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GENERALIZED DISTRIBUTION FOR BI-UNIVALENT FUNCTIONS DEFINED BY ERROR AND POISSON DISTRIBUTION VIA BELL NUMBER

[1] Gbolagade, A. M. and [2] Awolere, I. T.
[1]Department of Mathematics and Computing Science Education, Emmanuel Alayande University of Education, Oyo
[2]Department of Mathematical Sciences, Olusegun Agagu University of Science and Technology, Okitipupa, Ondo State, Nigeria
Corresponding Author's E-mail: gbolagadeam@eauedoyo.edu.ng
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Abstract
In the field of geometric function theory, generalized distributions have revealed novel insights and applications, particularly in understanding the behaviour of various complex functions. This paper focuses on estimating bounds for bi-univalent functions within probability distribution series defined by error and Poisson distributions, particularly in relation to the Bell numbers. These distributions are utilized to establish coefficient bounds, which hold significance for both the structural properties of bi-univalent functions and their applications in probability theory. By extending these methods, the study contributes to the broader framework of geometric function theory, where probability distributions offer new tools to analyze and interpret functional bounds. The findings have potential implications in areas requiring complex function estimation, including mathematical physics and statistical modeling.
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Introduction

Let Γ represents the class of analytic functions of the form

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+...(z\in\Delta,\Delta=\{z\in C:|z|<1\}) \tag{1}$$

and H denote the subclass of Γ, which are normalized by the condition

$f(0)=0$ and $f'(0)-1=0$.

Recall the quantities

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\eta \quad \text{and} \quad \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\eta$$

which are two familiar subclasses of univalent functions denoted by S and are known as starlike function of order $\eta(0\le\eta<1)$ and convex function of order η respectively.

Historically, Babalola (2013) was the first person to define a new subclass of λ-pseudostarlike function of order η satisfying the analytic condition

$$\mathrm{Re}\left(\frac{zf'(z)^{\lambda}}{f(z)}\right)>\eta$$

In fact, ever since the publication by Babalola (2013), a huge flood of publications has appeared and still appearing in the literature dealing with various subclasses of λ-pseudo starlike function (see, for details Awolere and Oladipo (2019), Awolere and Ibrahim-Tiamiyu (2017)).

Let the functions f and h be analytic in Δ, then f is said to be subordinate to h written as $f(z)\prec h(z)(z\in\Delta)$ if there exist a Schwarz function $w\in v$ of the form

$$w(z)=w_1z+w_2z^2+w_3z^3+w_4z^4+... \tag{2}$$

And $v=\{w:w(0)=0,\ |w(z)|<1,\ z\in\Delta\}$

such that

$f(z)=h(w(z)) \quad (z\in\Delta)$.

To be specific, if h is univalent in Δ, the conditions $f(0)=h(0),\ f(\Delta)\subset h(\Delta)$ will be identical to the above stated subordination condition.

A function $f\in\Gamma$ is bi-univalent in Δ if f^{-1} exists and it is univalent in Δ. The concept of bi-univalent was introduced in Lewin (1967) and coefficient bounds was estimated. Other researchers, the likes of Netanyahu (1969), Strivastava et al. (2010), Strivastava et al. (2013), Frasin et. al. (2011), Murusundamorthy (2015, 2017), Murusundamorthy et. al. (2015). Shuhai et. al. (2015) further the work on bi-univalency and they obtained very useful results, (see also Altınkaya and Yalçın (2017), Awolere and Oladipo (2019), Awolere and Ibrahim-Tiamiyu (2017), Baricz (2006,2008), Bulut and Magesh (2016), Gbolagade et. al. (2024), Hayami and Owa (2012), Laxmi and Sharma (2017), Strivastava et. al. (2010)) and literature therein. Furthermore, we note that for $f\in\Gamma$ there exists f^{-1} (inverse) satisfying

$$f^{-1}(f(z))=z, \quad (z\in\Delta)$$

$$f(f^{-1}(w))=w, \quad \left(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4}\right)$$

and

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4 \quad (3)$$

Here we denote by $\sum$ the class of bi-univalent function.

Generalized distribution which was introduced by Porwal (2018) are objects extending the notions of function, useful in making discontinuous functions more like smooth functions, describing discrete physical phenomena such as points charges and have applications in physics and engineering. He also investigated its geometric properties in relation to univalent functions and he denotes by T the sum of convergent series of the form

$$T=\sum_{k=0}^{\infty}a_k$$

where $a_k\geq 0$ for all $k\in N$. The generalized discrete probability mass function is given as

$$p(k)=\frac{a_k}{T},\ k=0,1,2,...$$

$p(k)$ is the probability mass function because $p(k)\geq 0$ and $\sum_k p_k=1$.

