right]^{\alpha}\right]^{\lambda-1} \quad (53)$$

The EGEE by Bukoye and Oyeyemi, (2018) has pdf defined as:

$$f(x)=\frac{\alpha\beta\lambda}{\theta}e^{-\frac{x}{\theta}}\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda-1}\left[1-\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-\frac{x}{\theta}}\right]^{\lambda}\right]^{\alpha}\right]^{\beta-1} \quad (54)$$

And

The EWED by Elgarhy, *et al*. (2017). has pdf given as:

$$f(x)=\alpha\gamma\beta\lambda\left[e^{\lambda x}-1\right]^{\beta-1}\exp\left[-\left[\alpha\left(e^{\lambda x}-1\right)^{\beta}-\lambda x\right]\right]\left[1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{\alpha-1} \quad (55)$$

The two datasets that were used as examples in the application demonstrate the new family of distributions' flexibility and 'best fit' compared to the above comparator distributions in modelling the data sets experimentally. The R programming language is used to carry out all of the computations.

Data Set 1

The first data set as listed below represents the data set shown below represents the civil engineering data with 85 hailing times, previously used by Kotz and Dorp (2004):

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

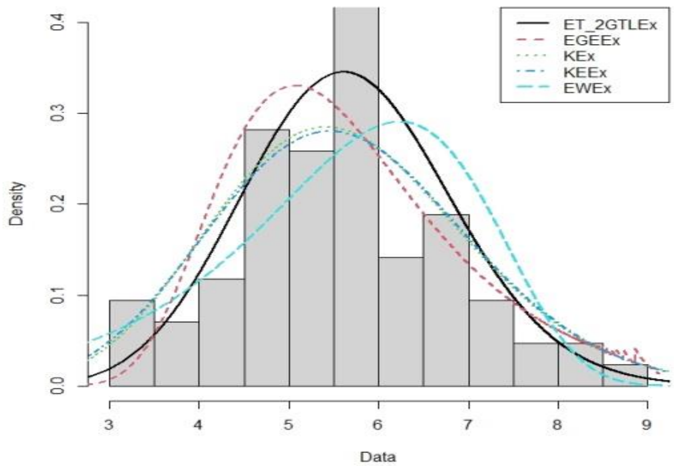

Fig. 3: Fitted pdfs for the ET$_2$GTLEx, EGEEx, KEx, KEEx, and EWEx, distributions to the data set 1

Table 1: MLE, Log-Likelihood and Goodness of fits statistics for Data 1

Distribution	β	α	θ	λ	γ	δ	LL	AIC
ET$_2$GTLEx	4.7976	6.0888	2.0676	-	-	0.3273	-133.3587	274.7174
EGEE	9.9583	4.5668	3.5730	3.3744	-	-	-135.235	278.47
KED	0.2888	11.7176	-	8.5072	-	-	-134.5189	275.0377
KEED	5.1259	3.0992	0.0590	7.4310	-	-	-134.9512	277.9023
EWED	0.3117	0.0060	-	0.2926	0.5932	-	-444.6284	897.2569

Table 1 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the Four comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value, though followed closely by the KED. The visual inspection of the fit presented in Figure 3, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution 'best fit' civil engineering data hailing times data set amongst the range of distributions considered.

Data Set 2

The second data set shown below represents the failure and service times for a windshield, previously used Kundu and Raqab (2009):

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483,1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881,1.262, 2.543, 5.140.

