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Analyzing Market Price Equilibrium Dynamics with Differential Equations: Incorporating Government Intervention and Market Forces

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Abstract: *This study seeks to investigate price stability in a dynamic market, where prices are subject to sudden impacts akin to those observed during the Covid-19 lockdown in 2020, as well as other influences introduced naturally or by price regulatory agencies. By examining functions derived from price observations, changes in prices, and changes in the rate of price changes, the study analyzes their stability amidst various influences. These influences are incorporated by examining factors affecting supply and demand quantities, which are modelled using a second-order linear differential equation;* $P''(t) + a_1P'(t) + a_0P(t) = f(t)$. *This study builds upon the research of Espinoza and Bob Foster, who analyzed a second-order differential equation with a constant inhomogeneity. It employs matrix methods to assess the stability of systems of differential equations. To analyze impulsive price changes modelled using the Dirac delta function and persistent price changes modelled with Heaviside's unit step function, the Laplace technique and its general inversion formula are applied. The study identifies conditions under which stability in the system can be maintained.*

Keywords*:Analysis, differential equation, market price, equilibrium dynamics, interventions*

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1.0 Introduction

Market price equilibrium, the point where supply and demand meet, is a fundamental concept in economics. However, achieving and maintaining this equilibrium can be a complex process influenced by various factors. This manuscript explores the dynamics of market price equilibrium using differential equations, incorporating the impact of government intervention and market forces.

Recent economic literature highlights the limitations of static models in capturing the dynamic nature of market prices. Studies by [Agent-Based Modeling in Economics: A Growing Approach (2016) by Cars Hommes] and [A Differential Equation Approach to Macroeconomic Modeling (2023) by Michael

Woodford] emphasize the need for models that account for the time-dependent interactions between supply and demand. Differential equations provide a powerful tool to analyze these interactions, allowing us to explore how prices evolve over time. Government intervention plays a crucial role in shaping market dynamics. [The Theory of Industrial Organization (2020) by Jean Tirole] outlines various policy instruments that governments can use to influence market behavior. This manuscript incorporates these interventions, such as price controls, subsidies, and taxes, into the differential equation framework.

By analyzing the interaction between government policies and market forces, we can gain valuable insights into how market prices respond to different stimuli. This approach has been shown to enhance the prediction of the impact of policy changes. For instance, (Bandara in 1991,. computable general equilibrium modelS for development policy analysis in LDCs). utilizes a similar framework to simulate the impact of environmental regulations on market prices. Furthermore, the model can reveal conditions that might lead to market disequilibrium, allowing for proactive policy adjustments. This aligns with the work of [Agent-Based Policy Analysis in Dynamic Markets (2021) by Leigh Tesfatsion], where she demonstrates how agent-based models (which can be incorporated into differential equation frameworks) can be used to identify potential imbalances and design effective policy interventions. Finally, the framework can be adapted to analyze different market structures, such as monopolies or oligopolies, and their response to government interventions. This is particularly relevant in today's increasingly complex market environments, as highlighted by [Market Power and Competition Policy in the Digital Age (2023) by Ariel Ezrachi and Maurice Stucke].

Considering the above facts, this manuscript is aimed at contributing to the growing body of research that utilizes differential equations to understand market dynamics. By incorporating government intervention and market forces, it provides a comprehensive framework for analyzing price equilibrium and its stability in a constantly evolving economic environment.

Differential equations play multifaceted roles in economics. They are crucial for establishing dynamic stability conditions in microeconomic models of market equilibriums and for tracking growth trajectories under diverse macroeconomic scenarios, as noted by Dowling (2001). They empower economists to derive functions describing growth rates, calculate point elasticity, and estimate demand functions. Additionally, they facilitate the estimation of capital functions from investment functions, as well as deriving total cost and revenue functions from marginal cost and revenue functions.

In the case when quantity demanded Q_d depends on price P only and quantity supplied Q_s depends only on price P**,** the relationships are captured by the equations;

$$Q_d = c + bP \quad (1)$$

$$Q_s = g + hP \quad (2)$$

where c, b, g and h are constants.

The equilibrium price is obtained when $Q_d = Q_s$ in such a situation, the equilibrium price is;

$$\bar{P} = \frac{c-g}{h-b}. \quad (3)$$

If both Q_d and Q_s depend on price, P and change in price, $\frac{dp}{dt}$ then we have

$$Q_s = c_1 + w_1 P + r_1 P' \quad \text{and} \quad Q_d = c_2 + w_2 P + r_2 P' \quad (4)$$

where c_i, w_i and r_i , $i = 1,2$ are constants.

In markets influenced by current prices and price trends (whether prices are rising or falling, and at what rate), economists require knowledge of several key variables. These include the current price $P(t)$**,** the first derivative representing the rate of change of price concerning time ($\frac{dP}{dt}$), and the second derivative, which indicates the rate of change of the rate of change of price,($\frac{d^2P}{dt^2}$). These variables are essential for analyzing supply and demand dynamics and for understanding market behaviours in response to price movements.

In this case, the formulae for Q_s *and* Q_d become

$Q_s = c_1 + c_2P + c_3P' + c_4P''$ *and* $Q_d = b_1 + b_2P + b_3P' + b_4P''$ (5)

for supply and demand respectively, where c_i, b_i, $i = 1,2,3,4$ are constants.

1.1 Dynamic Equilibrium

An equilibrium condition is attained when Q_d *and* Q_s, are equal implying that

$$c_1 + c_2P + c_3P' + c_4P'' = b_1 + b_2P + b_3P' + b_4P'' \quad (6)$$

from which with the following definition

$$a = (c_2 - b_2), b = (c_3 - b_3), c = (c_4 - b_4) \text{ and } d = (b_1 - c_1) \quad (7)$$

the equilibrium equation becomes

$$aP + bP' + cP'' = d \quad (8)$$

It's important to recognize that variables such as a, b, c *and* d can either be independent of time or functions dependent on various factors. These factors encompass public perception, the volume of money (denoted as V_m), the volume of credit (denoted as V_c), and other relevant considerations. Price itself can be a variable influenced by the relationship between the volume of money V_m, the volume of credit V_c and the availability of goods and services in the market. These dynamics illustrate the complex interplay of economic variables in determining market prices.

Following the Cobb-Douglass form, the price model in then is given as

$$p(t) = \gamma V_m^{\alpha} V_c^{\beta} P(t) \quad (9)$$

where α, β, γ are constants that may depend on the availability of goods and services and $p(t)$ being price change. This dependency makes the price volatile.

Considering the price change equation:

$$\lambda P + \mu P' + \sigma P'' = w \quad (10)$$

and applying equation (9) to equation (10), we get an equation of the form;

$$\lambda\gamma V_m^{\alpha} V_c^{\beta} P(t) + \mu\gamma V_m^{\alpha} V_c^{\beta} P'(t) + \sigma\gamma V_m^{\alpha} V_c^{\beta} P''(t) = w \quad (11)$$

which when compared to equation (8) gives the following

$$a = \lambda\gamma V_m^{\alpha} V_c^{\beta}, b = \mu\gamma V_m^{\alpha} V_c^{\beta}, c = \sigma\gamma V_m^{\alpha} V_c^{\beta}, w = -d \quad (12)$$

The volatility of price can be idealized as synchronous with that of certain springs. It is, reasonable to compare equation (8) to the general second-order linear ordinary differential equation of the type governing the motion of a mass of a spring given by

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = f(t) \quad (13)$$

where $f(t)$ is a prescribed function that influences motion. If $f(t) = 0$, the spring may move due to a slight disturbance of its weight.

Synchronizing our terms with those used in spring motion, we get

$$P + \frac{b}{a}P' + \frac{c}{a}P'' = \frac{d}{a}$$

which modifies to

$$P + a_1P' + a_0P'' = c_0. \quad (14)$$

with $a_1 = \frac{b}{a}, a_0 = \frac{c}{a}, c_0 = \frac{d}{a}$.