Next, let

$$\Omega(x)=\sum_{k=0}^{\infty}a_kx^k$$

then from $T=\sum_{k=0}^{\infty}a_k$ series Ω is a convergent for $|x|<1$ and $x=1$.

Porwal (2018) provided information on various definitions and derivations for illustration:

1. If X is a discrete random variable that takes values x_1, x_2, ... associated with probabilities $p_1, p_2,...$ then the expected X denoted by $E(X)$ is defined by

$$E(X)=\sum_{k=0}^{\infty}p_kx_k$$

2. The moment of a discrete probability distribution r^{th} about $x=0$ is defined by

$$\mu_r'=E(X^r)$$

where μ_r' is the mean of the distribution and the variance is given as $\mu_2'-(\mu_1')^2$

3. Moment about origin is given as

$$Mean=\mu_r'=\frac{\Omega'}{T}$$

Variance

$$=\mu_2'-(\mu_1')^2=\frac{1}{T}\left[\Omega''(1)+\Omega'(1)-\frac{(\Omega'(1)^2)}{T}\right].$$

The moment generating function of a random variable X is denoted by $M_x(t)$ and defined by

$$M_x(t)=E(e^{xt})$$

and the moment generating function of generalized discrete probability is given as

$$M_x(t)=\frac{\Omega(e^t)}{T}.$$

For the special value of a_k, various well known discrete probability distributions can be obtained. Presently, we are focusing on the polynomial whose coefficients are probabilities of the generalized distribution introduced and investigated in Porwal (2018) which has the form

$$K_\psi(z)=z+\sum_{k=2}^{\infty}\frac{a_{k-1}}{T}z^k \quad (4)$$

where $T=\sum_{k=0}^{\infty}a_k$, $a_k\geq 0$ for all $k\in N$.

A variable X is said to be Poisson distributed if it takes the values *0, 1, 2, 3, 4, ...* with the probabilities

$$e^{-m},\frac{m^2e^{-m}}{2!},\frac{m^3e^{-m}}{3!},\frac{m^4e^{-m}}{4!}\ldots$$

respectively, where *m* is called the parameter. Thus

$$p(X=r)=\frac{m^re^{-m}}{r!},\ r=0,1,2,3,4,\ldots$$

Very recent, Murugusundaramoorthy (2017), Porwal (2014, 2016), Porwal et. al. (2016) introduced a power series whose coefficients are probabilities of Poisson distribution

$$K(m,z)=z+\sum_{k=2}^{\infty}\frac{m^{k-1}e^{-m}}{(k-1)!}z^k,\ z\in\Delta \quad (5)$$

where $m>0$, which is also reported by Frazin (2019).

In real life situations, (5) has many applications as appear in literature. It's useful for the control of software defects, modelling of distribution of overlapping word occurrences, modelling of DNA substitution and has application in traffic accident data in Anwar and Ahmad (2014).

Special functions such as activation, error and Bessel functions defined by Ʊ have been studied to establish certain geometric properties like univalency, starlikeness and convexity in several publications [see Altınkaya and Olatunji (2019), Baricz (2006, 2008)]. But concern of this paper is that of error function which was normalized by Ramanchandran et. al (2017) as

$$Er\ f(z)=\frac{\sqrt{\pi z}}{2}erf(\sqrt{z})=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(k-1)!}z^k \quad (6)$$

Originally before normalization the error function is written by Abramowitz and Stegun (1965) as

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z\exp(-t^2dt)=\frac{2}{\sqrt{\pi}}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}z^{2k+1}}{(2k+1)k!} \quad (7)$$

The function defined in (7) also is important in estimating the probability of observing a particle in a specified region by Abramowitz and Stegun (1965). The error function, which appears frequently in heat conduction and diffusion is a part of transport phenomena that deal with the flow of physical quantity in a medium. This has significance applications in many disciplines like physics, chemistry, biology, thermo mechanics and mass flow. See Alzer (2010) for properties and inequalities of error function and Elbert and Laforgia (2008) for the properties of complementary error functions.