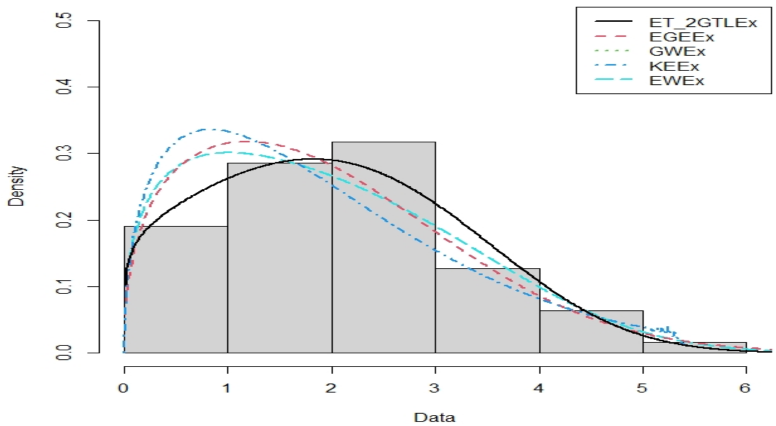

Fig. 5: Fitted pdfs for the ET_2GTLEx, EGEEx, KEx, KEEx, and EWEx, distributions to the data set 2

Table 2: MLE, Log-Likelihood and Goodness of fits statistics for Data set 2

Distribution	β	α	θ	λ	γ	δ	LL	AIC
ET_2GTLEx	4.6739	4.2458	0.1610	-	-	0.4472	- 98.1229	204.2458
EGEE	0.3570	8.1041	3.8821	4.0832	-	-	-99.49782	206.9956
KED	0.2445	1.8150	-	4.3041	-	-	-101.8542	209.7085
KEED	2.0812	6.3446	-	0.0409	1.5193	-	-99.61453	207.2291
EWED	0.3813	0.8037	-	1.0111	3.1110	-	-99.76702	207.534

Table 2 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the four comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value, though followed closely by the EGEE. The visual inspection of the fit presented in Figure 3, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution ‘best fit’ failure and service times for a windshield data set amongst the range of distributions considered.

8. 0 Conclusion

This study introduced the Exponentiated Type II Generalized Topp-Leone-G (ET2GTL-G) family of distributions, designed to offer greater flexibility in modeling complex lifetime data. The ET2GTL-G distribution integrates the cumulative distribution function of the Exponentiated-G family with a Type II Generalized Topp-Leone link function, resulting in a versatile model capable of representing both monotonic and non-monotonic hazard functions. Key statistical properties, such as moments, moment generating function, probability weighted moments, quantile function, reliability function, hazard function, and the distribution of order statistics, were derived and analyzed comprehensively.

Using the maximum likelihood estimation method, the parameters of the ET2GTL-G family were estimated and applied to real-life datasets: civil engineering hailing times and failure/service times for a windshield. The comparative analyses revealed that the ET2GTL-G distribution outperformed existing distributions, such as the Kumaraswamy Extension Exponential, Kumaraswamy Exponential, Exponential Generalized Exponentiated Exponential, and Exponentiated Weibull-Exponential distributions. The ET2GTL-G distribution reported lower Akaike Information Criterion (AIC) values, indicating a better fit for the data.

The results underscore the potential of the ET2GTL-G family to enhance the modeling of lifetime data, contributing significantly to fields like medical and engineering. Future research should focus on exploring the application of the ET2GTL-G family to a broader range of datasets and investigating the potential for further extensions or modifications to improve its flexibility and applicability in other domains.

9. 0 References

Adepoju, K. A., & Chukwu, O. I. (2015). Maximum Likelihood Estimation of the Kumaraswamy Exponential Distribution with Applications, *Journal of Modern Applied Statistical Methods*, 14, 1, pp. 208-214.

Ahmad, Z., Elgarhy, M., & Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, 5, pp. st is aour brea1-18.

Alzaatreh A. Famoye C. & Lee C. (2014). The Gamma-Normal distribution: Properties and Applications. *Computational Statistics & Data Analysis* 69, pp. 67-80,

Alzaatreh, A., Lee, C. & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, pp. 63-79.

Bello O.A., Doguwa S. I., Yahaya A. & Haruna M.J. (2021). A Type I Half Logistic Exponentiated –G family of distribution; properties and applications. *Communication in Physical Sciences*, 7, 3, pp. 147-163.