Equation (14) is the type derived by Espinoza (2009) and studied by Bob and Foster (2016). Our aim is to extend the above to an equation of the form

$$P''(t) + a_1P'(t) + a_0P(t) = f(t) \quad (15)$$

subject to initial conditions

$$P(0) = P_0 \text{ and } P'(0) = P_0' \quad (16)$$

The equation above aligns with the dynamics of linear spring motion, drawing parallels between economic concepts and physical phenomena:

In this analogy, the price $P(t)$ corresponds to the displacement of the spring, indicating its current position. The first derivative $P'(t)$, representing the change in price over time corresponds to the velocity of the spring's motion.

The second derivative $P''(t)$, which denotes the change in the rate of change of price, mirrors the need to resist or dampen price changes. This resistance can be likened to damping forces in spring motion and is analogous to regulatory measures implemented by authorities to stabilize price fluctuations.

The term $a_0P(t)$ in the equation mirrors the simplest form of Hooke's Law in physics, where $a_0$represents a constant capturing of the restoring force or internal mechanisms within the market that resist price changes. This constant reflects market conditions and factors that naturally oppose abrupt changes in price levels.

Equation (15) can be used to model various phenomena such as

(a) Spring volatility (b) Electrical circuit (c) Buoyancy (floating) (d) Price equilibrium, for which in each model the constant $a_1 and\ a_0$ have special significance and interpretation. But in our case, it is concerning concerning price equilibrium and reduces to equation (14) when $\mathrm{f}(t) = c_0$. So equation (15) is an extension of the works of Espinoza (2006) and Bob and Foster (2016), where $P(t)$ is the price, $P'(t)$is the change in price and so $\mathrm{P}'(0)$ is an initial change in price, a_1 is called friction or damping coefficient and in general, we require that $a_o \neq 0$.

The volatility is classified as the following

(a) Free volatility if $f(t) = 0$
(b) Forced volatility if $f(t) \neq 0$
(c) Undamped volatility if $a_1 = 0$
(d) Damped volatility if $a_1 \neq 0$
(e) Free but damped volatility if $f(t) = 0\ and\ a_1 \neq 0$
(f) Free and undamped volatility if $f(t) = 0\ and\ a_1 = 0$
(g) Forced and damped v volatility if $f(t) \neq 0\ and\ a_1 \neq 0$
(h) Forced and undamped volatility if $f(t) \neq 0 and\ a_1 = 0\ if\ a_0 \neq 0$

If $a_0 \neq 0\ and\ a_1 \neq 0$ the characteristics roots of equation (15) are obtained by setting

$$\mathrm{P} = e^{\lambda t} \tag{17}$$

to get

$$P'(t) = \lambda e^{\lambda t}, and\ P''(t) = \lambda^2 e^{\lambda t}\ . \tag{18}$$

Applying (18) to (19)

$$P''(t) + a_1P'(t) + a_0P(t) = 0 \tag{19}$$

leads to the quadratic equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{20}$$

The roots of equation (20) are

$$\lambda_1 = \frac{1}{2}\left(-a_1 + \sqrt{a_1^2 - 4a_0}\right) \tag{21}$$

and

$$\lambda_2 = \frac{1}{2}\left(-a_1 - \sqrt{a_1^2 - 4a_0}\right). \tag{22}$$

Let

$$\Delta = a_1^2 - 4a_0, \tag{23}$$

then we have three cases of the characteristics equation that are

Case I: $\Delta > 0$

The roots are distinct $(\lambda_1 \neq \lambda_2)$

Case II: $\Delta = 0$

The roots are repeated or coincide $(\lambda_1 = \lambda_2)$ Consequently, the equation (20) is a perfect square

Case III: $\Delta < 0$

The roots are distinct complex conjugates therefore

(i) Distinct roots lead to the case called damped volatility
(j) Coincident roots are called critically damped volatility
(k) Complex conjugate roots are called oscillatory damped volatility

The solution to equation (15) under the conditions outlined in equation (16), along with its mechanical interpretations, can be explored through Boyce and Daprima (1977). Asymptotic stability in systems and methods for solving them have been extensively studied:

Hovhannisyan (2004) demonstrated asymptotic stability for second-order linear differential equations using techniques involving asymptotic representations of solutions and error estimates.

Carauste (2011) focused on local asymptotic stability analysis for mathematical models of hematopoietic with delay, emphasizing models where coefficients depend on delays (delay-dependent coefficients).

Stability analysis of hematopoietic mathematical models, which involve differential equations with delay, requires identifying eigenvalues of characteristic equations typically represented as exponential polynomial functions with delay-dependent coefficients. This analysis is more intricate compared to standard differential equations; as it necessitates determining conditions under which all eigenvalues have negative real parts.

Three models of increasing complexity are typically considered, each requiring specific tools and methods: The primary approach involves reducing the stability analysis problem to find the roots of a real function. These roots indicate critical values of the delay where stability may transition.

This method offers advantages in systematically pinpointing stability thresholds but comes with its limitations, particularly in managing the complexity introduced by delay-dependent coefficients and higher-order differential equations.

Understanding these methodologies is crucial for effectively analyzing and predicting stability in dynamic systems governed by differential equations with delays, such as those encountered in hematopoietic models.

Fadali, (2009) showed that in the absence of pole-zero cancellation, an LTI digital system is asymptotically stable if their transfer function poles are in the open unit disc and marginally stable if the poles are in a closed unit disc with no repeated poles on the unit circle

Rily, Hobson, and Bence, (2002) posited that the δ-function is different from functions encountered in the physical sciences and can be used to discuss rigorous mathematical situations. They held that δ-function can be visualized as a very sharp narrow pulse (in space, time, density, etc) which produces an integrated effect of definite magnitude.

For many practical purposes, effects which are not strictly described by a δ-function may be analyzed by the use of the delta function, if they take place in an interval much shorter than the response interval of the system on which they act. For example, the idealized notion of an instantaneous impulse of magnitude T applied at time c can be represented by

$f(t) = \mathrm{T}\delta(\mathrm{t}-\mathrm{c}).$

Many physical situations are described by a δ-function in space rather than in time. Formulas have been developed to check the asymptotic stability for general linear second-order non-homogeneous differential equations. These formulas have shown their effectiveness in scenarios such as a spring moving freely and a spring influenced by an impulse force.

In this project, we proved the asymptotic stability for a system of two linear first-order homogeneous and non-homogeneous ordinary differential equations. We used an impulsive system with impulse magnitude T to demonstrate how the proposed method works.

2.0 Some Preliminary Results

In this segment, we briefly discuss the Laplace transform, Cramer's rule, Dirac function and Heaviside's step function which give results about the stability of linear systems. We shall also apply known results on stability from the theory of matrices.

A system involving two linear homogenous first-order ordinary differential equations can be written in matrix form as:

$$Y' = AY, Y(0) = Y_0 \,, \tag{24}$$

where

$$Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \,,\; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,,\; Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}. \tag{25}$$

Equation (24) using equation (25) can be written separately as

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t), \quad y_1(0) = y_{01} \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t), \quad y_2(0) = y_{02} \end{aligned}\right\}. \tag{26}$$

Also a system of two linear non-homogenous first order differential equations is written as

$$Y' = AY + F, \quad Y(0) = Y_0 \,, \tag{27}$$

where,

$$Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \,,\; A = \begin{pmatrix} - & b \\ c & d \end{pmatrix} \,,\; F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tag{28}$$

which can be written in a system form as

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t) + f_1(t), \quad y_1(0) = y_{01} \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t) + f_2(t), \quad y_2(0) = y_{02} \end{aligned}\right\} \tag{29}$$

We intend to study and investigate the conditions for the dynamic market price equilibrium of equation (15) with the initial condition, equation (16) by converting equation (15) to matrix form and evaluating the eigenvalues of the resultant matrix.