The motivation of this work is to define a new subclass of bi-univalent functions in terms of error and Poisson distribution functions associated with Bell numbers based on subordination principle and the coefficient bound would be estimated. Varying the parameters, corollary will be established

Method and Tools of Estimation

For $h(z)\in\Gamma$, given by $h(z)=z+h_1z^2+h_2z^3+\ldots$, the Hadarmard product (or convolution) of $f(z)$ and $h(z)$ is defined

$$(f*h)(z)=z+\sum_{k=2}^{\infty}a_k h_k=(h*f)(z) \quad (8)$$

Furthermore, the application of concept of convolution defined in (8) to (7) and using the functions of the forms (4) and (5) yields

$$K\,Er\ f_{\varphi}(z,m)=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}m^{k-1}e^{-m}}{(2k-1)(k-1)!^{2}}\frac{a_{k-1}}{H}z^{k} \quad (9)$$

For a fixed non-negative integer *k*, the Bell numbers B_k count the possible disjoint partitions of a set with *n* elements into non-empty subsets or, equivalently, the number of equivalence relations on it. The Bell numbers B_n satisfy a recurrence relation involving binomial coefficients $B_{k+1}=\sum_{n=0}^{k}C_n B_k$. Obviously, $B_0=B_1=1,\ B_2=2,\ B_3=5,\ B_4=15,\ B_5=52,\ B_6=203.$ For details (see Bell (1934, 1938), Canifield (1995), Najafzadeh et. al. (2022), Oyekan et. al. (2023), Qi (2017)). Kumar et al. (2019) investigated the function

$$Q(z)=e^{e^{z}-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!}=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\ldots \quad (10)$$

which is starlike with respect to unity and having Bell number as its coefficients. In this investigation, we employ the concept of subordination principle to define our class of analytic functions motivated largely by recent work of Altinkaya and Olatunji (2019), Oladipo (2019) and previous works of Babalola (2013), and that of Awolere et. al. (2019). Hence, the present work investigates bounds for bi-univalent function for generalized distribution associated with error function and Poisson distribution via Bell number.

Lemma 1 [Jahangiri et. al. (2018)]: Let $w(z)=w_1z+w_2z^2+w_3z^3+w_4z^4+\ldots\in\Gamma$ be so that $|w(z)|<1$ in ε. If t is a complex number, then $\left|w_{2\backslash}+tw_1^2\right|\le\max(1,|t|)$.

The inequality is sharp for the function $w(z)=z$ and $w(z)=z^2$.

Thus, we will obtain the coefficients bounds $\left|\frac{a_1}{H}\right|$ and $\left|\frac{a_2}{H}\right|$ for class $P_oE_{\mathrm{T}}^{\lambda}(m,\Lambda,b,Q)$ which defined in definition 1.

Definition 1: For real numbers $0\le\Lambda<1,\ \lambda\ge1, m>0, b\ne0$ and $Q(z)$ as defined by (1) one may say that $KEr\ f_{\varphi}\in\Omega$ is in the class $P_oE_{\mathrm{T}}^{\lambda}(m,\Lambda,b,Q)$ if

$$1+\frac{1-\Lambda}{b}\left\{\frac{\left[\left(zKErf_{\varphi}(z,m)\right)'\right]^{\lambda}}{KErf_{\varphi}(z,m)}-1\right\}\prec Q(z) \quad (11)$$

and

$$1+\frac{1-\Lambda}{b}\left\{\frac{w\left[\left(KErf_{\varphi}^{-1}(w,m)\right)'\right]^{\lambda}}{KErf_{\varphi}^{-1}(w,m)}-1\right\}\prec Q(w) \quad (12)$$

we note that by varying different values of λ, m, Λ and b in the above definition, we can obtain the following subclasses.

i. $E_r^{I}(m,\Lambda,b,Q)=E_r(m,\Lambda,b,Q)$

$$1+\frac{1-\Lambda}{b}\left\{\frac{z\left(KErf_{\varphi}(z,m)\right)'}{KErf_{\varphi}(z,m)}-1\right\}\prec Q(z)$$

and

$$1+\frac{1-\Lambda}{b}\left\{\frac{w\left(KErf_{\varphi}^{-1}(w,m)\right)'}{KErf_{\varphi}^{-1}(w,m)}-1\right\}\prec Q(w)$$