Bukoye, A., & Oyeyemi, G. M. (2018). On Development of Four-Parameters Exponentiated Generalized Exponential Distribution. *Pak. J. Statist*, 34, 4, pp. 331-358.

Cordeiro, G. M., Saboor, A., Khan, M. N.,Ozel, G., & Pascoa, M. A. (2016). The Kumaraswamy Exponential{Weibull Distribution: Theory and Applications. Hacettepe journal of mathematics and statistics, 45, 4, pp. 1203-1229.

Cordeiro, G. M., Ortega, E. M. M. & Christian da Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11, pp. 1-27.

David, H. A. (1970). Order Statistics, Second edition. Wiley, New York

Elbatal, I., Louzada, F., & Granzotto, D. C. (2018). A new lifetime model: The Kumaraswamy extension exponential distribution. *Biostatistics and Bioinformatics*, *2*, pp.1-9.

Elgarhy, M., Shakil, M., & Kibria, G. (2017). Exponentiated Weibull-Exponential Distribution with Applications. *Applications and Applied Mathematics Journal (AAM)*, *12*(2), 5.

Eugene, N., Lee, C. & Famoye, F. (2002). The beta-normal distribution and its applications. *Communications in Statistics Theory and Methods*, 31, pp. 497- 512.

Greenwood, J. A., Landwehr, J. M., & Matalas, N. C.(1979). Probability weighted moments: Definitions and relations of parameters of several distributions expressible in inverse form. Water Resources Research, 15, pp. 1049-1054.

Gupta, R.C., Gupta, P.L., & Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. *Commun. Stat. Theory Methods*, 27, pp. 887-904.

Hassan, A. S. & Elgarhy, M. (2016). Kumaraswamy Weibull- generated family of distributions with applications. *Advances and Application in Statistics*, 48, 205-239.

Hassan, A. S., Elgarhy, M., & Ahmad, Z. (2019). Type II Generalized Topp-Leone Family of distributions: Properties and Applications. *Journal of data science*, 17, 4, pp. 4-56

Ibrahim S, Doguwa S. I, Isah A & Haruna J. M. (2020). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease Data. *Journal of Biostatistics and Epidemiology* 6, 1, pp. 37-48.

Nofal, Z. M., Afify, A. Z., Yousof, H. M., & Cordeiro, G. M. (2017). The Generalized Transmuted-G Family of distributions. *Communications in Statistics-Theory and Methods*, *46*, *8*, *pp.* 4119-4136.

Tahir, M. H., Zubair, M., Mansoor M., Cordeiro G. M., Alizadeh, M. & Hamedani, G. G. (2016). A New Weibull-G Family of Distributions. *Hacettepe Journal of Mathematics and Statistics*, 45, 2, pp. 629-647.

Tahir, M. H., Cordeiro, G. M., Mansoor, M., & Zubair, M. (2015). The Weibull-Lomax distribution: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 44, pp. 461-480.

Torabi, H., & Montazari, N. H. (2014). The logistic-uniform distribution and its application. *Communications in Statistics Simulation and Computation* 43, pp. 25512569.

Yahaya A. & Doguwa S. I. S. (2021). On Theoretical Study of Rayleigh-Exponentiated Odd Generalized-X Family of Distributions. *Transactions of the Nigerian Association of Mathematical Physics.* 14, *pp.* 143 –154.

Compliance with Ethical Standards Declaration

Ethical Approval

Not Applicable

Competing interests

The authors declare that they have no known competing financial interests

Funding

The author declared no source of external funding

Availability of data and materials

Data would be made available on request**.**

Authors' Contribution

Each of the writers contributed equally to this study. Kolawole Ismail Adekunle was in charge of protocol development, statistical analysis, study design, and first draft manuscript drafting. Abubakar Yahaya, Sani Ibrahim Doguwa, and Aliyu Yakubu oversaw the data analysis and carried out the literature review. The final paper was reviewed and approved by all authors.