Let the coefficient matrix of equation (15) be $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

We now give the following matrix theoretic conditions for the system with coefficient matrix A to be stable:

The Laplace transforms

The Laplace transform of a function $f(t)$ defined for all real numbers $t \geq 0$, is the function, $\bar{f}(s)$, which is defined by:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s) \tag{30}$$

where **s** is the transform parameter. That is,

$$L\{y_1(t)\} = \int_0^\infty e^{-st} y_1(t)dt = \bar{y}_1(s) \,, \tag{31}$$

also,

$$L\{f_2(t)\} = \int_0^\infty e^{-st} f_2(t)dt = \bar{f}_2(s) \tag{32}$$

Properties of Laplace Transform

1. The Laplace transform of the derivative of $y_1(t)$ is given by

$$L\{y'_1(t)\} = sL\{y_1(t)\} - y_1(0) = s\bar{y}_1(s) - y_1(0). \tag{33}$$

2. Inversion formula for Laplace Transform

Once the Laplace transform of $f(t)$ is derived as $\bar{f}(s)$, $f(t)$ is recovered by use of the inversion formula for the Laplace transform given

$f(t) = \frac{1}{2\pi i}\int_{\sigma-\infty}^{\sigma+i\infty} \bar{f}(s)e^{st}\,ds,$ (34)

where σ the radius of the circle is exclusively centred at and enclosing a pole of $\bar{f}(s)$ so that no other pole of $\bar{f}(s)$ is within the circle, Dass (2016).

(b) Dirac Delta Function (The unit impulse)

The Dirac delta function ($\delta\ function$) is a generalized function or distribution introduced by the physicist, Paul Dirac. It is used to model the density of an idealized point mass or point charge as a function equal to zero everywhere except for zero where it is infinite and whose integral over the entire real line is equal to one.

If $f(t)$ represents a function then the effect of Dirac delta function $\delta(t)$ on f(t)at a point c is defined by the integral

$\int_{-\infty}^{\infty} f(t)\delta(t-c)dt = f(c), \quad (f(t)\ at\ t = c)$ (35)

This is the fundamental property of the delta function.

The Dirac delta function can be defined as a piece-wise function in the form

$\delta(t-c) = \begin{cases} 0, t \neq c \\ \infty, t = c \end{cases}$ (36)

From the definition and (15), if $f(t) = 1, then\ f(c) = 1$ and

$\int_{-\infty}^{\infty} \delta(t-c)dt = 1$ for all c such that $-\infty < c < \infty$. (37)

Laplace Transform of $\boldsymbol{\delta(t-c)}$

If $p < c < q$, then

$\int_{p}^{q} f(t)\delta(t-c)dt = f(c)$ provided $p < c < q$, (38)

therefore, if $p = 0\ and\ q = \infty$ then

$\int_{0}^{\infty} f(t).\delta(t-c)dt = f(c)\ (0 < c < \infty)$, (39)

hence, if $f(t) = e^{-st}$, $f(c) = e^{-cs}$

which leads to

$\int_{0}^{\infty} e^{-st}.\delta(t-c)dt = \mathcal{L}\{\delta(t-c)\} = e^{-sc}$. (40)

For $c = 0$, equation (40) becomes

$\int_{0}^{\infty} e^{-st}\delta(t)dt = \mathcal{L}\{\delta(t)\} = e^{0} = 1$.

So, the Laplace transform of the delta function $\delta(t)$ is 1. That is $\mathcal{L}\{\delta(t)\} = 1$.

For the general case of $\mathcal{L}\{f(t).\delta(t-c)\}$. we have

$\mathcal{L}\{f(t).\delta(t-c)\} = \int_{0}^{\infty} e^{-st}f(t)\delta(t-c)dt$

$= f(c)e^{-st}$ (41)

Definition 1: A function $f(t)$ is called an impulsive function of magnitude T, if it can be written in terms of Dirac delta function as

$f(t) = T\delta(t-c)$. $T\delta(t-c)$ represents an impulse or a force of possibly large magnitude, T that acts over a short time, $t = c$

The Laplace transform of this impulsive function $f(t) = T\delta(t-c)$

is given by

$\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st}T\_(t-c)dt$

$= \mathrm{T}\int_{0}^{\infty} \mathrm{e}^{-\mathrm{st}}\,\delta(\mathrm{t}-\mathrm{c})\mathrm{dt} = Te^{-cs}$.

That is,

$\bar{f}(s) = Te^{-cs}$ where $\bar{f}(s) = \overline{T\delta(t-c)}$. (42)

(c) Heaviside's Unit Step Function

Heaviside's unit step function will be denoted by $H_c(t)$ (it is denoted $H(t-c)$ in some works) and defined by

$$H_c(t) = \begin{cases} 0, & t < c \qquad t \in (-\infty, c) \\ & \qquad\qquad c \geq 0 \\ 1, & t \geq c \qquad t \in [c, \infty] \end{cases} \tag{43}$$

Hence,

$$\lim_{t\to\infty} H_c(t) = 1, and \quad \lim_{t\to-\infty} H_c(t) = 0$$

Note that $H_c(t)$ is discontinuous at $t = c$ because $\lim_{t\to c^+} H_c(t) \neq \lim_{t\to c^-} H_c(t)$

The unit step function is very helpful in dealing with functions having jump discontinuities.

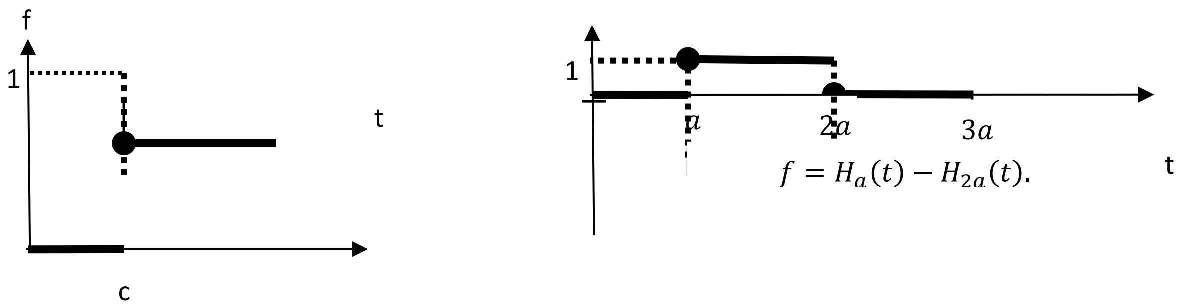

Fig 1. $f = H_c(t)$ (b) square wave between a and $2a$

The unit step function can be used as a building block in the construction of other functions. Suppose Q is a constant that persists for time, $t > c$ then this situation can be expressed as

$$f(t) = QH_c(t)\,, \tag{44}$$

therefore

$$f(t) = QH_c(t) = \begin{cases} 0, & t < c \\ Q, & t \geq c \end{cases} \,. \tag{45}$$

The Laplace transform of $f(t)$ is given by

$$L\{f(t)\} = L\{QH_c(t)\}$$

$$= \int_0^\infty e^{-st}QH_c(t)dt = \bar{f}(s) = Q\int_0^\infty e^{-st}H_c(t)dt$$

$$= Q\int_0^c e^{-st}H_c(t)dt + \int_0^\infty e^{-st}H_c(t)dt = Q\int_0^\infty e^{-st}dt$$

$$= -\frac{Q}{s}e^{-st}\Big|_{t=c}^{t=\infty}$$

$$= \frac{Q}{s}e^{-sc},\ s > 0 \tag{46}$$

Relationship between Dirac delta function and Heaviside's step function

Theorem 1 (Espinoza, 2009)

The Heaviside's function $H_0(t)$ is related to the Dirac delta function $\delta(t-0) = \delta(t)$ through

$$\frac{d}{dt}H_0(t) = \delta(t). \tag{47}$$

Espinoza, 2009.