ii. $E_r^{\lambda}(m,0,b,Q)=E_r^{\lambda}(m,b,Q)$

$$1+\frac{1}{b}\left\{\frac{\left[\left(z\,KErf_{\varphi}(z,m)\right)'\right]^{\lambda}}{KErf_{\varphi}(z,m)}-1\right\}\prec Q(z)$$

and

$$1+\frac{1}{b}\left\{\frac{w\left[\left(KErf_{\varphi}^{-1}(w,m)\right)'\right]^{\lambda}}{KErf_{\varphi}^{-1}(w,m)}-1\right\}\prec Q(w)$$

iii. $E_r^{\lambda}(1,0,b,Q)=E_r^{\lambda}(\Lambda,b,Q)$

$$1+\frac{1}{b}\left\{\frac{\left[z\left(KErf_{\varphi}(z)\right)'\right]^{\lambda}}{KErf_{\varphi}(z,m)}-1\right\}\prec Q(z)$$

and

$$1+\frac{1}{b}\left\{\frac{w\left[\left(KErf_{\varphi}^{-1}(w)\right)'\right]^{\lambda}}{KErf_{\varphi}^{-1}(w)}-1\right\}\prec Q(w)$$

Results

Theorem 1: Let $b\neq 0$ and $Q(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\ldots$ If $KErf_{\varphi}$ given by (9) belongs to the class $P_oE_{\mathrm{T}}^{\lambda}(m,\Lambda,b,Q)$, then we get

$$\left|\frac{a_1}{H}\right|\leq \min\left\{\frac{3|b|e^m}{(1-\Lambda)(2\lambda-1)m},\frac{3|b|\sqrt{2e^m}}{(1-\Lambda)(2\lambda-1)m},A\right\} \tag{13}$$

$$\left|\frac{a_2}{H}\right|\leq \min\left\{\left(\frac{6\sqrt{5}|b|e^m}{(1-\Lambda)(2\lambda-1)m}\right)^2+\frac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2},B\right\} \tag{14}$$

where

$$A=\frac{6\sqrt{5}|b|e^m}{m\sqrt{\left|\begin{array}{l}20(1-\Lambda)\left[b\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right]+\\ 9b(1-\Lambda)(3\lambda-1)e^m\end{array}\right|}}$$

$$B=\frac{180|b|^2e^{2m}}{\left|20(1-\Lambda)\left[b\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right]+9b(1-\Lambda)(3\lambda-1)e^m\right|}+\frac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2}$$

Proof: Let $KErf_{\varphi}\in P_oE_{\mathrm{T}}^{\lambda}(m,\Lambda,b,Q)$ then there exist two Schwarz functions $u,v\in\Gamma$ of the form (2) such that

$$1+\frac{1-\Lambda}{b}\left\{\frac{\left[\left(z\,KErf_{\varphi}(z,m)\right)'\right]^{\lambda}}{KErf_{\varphi}(z,m)}-1\right\}=Q(u(z)) \tag{15}$$

and

$$1+\frac{1-\Lambda}{b}\left\{\frac{w\left[\left(KErf_{\varphi}^{-1}(w,m)\right)'\right]^{\lambda}}{KErf_{\varphi}^{-1}(w,m)}-1\right\}=Q(v(w)) \tag{16}$$

By (15) and (16) we observe that

$$(1-\Lambda)(2\lambda-1)E_2\frac{a_1}{H}=bu_1, \tag{17}$$

$$(1-\Lambda)(3\lambda-1)E_3\frac{a_2}{H}+(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2\frac{a_1^2}{H^2}=bu_2+bu_1^2 \tag{18}$$

$$-(1-\Lambda)(2\lambda-1)E_2\frac{a_1}{H}=bv_1 \tag{19}$$

$$\left[(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+2(1-\Lambda)(3\lambda-1)E_3\right]\frac{a_1^2}{H^2}-(1-\Lambda)(3\lambda-1)E_3\frac{a_2}{H}=bv_2+bv_1^2 \tag{20}$$

where

$$A_k=\frac{(-1)^{k-1}m^{k-1}e^{-m}}{(2k-1)(k-1)!^2} \tag{21}$$

From (17) and (19) we get

$$u_1=-v_1 \tag{22}$$

And

$$2(1-\Lambda)^2(2\lambda-1)^2E_2^2\frac{a_1^2}{H^2}=b^2\left(u_1^2+v_1^2\right) \tag{23}$$