If $H_c$is the Heaviside's unit step function and $\delta(t)$ is the Dirac delta function, then

$H_c(t) = \int_{-\infty}^{t} \delta(w - c)dt, \quad c \geq 0 .$ (48)

The interval of persistence of a constant Q

Theorem 2 (Dowling, 2001)

The constant Q persist in an interval $(c, \alpha c)$, where $1 < \alpha < \infty$, if we define

$h(t) = H_c(t) - H_{\alpha c}(t)$ (49)

then

$Qh(t) = Q \quad for\ t \in (c, \alpha c).$

That is Q persists for $c \leq t \leq \alpha c$

Lemma 1

The Laplace transform of

$Qh(t),\ c \leq t < \alpha c,\ \alpha > 1\ ,\quad c > 0,$

is given by

$\int_0^\infty e^{-st} Qh(t)dt = -\frac{Q}{s}[e^{-\alpha cs} - e^{-cs}],\ \alpha > 1, c > 0, s > 0 .$ (50)

III. General Matrix Theory for Stability of Linear Systems

In this section, a matrix theoretic approach to the stability of linear systems will be applied to get the solutions of

(i) the homogenous system $Y' = AY$ and

(ii) the non-homogenous system $Y' = AY + F$ in terms of the eigenvalues, components of A and the initial conditions only.

1. The Homogeneous Case

Theorem 3 (Boyce and Diprima (1977))

We state without proof the main stability theorem for autonomous systems represented by matrices;

The critical point (0, 0) of the linear system

$\frac{dx}{dt} = ax + by,\ \frac{dy}{dt} = cx + dy$ (a, b, c and d are constants)

which in matrix form is: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

The system is

1) asymptotically stable if the roots $\lambda_1, \lambda_2$of the characteristics equation $\lambda^2 - (a + b)\lambda + ad - bc = 0$ are real and negative or have negative real parts.
2) stable, but not asymptotically stable, if $\lambda_1 and\ \lambda_2$ are pure imaginary
3) unstable if $\lambda_1 and\ \lambda_2$ are real and either positive or if they have positive real parts.

Theorem 4

Let

$Y' = AY$

be a system of two homogenous first-order ordinary differential equations with

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad - bc \neq 0, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},\quad Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}$

where a, b, c, and d are constants. That is,

$\frac{dy_1(t)}{dt} = ay_1(-) + by_2(t)$

$\frac{dy_2(t)}{dt} = cy_1(t) + dy_2(t)$

whose distinct eigenvalues are $\lambda_1 \; and \; \lambda_2$ and initial values are

$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$

then,

$$y_1(t) = \frac{[(\lambda_1 - d)y_{01} + by_{02}]e^{\lambda_1 t}}{\lambda_1 - \lambda_2} + \frac{[(\lambda_2 - d)y_{01} + by_{02}]e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \quad \lambda_1 \neq \lambda_2 \tag{51}$$

$$y_2(t) = \frac{[(\lambda_1 - a)y_{02} + cy_{01}]e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{[(\lambda_2 - a)y_{02} + cy_{01}]e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \quad \lambda_1 \neq \lambda_2 \tag{52}$$

hence, if $\lambda_1 < 0 \; and \; \lambda_2 < 0 \lambda_1 \neq \lambda_2$, then

$\lim_{t\to\infty} y_1(t) = 0$ and $\lim_{t\to\infty} y_2(t) = 0$.

In this case, the system is asymptotically stable.

Using the inverse Laplace transform of $y_j(t)$ defined in (34) and by the Bromwich integral

$$y_j(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \bar{y}_j(s) e^{st} \text{ds}. \tag{53}$$

The application of Cauchy's residue theorem gives

$y_j(t) = sum \; of \; residues \; of \; \bar{y}_j(s)e^{st}$ at the poles of $\bar{y}_j(s)$ such that for $j = 1$;

$$y_1(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \bar{y}_1(s) e^{st} \text{ds} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{(s-d)y_{01} + by_{02}}{(s-\lambda_1)(s-\lambda_2)} e^{st} \text{ds}.$$

To use Cauchy's residue theorem to evaluate the Bromwich integral, we note that the integrand has simple poles at $s = \lambda_1 \; and \; s = \lambda_2$.

Theorem 5: Cauchy's residue theorem (Rade and Westergreen (2004))

At $s = k$ the general residue formula is given by the following equation:

For a pole of order m:

$$\text{Res}\left(s = k, \bar{f}(s)\right) = \lim_{s\to k} \frac{1}{(m-1)!}\left(\frac{d}{ds}\right)^{m-1}\left\{(s-k)^m \bar{f}(s)\right\}, \tag{54}$$

where $Res(s = k, \bar{f}(s))$ means residue of $f(s)$ at the pole $s = k$,

therefore at $s = \lambda_1$

$$Res\left(s = \lambda_1, \bar{y}_1(s)\right) = \lim_{s\to\lambda_1} (s - \lambda_1)\left[\frac{(s-d)y_{01} + by_{02}}{(s-\lambda_1)(s-\lambda_2)}\right] e^{st}$$

$$= \left[\frac{(\lambda_1 - d)y_{01} + by_{02}}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t}.$$

At $s = \lambda_2$

$$Res\left(s = \lambda_2, \bar{y}_1(s)\right) = \lim_{s\to\lambda_2} (s - \lambda_2)\left[\frac{(s-d)y_{01} + by_{02}}{(s-\lambda_1)(s-\lambda_2)}\right] e^{st}$$

$$= \left[\frac{(\lambda_2 - d)y_{01} + by_{02}}{(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t},$$

hence, $y_1(t)$ becomes,

$$y_1(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(s-d)y_{01} + by_{02}}{(s-\lambda_1)(s-\lambda_2)} e^{st} \text{ds} = Res\left(s = \lambda_1, \bar{y}_1(s)\right) + Res\left(s = \lambda_2, \bar{y}_2(s)\right),$$

Which modifies to

$$y_1(t) = \left[\frac{(\lambda_1 - d)y_{01} + by_{02}}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} + \left[\frac{(\lambda_2 - d)y_{01} + by_{02}}{(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t} \quad , \lambda_1 \neq \lambda_2 \, . \tag{55}$$

For $j = 2$

$$y_2(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \bar{y}_2(s) e^{st} \text{ds} = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(s-a)y_{02} + cy_{01}}{(s-\lambda_1)(s-\lambda_2)} e^{st} \text{ds}.$$

Cauchy's Residue theory is used to evaluate the contour integral, whose integrand has simple poles at $s = \lambda_1 \ and \ s = \lambda_2$.

At $s = \lambda_1$

$$Res\big(s = \lambda_1, \bar{y}_2(s)\big) = \lim_{s\to\lambda_1}(s-\lambda_2)\left[\frac{(s-a)y_{01}+cy_{02}}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \lim_{s\to\lambda_1}\left[\frac{(s-a)y_{01}+cy_{02}}{(s-\lambda_2)}\right]e^{st}$$

$$= \left[\frac{(\lambda_1-a)y_{01}+cy_{02}}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t}$$

At $s = \lambda_2$

$$Res\big(s = \lambda_2, \bar{y}_2(s)\big) = \lim_{s\to\lambda_2}(s-\lambda_2)\left[\frac{(s-a)y_{01}+cy_{02}}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \lim_{s\to\lambda_2}\left[\frac{(s-a)y_{01}+cy_{02}}{(s-\lambda_1)}\right]e^{st}$$

$$= \left[\frac{(\lambda_2-a)y_{01}+cy_{02}}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t}\ .$$

Hence, $y_2(t)$ becomes,

$$y_2(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{(s-a)y_{01}+cy_{02}}{(s-\lambda_1)(s-\lambda_2)}e^{st}\mathrm{ds} = Res\big(s=\lambda_1,\bar{y}_2(s)\big) + Res\big(s=\lambda_2,\bar{y}_2(s)\big)$$

from which we have

$$y_2(t) = \left(\frac{(\lambda_1-a)y_{01}+cy_{02}}{\lambda_1-\lambda_2}\right)e^{\lambda_1 t} + \left(\frac{(\lambda_2-a)y_{01}+cy_{02}}{\lambda_2-\lambda_1}\right)e^{\lambda_2 t} \qquad (56)$$

The Non-Homogeneous Case

Theorem 6

Let $Y' = AY + F$

be a system of two autonomous non-homogenous first-order differential equations with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}, F = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix},$$

and a, b, c, d are constants, then, the expansion of $Y' = AY + F$ yields;