Also by (18) and (20) we establish that

$$\left\{2(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+2(1-\Lambda)(3\lambda-1)E_3,\right\}\frac{a_1^2}{H^2}=b\left(u_2+v_2\right)+b\left(u_1^2+v_1^2\right) \tag{24}$$

Therefore by (17), (22). (23) and (24) we find out that

$$\frac{a_1}{H}=\frac{bu_1}{(1-\Lambda)(2\lambda-1)E_2} \tag{25}$$

$$\frac{a_1^2}{H^2}=\frac{b^2\left(u_1^2+v_1^2\right)}{2(1-\Lambda)^2\left(4\lambda^2-4\lambda+1\right)E_2^2\Lambda} \tag{26}$$

$$\frac{a_1^2}{H^2}=\frac{b^2\left(u_2+v_2\right)}{2\left\{\begin{array}{l} b(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+b(1-\Lambda)(3\lambda-1)E_3 \\ -(1-\Lambda)^2(2\lambda-1)E_2^2 \end{array}\right\}} \tag{27}$$

Since $|u_i|\le 1$ and $|v_i|\le 1$ from Lemma 1 and (21), it follows from (25), (26) and (27) that

$$\left|\frac{a_1}{H}\right|\le\frac{3|b|e^m}{(1-\Lambda)(2\lambda-1)m}$$

$$\left|\frac{a_1}{H}\right|\le\frac{3|b|\sqrt{2}e^m}{(1-\Lambda)(2\lambda-1)m}$$

$$\left|\frac{a_1}{H}\right|\le\frac{6\sqrt{5}|b|e^m}{m\sqrt{\left|\begin{array}{l} 20(1-\Lambda)\left[b\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right] \\ +9b(1-\Lambda)(3\lambda-1)e^m \end{array}\right|}}$$

which gives us the desired estimate on $\frac{a_1}{H}$ as asserted by (13).

Next, by (18), (20), (22) and (23) we have

$$\frac{a_2}{H}=\frac{b^2\left(u_1^2+v_1^2\right)}{2(1-\Lambda)^2(2\lambda-1)^2E_2^2}+\frac{b\left(u_2-v_2\right)}{2(1-\Lambda)(3\lambda-1)E_3} \tag{28}$$

Also, from (18), (20), (22) and (27) we obtain

$$\frac{a_2}{H}=\frac{b^2\left(u_2+v_2\right)}{2\left\{b(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+b(1-\Lambda)(3\lambda-1)E_3-(1-\Lambda)^2(2\lambda-1)^2E_2^2\right\}}+\frac{b\left(u_2+v_2\right)}{2(1-\Lambda)(3\lambda-1)E_3} \tag{29}$$

Once again, since $|u_i|\le 1$ and $|v_i|\le 1$ from Lemma 1 and using (21), it infers from (28) and (29) that

$$\left|\frac{a_2}{H}\right|\le\left(\frac{6\sqrt{5}|b|e^m}{(1-\Lambda)(2\lambda-1)m}\right)^2+\frac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2} \tag{30}$$

and

$$\left|\frac{a_2}{H}\right|\le\frac{180|b|^2e^{2m}}{\left|20(1-\Lambda)\left[b\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right]+9b(1-\Lambda)(3\lambda-1)e^m\right|}+\frac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2} \tag{31}$$

Corollary 1: Let $b\neq 0$ and $Q(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\ldots$ If $KErf_\varphi$ given by (9) belongs to the class $P_oE_{\mathrm{T}}^{\lambda}(m,0,b,Q)$, then we get

$$\left|\frac{a_1}{H}\right|\le\min\left\{\frac{3|b|e^m}{(2\lambda-1)m},\frac{3|b|\sqrt{2}e^m}{(2\lambda-1)m},A_1\right\} \tag{32}$$

$$\left|\frac{a_2}{H}\right|\le\min\left\{\left(\frac{6\sqrt{5}|b|e^m}{(2\lambda-1)m}\right)^2+\frac{20|b|e^m}{(3\lambda-1)m^2},B_1\right\} \tag{33}$$

where

$$A_1=\frac{6\sqrt{5}|b|e^m}{m\sqrt{\left|20\left[b\left(2\lambda^2-4\lambda+1\right)-\left(4\lambda^2-4\lambda+1\right)\right]+9b(3\lambda-1)e^m\right|}}$$

$$B_1=\frac{180|b|^2e^{2m}}{\left|20\left[b\left(2\lambda^2-4\lambda+1\right)-\left(4\lambda^2-4\lambda+1\right)\right]+9b(3\lambda-1)e^m\right|}+\frac{20|b|e^m}{(3\lambda-1)m^2}$$

Theorem 2: Let $b\neq 0$ and $Q(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\ldots$ If $KErf_\varphi$ given by (9) belongs to the class $P_oE_{\mathrm{T}}^{\lambda}(m,\Lambda,b,Q)$, then we get

$$\left|\frac{a_2}{H}-\varphi\frac{a_1^2}{H^2}\right|\le\begin{cases}\dfrac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2}, & |\varphi-1|\le M\\[2ex] \dfrac{180|b|^2|\varphi-1|e^{2m}}{\left|20m^2b(1-\Lambda)\left[\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right]+9bm^2(1-\Lambda)(3\lambda-1)e^m\right|}, & |\varphi-1|\ge M\end{cases}$$

where

$$M=\frac{\left|\begin{array}{l}20m^2b(1-\Lambda)\left[\left(2\lambda^2-4\lambda+1\right)-(1-\Lambda)\left(4\lambda^2-4\lambda+1\right)\right]+\\ 9bm^2(1-\Lambda)(3\lambda-1)e^m\end{array}\right|}{9m^2|b|(3\lambda-1)e^m}$$

Proof: From Theorem 1, we have

$$\frac{a_2}{H}-\varphi\frac{a_1^2}{H^2}=\frac{(1-\varphi)b^2(u_2+v_2)}{2\left\{\begin{array}{l}b(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+b(1-\Lambda)(3\lambda-1)E_3\\ -(1-\Lambda)^2(2\lambda-1)^2E_2^2\end{array}\right\}}+\frac{b(u_2-v_2)}{2(1-\Lambda)(3\lambda-1)E_3}$$

$$=b\left[\left(h(\phi)+\frac{1}{2(1-\Lambda)(3\lambda-1)E_3}\right)u_2+\left(h(\phi)-\frac{1}{2(1-\Lambda)(3\lambda-1)E_3}\right)v_2\right]$$

where

$$h(\varphi)=\frac{(\varphi-1)b}{2\left\{\begin{array}{l}b(1-\Lambda)\left(2\lambda^2-4\lambda+1\right)E_2^2+b(1-\Lambda)(3\lambda-1)\\ E_3-(1-\Lambda)^2(2\lambda-1)^2E_2^2\end{array}\right\}}$$

Then in view of (21), we can establish that

$$\left|\frac{a_2}{H}-\phi\frac{a_1^2}{H^2}\right|\le\begin{cases}\dfrac{20|b|e^m}{(1-\Lambda)(3\lambda-1)m^2}, & 0\le h(\phi)\le\dfrac{10e^m}{(1-\Lambda)(3\lambda-1)m^2}\\[2ex] 2|b||h(\phi)|, & |h(\phi)|\ge\dfrac{10e^m}{(1-\Lambda)(3\lambda-1)m^2}\end{cases}\qquad(34)$$

Corollary 2: Let $b\neq 0$ and $Q(z)=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+\ldots$ If $KErf_\varphi$ given by (9) belongs to the class $P_oE_{\mathrm{T}}^{\lambda}(m,0,b,Q)$, then we get

$$\left|\frac{a_2}{H}-\varphi\frac{a_1^2}{H^2}\right|\le\begin{cases}\dfrac{20|b|e^m}{(3\lambda-1)m^2}, & |\varphi-1|\le M_1\\[2ex] \dfrac{180|b|^2|\varphi-1|e^{2m}}{\left|20m^2b\left[\left(2\lambda^2-4\lambda+1\right)-\left(4\lambda^2-4\lambda+1\right)\right]+9bm^2(3\lambda-1)e^m\right|}, & |\varphi-1|\ge M_1\end{cases}\qquad(35)$$

where

$$M_1=\frac{\left|20m^2b\left[\left(2\lambda^2-4\lambda+1\right)-\left(4\lambda^2-4\lambda+1\right)\right]+9bm^2(3\lambda-1)e^m\right|}{9m^2|b|(3\lambda-1)e^m}$$

Conclusion

Our study introduces a novel application of bi-univalent analytic functions, incorporating convoluted error and Poisson distribution functions to investigate the establishment of coefficient bounds. Notably, our findings align with previous researches by Oyekan et. al. (2023) and Najafzadeh et. al. (2022), demonstrating the robustness of the Bell number framework in unifying these distinct approaches.

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