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t) + f_1(t) \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t) + f_2(t) \end{aligned}\right\}. \qquad (57)$$

If $\lambda_1 \ and \ \lambda_2$ are distinct eigenvalues of A, then

$$\left.\begin{aligned} y_1(t) = &\left[\frac{y_{01}(\lambda_1-d)+by_{02}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_1(\lambda_1)(\lambda_1-d)+b\bar{f}_2(\lambda_1)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t} + \\ &\left[\frac{y_{01}(\lambda_2-d)+by_{02}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_1(\lambda_1)(\lambda_2-d)+b\bar{f}_2(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} \\ &\qquad\qquad and \\ y_2(t) = &\left[\frac{y_{02}(\lambda_1-d)+cy_{02}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_1(\lambda_1)(\lambda_1-d)+c\bar{f}_2(\lambda_1)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t} + \\ &\left[\frac{y_{01}(\lambda_2-d)+by_{02}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_1(\lambda_1)(\lambda_2-d)+b\bar{f}_2(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} \end{aligned}\right\}. \qquad (58)$$

Hence, if $\lambda_1 < 0 \ and \ \lambda_2 < 0$, $(\lambda_1 \neq \lambda_2)$ then,

$\lim_{t\to\infty} y_1(t) = 0 \ and \ \lim_{t\to\infty} y_2(t) = 0$.

In this case the system is asymptotically stable.

For the inversion formula for the Laplace transform of $y_j(t)$ define by

$$y_j(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{y}_j(s)e^{st}\mathrm{ds}\ ,$$

we have the following;

for $j = 1$

$$y_1(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}\text{ds} .$$

To use Cauchy's residue theorem to evaluate the Bromwich integral, whose integrand has simple poles at, $s = \lambda_1$ *and* $s = \lambda_2$; we note that at, $s = \lambda_1$:

If $\overline{f}_1(s)$ *and* $\overline{f}_2(s)$are continuous in their domains, then

$$Res\big(s = \lambda_1, \bar{y}_1(s)\big) = \lim_{s\to\lambda_1}(s-\lambda_1)\left[\frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \left[\frac{y_{01}(\lambda_1-d)+by_{02}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_1(\lambda_2)(\lambda_2-d)+b\bar{f}_2(\lambda_2)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t}, \tag{59}$$

also at, $s = \lambda_2$

$$Res\big(s = \lambda_2, \bar{y}_1(s)\big) = \lim_{s\to\lambda_2}(s-\lambda_2)\left[\frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \left[\frac{y_{01}(\lambda_2-d)+by_{02}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_1(\lambda_2)(\lambda_2-d)+b\bar{f}_2(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t}. \tag{60}$$

That

$$y_1(t) = Res\big(s = \lambda_2, \bar{y}_1(s)\big) + Res(s = \lambda_2, \bar{y}_1(s))$$

implies

$$y_1(t) = \left[\frac{y_{01}(\lambda_1-d)+by_{02}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_1(\lambda_1)(\lambda_1-d)+b\bar{f}_2(\lambda_1)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t} +$$

$$\left[\frac{y_{01}(D_2-d)+by_{02}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_1(\lambda_2)(\lambda_2-d)+b\bar{f}_2(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t}\lambda_1 \neq \lambda_2 \tag{61}$$

For $j = 2$, we have

$$y_2(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\bar{y}_2(s)e^{st}\text{ds}$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\frac{y_{02}(s-a)+cy_{01}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_2(s)(s-a)+c\bar{f}_1(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}\text{ds} .$$

Here the integrand has simple poles at $s = \lambda_1$ *and* $s = \lambda_2$.

Using Cauchy's Residue theorem at $s = \lambda_1$, with $\overline{f}_1(s)$ *and* $\overline{f}_2(s)$continuous, we have

$$Res\big(s = \lambda_1, \bar{y}_2(s)\big) = \lim_{s\to\lambda_1}(s-\lambda_1)\left[\frac{y_{02}(s-a)+cy_{01}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_2(s)(s-a)+c\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \left[\frac{y_{02}(\lambda_1-a)+cy_{02}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_2(\lambda_1)(\lambda_2-a)+c\bar{f}_1(\lambda_1)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t}\lambda_1 \neq \lambda_2 . \tag{62}$$

At $s = \lambda_2$,

$$Res\big(s = \lambda_2, \bar{y}_2(s)\big) = \lim_{s\to t_2}(s-\lambda_2)\left[\frac{y_{02}(s-a)+cy_{01}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_2(s)(s-a)+c\bar{f}_1(s)}{(s-\lambda_1)(s-\lambda_2)}\right]e^{st}$$

$$= \left[\frac{y_{02}(\lambda_2-a)+cy_{02}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_2(\lambda_2)(\lambda_2-a)+c\bar{f}_1(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t}$$

thus,

$$y_2(t) = Res\big(s = \lambda_1, \bar{y}_1(s)\big) + Res(s = \lambda_2, \bar{y}_1(s)) ,$$

which gives

$$y_2(t) = \left[\frac{y_{02}(\lambda_1-a)+cy_{01}}{(\lambda_1-\lambda_2)} + \frac{\bar{f}_2(\lambda_1)(\lambda_1-a)+c\bar{f}_1(\lambda_1)}{(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t} +$$

$$\left[\frac{y_{02}(\lambda_2-a)+cy_{01}}{(\lambda_2-\lambda_1)} + \frac{\bar{f}_2(\lambda_2)(\lambda_2-a)+c\bar{f}_1(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t}\ ,\lambda_1 \neq \lambda_2. \tag{63}$$

When $\overline{f}_1(s)$ *and* $\overline{f}_2(s)$ are not continuous their poles are incorporated to get

$$y_1(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\left(\frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)}\right)e^{st}\mathrm{ds}+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right)e^{st}\mathrm{ds}\right]$$
$$=\frac{y_{01}(\lambda_1-d)+by_{02}}{\lambda_1-\lambda_2}e^{\lambda_1 t}+\frac{y_{02}(\lambda_2-d)+by_{02}}{\lambda_2-\lambda_1}e^{\lambda_2 t}+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}\right)e^{st}\mathrm{ds}. \quad (64)$$

and

$$y_2(t)=\frac{y_{02}(\lambda_1-a)+cy_{01}}{\lambda_1-\lambda_2}e^{\lambda_1 t}+\frac{y_{02}(\lambda_1-a)+cy_{01}}{\lambda_2-\lambda_1}e^{\lambda_2 t}+\frac{1}{2\pi i}\int_{c-i\infty}^{c\pm i\infty}\left(\frac{\bar{f}_2(s)(s-a)+c\bar{f}_1(s)}{(s-\lambda_1)(s-\lambda_2)}\right)e^{st}\mathrm{ds}. \quad (65)$$

IV. Results in Closed Form and Their Analysis

In this section, the results obtained are presented and analyzed.

Basic Equations

Let $P(t)$ be the price of a commodity. We have remarked that the works of Espinoza (2009) and Bob Foster (2016) showed that the model equation that captures a relation between price $P(t)$, change in price, $P'(t)$ and change in the change in price, $P''(t)$ is of the form

$$P''(t)+a_1P'(t)+a_0P(t)=0.$$

Their solutions and techniques are similar to those of dynamic spring-mass mechanical systems. The constant a_0 cannot be zero in the equation because it represents restoring market forces. It is the aggregate of dynamic market forces that resist the collapse of price. It is therefore pertinent to liken it to Hooke's Law of spring-mass elastic mechanical systems where price, $P(t)$ agrees with displacement.

Hooke's law simply states that "force is proportional to elongation" in elastic materials. Let $g_1(t)$ be the applied force, then the Hooke's law says

$$g_1(t)=-kP(t)$$

where $P(t)$ is the price due to market forces and k is a constant. The second derivative, $P''(t)$, which Esponoza (2009) and Bob Foster (2016) called a change in the change in price, is similar to the inertia effect determined by Newton's law of motion given by

$$g_2(t)=m\frac{d^2P}{dt^2}$$

where m is mass and is constant.

Equating the forces, we have

$$\frac{d^2P}{dt^2}+\frac{k}{m}P=0 \quad (66)$$

If the price is modelled by (66) then it will oscillate with known amplitude and frequency. To check the oscillation of price, regulatory government policies may be enacted in the form of a check in the change in price given as

$$g_3(t)=a_1\frac{dp}{dt}.$$

In dynamic mechanical systems a_1 maybe a form of resistance that depends on external factors. The idea of introducing a_1 is to slow the price change down.

$g_3(t)$ is often called damping force and a_1 is called the damping constant. When $a_1\frac{dp}{dt}$ is added to equation (66) the equation of price change becomes the homogeneous equation.

$P''(t)+a_1P'(t)+a_0P(t)=0,$ subject to $P(0)=\mu\ and\ P'(0)=\xi$, where $a_0=\frac{k}{m}$.

Next, we assume that the price system is also subject to external forces denoted by $\mathrm{f}(t)$. Our analysis will then be based on the solution of the non-homogeneous second-order linear differential equation given in (18) as

$$P''(t)+a_1P'(t)+a_0P(t)=f(t)$$
$$P(0)=\mu\ and\ P'(0)=\xi.$$

To obtain the solution of the system that will guarantee the stability of price, matrix methods as discussed earlier and assume that $F(t)$ is in the form

$$f(t)=T\delta(t-c)+Q[H_c(t)-H_{\alpha c}(t)]+R[H_c(t)+H_{2c}(t)+H_{3c}(t)]. \quad (67)$$

The first term on the right of equation (67) is an impulse market force of magnitude T comparable to what happened to prices during the universal coronavirus lockdown in 202

The Homogeneous Case

Theorem 7

Let

$Y' = AY$

be a system of two homogenous first-order ordinary differential equations with

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc \neq 0, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix},$

where a, b, c, and d are constants,

alternatively

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t) \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t) \end{aligned}\right\}, \tag{68}$$

whose distinct eigen-values are λ_1 *and* λ_2 and initial values are

$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$

then

$$\left.\begin{aligned} y_1(t) &= \frac{[(\lambda_1 - d)y_{01} + by_{02}]e^{\lambda_1 t}}{\lambda_1 - \lambda_2} + \frac{[(\lambda_2 - d)y_{01} + by_{02}]e^{\lambda_2 t}}{\lambda_2 - \lambda_1}; \\ y_2(t) &= \frac{[(\lambda_1 - a)y_{02} + cy_{01}]e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{[(\lambda_2 - a)y_{02} + cy_{01}]e^{\lambda_2 t}}{D_2 - \lambda_1},, \lambda_1 \neq \lambda_2 \end{aligned}\right\}, \tag{69}$$

hence, if $\lambda_1 < 0$ *and* $\lambda_2 < 0 \lambda_1 \neq \lambda_2$

then

$$\left.\begin{aligned} &\lim_{t\to\infty} y_1(t) = 0 \\ &\quad and \\ &\lim_{t\to\infty} y_2(t) = 0. \end{aligned}\right\}. \tag{70}$$

In this case, the system is asymptotically stable.

Proof of Theorem 7

Taking Laplace transform of both sides of the system (68), we have

$$\left.\begin{aligned} \mathcal{L}\{y_1'(t)\} &= \mathcal{L}\{ay_1(t) + by_2(t)\} \\ \mathcal{L}\{y_2'(t)\} &= \mathcal{L}\{cy_1(t) + dy_2(t)\} \end{aligned}\right\}, \tag{71}$$

by (68), we have;

$s\bar{y}_1(s) - y_1(0) = a\bar{y}_1(s) + b\bar{y}_2(s)$

$s\bar{y}_2(s) - y_2(0) = c\bar{y}_1(s) + d\bar{y}_2(s).$

The application of the initial conditions,

$y_1(0) = y_{01}$ *and* $y_2(0) = y_{02},$

Gave

$s\bar{y}_1(s) - y_{01} = a\bar{y}_1(s) + b\bar{y}_2(s)$

$s\bar{y}_2(s) - y_{02} = c\bar{y}_1(s) + d\bar{y}_2(s)$

which in matrix form is

$$\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}\begin{pmatrix} \bar{y}_1(s) \\ \bar{y}_2(s) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}. \quad (72)$$

The evaluation for $\bar{y}_1(s)$ *and* $\bar{y}_2(s)$ using the Crammer's rule we have

$$\Delta = (s-a)(s-d) - bc. \quad (73)$$

$$\Delta_1 = y_{01}(s-d) + by_{02}. \quad (74)$$

$$\Delta_2 = (s-a)y_{02} + cy_{01}. \quad (75)$$

Thus,

$$\bar{y}_1(s) = \frac{\Delta_1}{\Delta} = \frac{y_{01}(s-a)+cy_{01}}{(s-a)(s-d)-bc} \quad (76)$$

Expanding equation (72), for Δ we have

$$\Delta = (s-a)(s-d) - bc = s^2 - sd - sa + ad - bc$$
$$= s^2 - (a+d)s + (ad-bc) \quad (77)$$

The eigen-values of the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\lambda_1 = \frac{(a+d)+\sqrt{D}}{2} \text{ and } \lambda_2 = \frac{(a+d)-\sqrt{D}}{2} \text{ respectively,} \quad (78)$$

where

$$D = (a+d)^2 - 4(ad-bc). \quad (79)$$

Therefore sum of roots,

$$\lambda_1 + \lambda_2 = a + d, \quad (80)$$

and product of roots

$$\lambda_1\lambda_2 = (ad-bc) \quad (81)$$

Clearly

$$\Delta = (s-\lambda_1)(s-\lambda_2). \quad (82)$$

The application of equation (82) to equation (76) yields

$$\bar{y}_1(s) = \frac{(s-d)y_{01}+by_{02}}{(s-\lambda_1)(s-\lambda_2)} \quad \text{and} \quad \bar{y}_2(s) = \frac{(s-a)y_{02}+cy_{01}}{(s-\lambda_1)(s-\lambda_2)}. \quad (83)$$

Results for the Equilibrium of Price Influenced by Government Intervention a_1 Only

In this case the homogeneous initial value problem is:

$$P''(t) + a_1P'(t) + a_0P(t) = 0, \quad P(0) = \mu \text{ and } P'(0) = \xi$$

and to put the differential equation in matrix form we set:

$$P''(t) = -a_1P'(t) - a_0P(t).$$

Let $P'(t) = u$

then

$$u'(t) = -a_0P(t) - a_1u$$

and so the system in matrix form becomes

$$\begin{pmatrix} P \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}\begin{pmatrix} P \\ u \end{pmatrix}. \quad (84)$$

Comparing equation (84) with

$$Y' = AY, \; Y(0) = Y_1,$$

we have

$$Y = \begin{pmatrix} P \\ u \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, Y(0) = \begin{pmatrix} \mu \\ \xi \end{pmatrix} = \begin{pmatrix} P(0) \\ u(0) \end{pmatrix}$$

and the characteristic polynomial is given by the quadratic equation

$$\lambda^2 + a_1\lambda + a_0 = 0$$

whose solutions are

$$\left.\begin{aligned} \lambda_1 &= -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_0} \\ \lambda_2 &= -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_0}\,, \\ a_0 &\neq \left(\frac{a_1}{2}\right)^2 \end{aligned}\right\}. \quad (85)$$

Then

$\lambda_1 - \lambda_2 = \sqrt{a_1^2 - 4a_0}$

$\lambda_2 - \lambda_1 = -\sqrt{a_1^2 - 4a_0}$.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$a = 0, b = 1, c = -a_0, d = -a_1$,

then

$$\lambda_1 - a = -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_0}$$

$$\begin{aligned} \lambda_1 - d &= -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_0} - a_1 \\ &= \frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_0} \\ &= \frac{1}{2}\left[a_1 + \sqrt{a_1^2 - 4a_0}\right] \end{aligned}$$

$$\lambda_2 - a = -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_0}$$

$$\begin{aligned} \lambda_2 - d &= -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_0} + a_1 \\ &= \frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_0} \\ &= -\frac{1}{2}\left[a_1 - \sqrt{a_1^2 - 4a_0}\right] \end{aligned}$$

The Non-Homogeneous Case

Theorem 8:

Let $Y' = AY + F$

be a system of two autonomous non-homogenous first-order differential equations with

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}, F = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$

$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$, a, b, c, d are constants,

then, expanding $Y' = AY + F$ yields

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t) + f_1(t) \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t) + f_2(t) \end{aligned}\right\}. \quad (86)$$

If λ_1 *and* λ_2 are distinct eigenvalues of A, then

$$y_1(t) = \left[\frac{y_{01}(\lambda_1 - d) + by_{02}}{(\lambda_1 - \lambda_2)} + \frac{\overline{f}_1(\lambda_1)(\lambda_1 - d) + b\overline{f}_2(\lambda_1)}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} + \left[\frac{y_{01}(\lambda_2 - d) + by_{02}}{(\lambda_2 - \lambda_1)} + \frac{\overline{f}_1(\lambda_1)(\lambda_2 - d) + b\overline{f}_2(\lambda_2)}{(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t} \quad (87)$$

and

$$y_2(t) = \left[\frac{y_{02}(\lambda_1 - d) + cy_{02}}{(\lambda_1 - \lambda_2)} + \frac{\overline{f}_1(\lambda_1)(\lambda_1 - d) + c\overline{f}_2(\lambda_1)}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} +$$

$$\left[\frac{y_{02}(\lambda_2-d)+cy_{01}}{(\lambda_2-\lambda_1)}+\frac{\overline{f}_1(\lambda_2)(\lambda_2-d)+c\overline{f}_2(\lambda_2)}{(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} \quad (88)$$

Hence, if $\lambda_1 < 0 \; and \; \lambda_2 < 0, \;\; (\lambda_1 \neq \lambda_2)$
then,
$\lim_{t\to\infty} y_1(t) = 0 \; and \; \lim_{t\to\infty} y_2(t) = 0$. in this case, the system is asymptotically stable.

Proof of Theorem 8

The forced volatility case: the case with force non-zero markets force, $f(t)$.
from (86) , we have the non-homogeneous system:

$$\left.\begin{array}{l} y_1'(t) = ay_1(t) + by_2(t) + f_1(t), \;\; y_1(0) = y_{01} \\ y_2'(t) = cy_1(t) + dy_2(t) + f_2(t), \;\; y_2(0) = y_{02} \end{array}\right\}. \quad (89)$$

Taking the Laplace transform of equation (89), we get

$$\left.\begin{array}{l} \mathcal{L}\{y_1'(t)\} = \mathcal{L}\{ay_1(t) + by_2(t) + f_1(t)\} \\ \mathcal{L}\{y_2'(t)\} = \mathcal{L}\{cy_1(t) + dy_2(t) + f_2(t)\} \end{array}\right\}. \quad (90)$$

By equation (24), and applying the initial conditions we have;

$$\left.\begin{array}{l} (s-a)\bar{y}_1(s) - b\bar{y}_2(s) = y_{01} + \bar{f}_1(s) \\ -c\bar{y}_1(s) + (s-d)\bar{y}_2(s) = y_{02} + \bar{f}_2(s\,) \end{array}\right\}, \quad (91)$$

which in matrix form is

$$\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}\begin{pmatrix} \bar{y}_1(s) \\ \bar{y}_2(s) \end{pmatrix} = \begin{pmatrix} y_{01} + \bar{f}_1(s) \\ y_{02} + \bar{f}_2(s) \end{pmatrix}. \quad (92)$$

$\bar{y}_1(s)$ and $\bar{y}_2(s)$are obtained using Crammer's rule to be

$$\left.\begin{array}{l} \bar{y}_1(s) = \frac{\Delta_1}{\Delta} = \frac{y_{01}(s-d)+by_{02}+\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{s^2-(a+d)s+ad-bc} \\ \bar{y}_2(s) = \frac{\Delta_2}{\Delta} = \frac{y_{02}(s-a)+cy_{01}+\bar{f}_1(s)(s-a)+b\bar{f}_1(s)}{s^2-(a+d)s+ad-bc} \end{array}\right\}. \quad (93)$$

By the same procedure used in equations (80)-(82) we

$$\Delta = (s-\lambda_1)(s-\lambda_2)$$

from which the values of $\bar{y}_1(s)$ and $\bar{y}_2(s)$ are obtained as given below,

$$\bar{y}_1(s) = \frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)}, \quad (94)$$

and

$$\bar{y}_2(s) = \frac{y_{02}(s-a)+cy_{01}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_2(s)(s-a)+c\bar{f}_1(s)}{(s-\lambda_1)(s-\lambda_2)}. \quad (95)$$

Results for the Equilibrium of Price influenced by Government Intervention and other Market Forces

This section investigated the equilibrium dynamic price when the price system is subjected to aggregate mixed external market forces of the type $F(t)$ given the form;

$$F(t) = T\delta(t-a) + Q[H_\sigma(t) - H_{\alpha\sigma}(t)] + R_1H_q(t) + R_2H_{\frac{3}{2}q}(t) + R_3H_{2q}(t), \;\; \alpha > 1. \quad (96)$$

The terms have the following meanings:

i) $T\delta(t-a)$means a market force of magnitude T that is impulsive; and lingers for a very short time. This type was witnessed during the covid-19 lockdown in 2020.

ii) $Q[H_\sigma(t) - H_{\alpha\sigma}(t)]$means a market force of magnitude Q that persists during the time interval $\sigma \leq t \leq \alpha\sigma, \;\; \alpha > 1$ and stops thereafter. It is a positive square wave-like force which debilitates prices.

iii) $R_i H_{\beta e}(t),\ i = 1,2,3 \quad \beta = 1, \frac{3}{2}, 2$ represents market forces for which

$R_1 H_q(t)$means the market force of magnitude R_1 that starts at time $t = q$ and stops at time $t = \frac{3}{2}q$

$R_2 H_{\frac{3}{2}q}(t)$means a market force of magnitude R_1 that begins at time $t = \frac{3}{2}q$ and ends at $t = 2q$.

$R_3 H_{2q}(t)$means a market force of magnitude R_3 that begins at time $t = 2q$ and persists for a long time. These forces are the type experienced in the public power sector in Nigeria, where $R_1 > R_2 > R_3$ (ever-increasing). It is also the type that models the OPEC price of crude oil. In the case of OPEC, oil prices could fall, in which case one R_i could be less than an R_j.

The appropriate non-homogeneous system to be used to study the presence of these forces is given in **Theorem 8**.

The matrix form appropriate for the analysis is

$$\begin{pmatrix} P \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} P \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \begin{pmatrix} P(0) \\ u(0) \end{pmatrix} = \begin{pmatrix} \mu \\ \xi \end{pmatrix}. \tag{97}$$

The Laplace transform of $f(t)$ is obtained as follows;

$$L\big(f(t)\big) = L\big(T\delta(t-a)\big) + L(Q[H_b(t) - H_{\alpha b}(t)] + L(R_2 H_{\frac{3}{2}q}(t) + R_3 H_{2q}(t), \quad \alpha > 1$$

$$= TL\big(\delta(t-a)\big) + QL\big(H_b(t) - H_{\alpha b}(t)\big) + R_1 L\left(H_q(t)\right) + R_2 L\left(H_{\frac{3}{2}q}(t)\right) + R_3 L(H_{2q}(t))$$

Noting the following

$TL\big(\delta(t-a)\big) = Te^{-as}$; $QL\big(H_b(t) - H_{\alpha b}(t)\big) = \frac{Q}{s}[e^{-bs} - e^{-\alpha bs}], \alpha > 1$; $R_1 L\left(H_q(t)\right) = \frac{R_1}{s} e^{-qs}$; $R_2 L\left(H_{\frac{3}{2}q}(t)\right) = \frac{R_2}{s} e^{-\frac{3}{2}qs}$; $R_3 L\left(R_{2q}(t)\right) = \frac{R_3}{s} e^{-2qs}$,

the Laplace transform of $f(t)$ becomes

$L\big(f(t)\big) = \bar{f}(s),$

where

$$\bar{f}(s) = Te^{-as} + \frac{Q}{s}[e^{-bs} - e^{-\alpha bs}] + \frac{R_1}{s} e^{-qs} + \frac{R_2}{s} e^{-\frac{3}{2}qs} + \frac{R_3}{s} e^{-2qs} + \cdots + R_n e^{-ks}. \tag{98}$$

Because the first and second terms on the right of (4.3.3) are everywhere continuous, though the second term has a removable discontinuity

at $s = 0$,

we write

$\bar{f}(s) = \bar{f}_2(s) = \bar{f}_3(s) + \bar{f}_4(s)$

where

$$\bar{f}_3(s) = Te^{-as} + \frac{Q}{s}[e^{-bs} - e^{-\alpha bs}] \tag{99}$$

which is a continuous function and

$$\bar{f}_4(s) = R_1 e^{-qs} + R_2 e^{-\frac{3}{2}qs} + R_3 e^{-2qs} + \ldots + R_n e^{-kqs} \tag{100}$$

has a simple pole at $s = 0$ with residue given by

$$Res\left(s = 0;\ \bar{f}_4(s)\right) = \lim_{s \to 0} s\, f(s)$$

$$= \lim_{s \to 0} (R_1 e^{-qs} + R_2 e^{-\frac{3}{2}qs} + R_3 e^{-2qs} + \ldots + R_n e^{-ks})$$

$$= R_1 + R_2 + R_3 + \cdots + R_n. \tag{101}$$

R_i is zero as soon as R_{i+1} comes into play.

This is what happens in public electricity charges or even in oil prices; the old price is discarded as soon as new prices are pronounced, therefore it is R_n that remains in equation (101). This makes

$\bar{f}_4(s) = \frac{1}{s} R_n e^{-ks}$

where k is a highest real number associated with the last increase in price.

$\bar{f}_4(s)$ has a simple pole at $s = 0$ and the solution of the non-homogenous ordinary differential equation is obtainable from (64) and (65)

$\bar{f}_1(s) = 0$ *and* $\bar{f}_2(s) = f_3(s) + \bar{f}_4(s)$

with

$y_1(t) = P(t), y_{01} = \mu, y_{02} = \xi,\ u(t) = P'(t),$

The Non-Homogeneous Case

Theorem 9

Let $Y' = AY + F$

be a system of two autonomous non-homogenous first-order differential equations with

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix},\ F = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$

$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix},$ a, b, c, d are constants.

The expansion of $Y' = AY + F$ yields

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= ay_1(t) + by_2(t) + f_1(t) \\ \frac{dy_2(t)}{dt} &= cy_1(t) + dy_2(t) + f_2(t) \end{aligned}\right\}. \tag{102}$$

If λ_1 *and* λ_2 are distinct eigen-values of A, then

$$\begin{aligned} y_1(t) = &\left[\frac{y_{01}(\lambda_1 - d) + by_{02}}{(\lambda_1 - \lambda_2)} + \frac{\bar{f}_1(\lambda_1)(\lambda_1 - d) + b\bar{f}_2(\lambda_1)}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} + \\ &\left[\frac{y_{01}(\lambda_2 - d) + by_{02}}{(\lambda_2 - \lambda_1)} + \frac{\bar{f}_1(\lambda_1)(\lambda_2 - d) + b\bar{f}_2(\lambda_2)}{(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t}, \end{aligned} \tag{103}$$

and

$$\begin{aligned} y_2(t) = &\left[\frac{y_{02}(\lambda_1 - d) + cy_{02}}{(\lambda_1 - \lambda_2)} + \frac{\bar{f}_1(\lambda_1)(\lambda_1 - d) + c\bar{f}_2(\lambda_1)}{(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} + \\ &\left[\frac{y_{02}(\lambda_2 - d) + cy_{01}}{(\lambda_2 - \lambda_1)} + \frac{\bar{f}_1(\lambda_2)(\lambda_2 - d) + c\bar{f}_2(\lambda_2)}{(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t}. \end{aligned} \tag{104}$$

Hence, if $\lambda_1 < 0$ *and* $\lambda_2 < 0$, $(\lambda_1 \neq \lambda_2)$

then,

$\lim_{t\to\infty} y_1(t) = 0$ *and* $\lim_{t\to\infty} y_2(t) = 0$. in this case, the system is asymptotically stable.

Proof of Theorem 9

From the forced volatility case (the case with force non-zero markets force) $f(t)$ and adopting the method leading to (91) and (102), we get

$$\left.\begin{aligned} (s - a)\bar{y}_1(s) - b\bar{y}_2(s) &= y_{01} + \bar{f}_1(s) \\ -c\bar{y}_1(s) + (s - d)\bar{y}_2(s) &= y_{02} + \bar{f}_2(s\,) \end{aligned}\right\}, \tag{105}$$

which in matrix form is

$$\begin{pmatrix} s-a & -b \\ -c & s-d \end{pmatrix}\begin{pmatrix} \bar{y}_1(s) \\ \bar{y}_2(s) \end{pmatrix} = \begin{pmatrix} y_{01}+\bar{f}_1(s) \\ y_{02}+\bar{f}_2(s) \end{pmatrix}. \quad (106)$$

Using Crammer's rule $\bar{y}_1(s)$ and $\bar{y}_2(s)$ were evaluated to,

$$\left.\begin{aligned} \bar{y}_1(s) &= \frac{\Delta_1}{\Delta} = \frac{y_{01}(s-d)+by_{02}+\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{s^2-(a+d)s+ad-bc} \\ \bar{y}_2(s) &= \frac{\Delta_2}{\Delta} = \frac{y_{02}(s-a)+cy_{01}+\bar{f}_1(s)(s-a)+b\bar{f}_1(s)}{s^2-(a+d)s+ad-bc} \end{aligned}\right\}. \quad (107)$$

Using the same procedure as in (80)-(82)

$\Delta = (s-\lambda_1)(s-\lambda_2)$,

which leads to

$$\left.\begin{aligned} \bar{y}_1(s) &= \frac{y_{01}(s-d)+by_{02}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_1(s)(s-d)+b\bar{f}_2(s)}{(s-\lambda_1)(s-\lambda_2)} \\ \bar{y}_2(s) &= \frac{y_{02}(s-a)+cy_{01}}{(s-\lambda_1)(s-\lambda_2)} + \frac{\bar{f}_2(s)(s-a)+c\bar{f}_1(s)}{(s-\lambda_1)(s-\lambda_2)} \end{aligned}\right\}. \quad (108)$$

5.0 Conclusion

This study analyzes the stability of market price equilibrium using a second-order differential equation framework. It incorporates the influence of government intervention and external market forces. The key finding is that for the system to remain stable, the government's moderating influence (represented by a constant) needs to be greater than twice the square root of the price's resistance to collapse (represented by another constant). This result aligns with previous findings in mechanical systems by Boyce and Diprima (1977).

The manuscript extends prior research by Espinoza (2009) and Bob Foster (2016) by introducing time-dependent external influences on the price dynamics. The analysis utilizes a matrix-theoretic approach, ensuring stability conditions for both price and its rate of change. The key takeaway is that a sufficiently strong government moderating force is crucial for achieving and maintaining a stable market equilibrium. The framework also acknowledges the impact of impulsive market forces and persistent market forces on the system's stability.

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This study was carried out in collaboration among the authors. Authors Augustine Osondu Friday Ador and Bright O. Osu designed the study, carried-out the analysis, investigated the basic properties and the first draft of the manuscript. Authors Silas Abahia Ihedioha, Isaac Mashingil Mankili and Franka Amaka Nwafor conducted the analyses of the study and handled the literature reviews and wrote the manuscript. All authors read and approved the final manuscript.